CATASTROPHE INSURANCE MODELLED WITH SHOT-NOISE PROCESSES.

THORSTEN SCHMIDT

Abstract. Shot-noise processes generalize compound Poisson processes in the following way: a jump (the shot) is followed by a decline (noise). This constitues a useful model for insurance claims in many circumstances: claims due to natural catastrophes or self-exciting processes exhibit similar features. We give a general account of shot-noise processes with time-inhomogeneous drivers and derive a number of useful results for modelling, estimation and pricing with shot-noise processes. Furthermore, we give some closed-form examples which are highly tractable and constitute a useful modelling tool for dynamic claims processes. The results can in particular be used for pricing CAT bonds, a traded risk-linked security.

Keywords: shot-noise processes; tail dependence; catastrophe derivatives; marked point process; minimum-distance estimation; self-exciting processes; CAT bonds

1. Introduction

An insurance company insures occurring claims in exchange for a regular premium. Numerous works study the determination of an optimal premium: for example, the premium should be high enough such that the ruin probability of the insurance company is sufficiently small. The claim sizes itself are often considered to be independent and identically distributed with arrival times being jump times from a Poisson process. A by now classical extension of this model considers renewal times, where the inter-arrival times are no longer exponential.

In this paper we extend this set-up further and study arrival times with random arrival rate. In particular we will consider arrival rates having shot-noise features. This could, for example, be used to model the claims arrivals after a catastrophe in a dynamic way: many claims will be reported right after the catastrophe, such that the arrival rate in the beginning is high. Further claims will be announced later and later corresponding to a decreasing arrival rate. Shot-noise arrival rates directly model such an effect. An alternative application appears when considering claims caused by a flood or hail: they typically admit spatial patterns with a centre where the majority of the claims are located and a decreasing number of claims with increasing distance from the centre. In a life insurance context, a natural catastrophe like a tsunami also leads to a similar patterns.

The main idea we follow here is to give a new view on insurance claims processes inspired by recent results in credit risk. In particular, we propose a model with multiple claim arrivals, i.e. claims can occur at the same time. This is an important issue for catastrophe modelling and for pricing CAT bonds. The size of CAT bond markets has been increasing tremendously over the last decade. Currently, it reaches an all-time high: the outstanding volume hit $19 billion dollars in October 2013[1].

Date: November 15, 2013.

[1]Sources: Artemis (2013) and Insurance Insider (2013).

Besides this, we consider shot-noise processes driven by inhomogeneous Poisson processes, such that seasonal effects can be taken into account.

Shot-noise processes are a well-known and well-studied object. Inspired by physical effects as in Schottky (1918) many applications have been proposed. Many references may be found, for example, in Parzen (1962), Lund et al. (1999) and Kühn (2004), among many others. Applications in the insurance context are given in Mikosch (2009) or in Dassios and Jang (2003). Shot-noise processes in credit risk are treated in Scherer et al. (2012) and in Jang et al. (2011).

More precisely, consider a Poisson process N with jump times $\sigma_1, \sigma_2, \dots$. If $\mathscr{L}$ is a non-decreasing function, then the time-transformed process

$$N(\mathscr{L}(t)), \quad t \geq 0$$

is a inhomogeneous Poisson process if $\mathscr{L}$ is absolutely continuous. If $\mathscr{L}$, however, has jumps, then N has multiple claim arrivals with positive probability. In this case it might happen that $\Delta N_t = N_t - N_{t-} > 1$, i.e. more than one claim arrives at time t.

It turns out that $\mathscr{L}$ can be replaced by a stochastic process, which is non-decreasing, and we will show how to incorporate shot-noise effects in here. The obtained results have a sufficient degree of generality, in particular, we will not need Markovianity of the shot-noise processes.

Besides the claim arrival process, one additionally needs a model for the loss magnitudes. In this article we choose to take i.i.d. claims sizes. This approach can be extended to a setting where all loss magnitudes between two events are stochastically dependent. This can be achieved, e.g., via an Archimedean dependence structure induced by taking our shot-noise process as mixing variable in a Marshall and Olkin (1967)-type conditionally i.i.d. model. For a related approach in this direction see Czado et al. (2012).

The structure of the paper is as follows: in Chapter 2 we introduce a general form of shot-noise processes and derive general results. In Chapter 2.1 we give the claims arrival process with a stochastic intensity process having a shot-noise structure. In Chapter 3 we study the pricing of catastrophe bonds while Chapter 4 we discuss the estimating of shot-noise processes. The closing Chapter 5 shows how to simulate shot-noise processes.

2. Shot-Noise Processes

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, i.e. $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-nullsets.

From a general viewpoint, non-life insurance can be described as follows: insurance claims are reported at the *arrival times* $0 < T_1 \leq T_2 \leq \cdots$. An arrival time is a $\mathbb{F}$-stopping time, such that the available information at time t, given by $\mathcal{F}_t$, contains the precise timing of all claims occurred up to t. The *size of claim* i is denoted by Z_i and we assume that the size of claims are immediately available, i.e. Z_i is $\mathcal{F}_{T_i}$-measurable for all $i \geq 1$. The *aggregated claim amount process* C accumulates arrived claim sizes and is given by

$$C_t = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leq t\}} Z_i, \qquad t \geq 0. \tag{1}$$

The sequence $(T_i, Z_i)_{i\geq 0}$ is a *marked point process* (MPP). We refer to Brémaud (1981) for a detailed exposition of the theory of point processes and marked point processes, which we follow here. If the claim sizes are non-zero, then there is a one-to-one correspondence between the marked point process $(T_i, Z_i)_{i\geq 0}$ and its dynamic representation $C = (C_t)_{t\geq 0}$

and we will use both interchangeably. There is a further useful tool to describe this MPP: the random measure defined by

$$M(\omega; dt, dz) = \sum_{i=1}^{\infty} \delta_{T_i(\omega), Z_i(\omega)}(dt, dz)$$

where $\delta_{(t,z)}$ denotes the Dirac-measure at the point (t, z). The compensator in the Doob-Meyer decomposition of M will play an essential rôle in the following. By $\mathcal{B}(\mathbb{R})$ we denote the Borel σ-algebra on the real line.

Fix $A \in \mathcal{B}(\mathbb{R})$ and note that $M_t := M([0, t], A)$, $t \geq 0$ is a *point process*. Assume that M is $\mathbb{F}$-adapted. If there exists a non-negative $\mathbb{F}$-progressive process ℓ such that $\int_0^t \ell(s)ds < \infty$ with probability one and for all non-negative, $\mathbb{F}$-predictable processes Y it holds that

$$\mathbb{E}\Big[\int_0^\infty Y_s dM_s\Big] = \mathbb{E}\Big[\int_0^\infty Y_s \ell_s ds\Big],$$

then ℓ is called the $\mathbb{F}$-*intensity* of M.

Definition 2.1. Consider a marked point process with associated random measure M. Suppose that for each $A \in \mathcal{B}(\mathbb{R})$, $M([0, t], A)$ has the $\mathbb{F}$-predictable intensity $(\ell_t(A))_{t\geq 0}$. Then $\ell_t(dz)$ is called $\mathbb{F}$-intensity kernel of M.

As the filtration often will be clear from the context, we will call ℓ simply intensity kernel or intensity of the marked point process. The intensity gives the compensator in the Doob-Meyer decomposition of the marked point process. More generally, we have that for a $\mathbb{F}$-predictable processes Y with

$$\mathbb{E}\Big[\int_0^t \int |Y(s, z)| \ell_s(dz) ds\Big] < \infty, \quad t \geq 0,$$

the process

$$\int_0^t \int Y(s, z)(M(ds, dz) - \ell_s(dz)ds), \quad t \geq 0$$

is a $\mathbb{F}$-martingale. Under weaker assumptions we of course only obtain local martingales, see Brémaud (1981), Corollary VIII.C4.

Example 2.1 (Cramér-Lundberg model)**.** Consider a Poisson process with jump times T_n, $n \geq 1$ and assume that Z_n, $n \geq 1$ are independent and identically distributed (i.i.d.), and independent of $M_t = \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}}$. Then the claims process C is a *compound Poisson process*. Together with its canonical filtration given by $\mathcal{F}_t = \sigma(M_t) \vee N$ where N denotes the $\mathbb{P}$-nullsets this model fits in our set-up.

Lundbergs exponential upper bound on the ruin probability is a classical result, see Mikosch (2009) Theorem 4.2.3, and ensures that if the insurer starts with a sufficiently high initial capital the ruin probability is small.

Example 2.2 (Stochastic discounting)**.** If the insurance company discounts the claim costs from arrival T_n to today t, the following modification of (1) is appropriate:

$$C_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leq t\}} h(t, T_n), Z_n, \qquad t \geq 0,$$

where $h(t, T)$ is a non-negative, measurable function, for example $h(t, T) = e^{-r(t-T)}$ or $h(t, T) = e^{-\int_T^t r(s)ds}$. Assuming non-negative interest rates implies that h is non-increasing in t. Moreover, $h(T, T) = 1$. The process C in this case is a special shot-noise process which we will study in the following section in detail. Remarkably, Bremaud (2000) shows that the Lundberg estimate still holds under $h(t,T) = g(t-T)$ with non-increasing function g if the claim sizes are in a certain sense not too heavy-tailed.

The Poisson process has a constant arrival rate, i.e. the expected number of claim arrivals over a time interval $[t, t+\Delta]$ is always the same. It has been a successful road in insurance mathematics to generalize the distribution of inter-arrival times by means of renewal processes. One of the main achievements was to show that similar results as in the Poissonian case hold in the limit when time grows large, see Mikosch (2009) for a detailed account and further references.

Our intention here is to introduce a dynamic dependence of the claim arrival rate with respect to a stochastic process. For example, in the case of a hurricane, the claims are not reported immediately to the insurer but arrive delayed over time. It is expected that most claims are reported soon after the catastrophe, while some claims will be reported late. Moreover, the height of the claim is often not known precisely when the claim is announced, such that a dynamic evolution of the claims process is necessary. An interesting question is the estimation of the total claim size of the catastrophe when only a fraction of the total claims is known. This question will be discussed in the section on estimation, Section 4.

Stochastic arrival rates. We study models allowing for factor-driven dynamics by borrowing heavily from current developments in credit risk, in particular reduced form modelling, see Filipović (2009) or Bielecki and Rutkowski (2002) for detailed accounts. A particular interesting example will be given in terms of general shot-noise processes.

Example 2.3 (Doubly stochastic setting)**.** Consider a non-decreasing process $\mathscr{L} = (\mathscr{L}_t)_{t\geq 0}$ starting at zero and i.i.d., standard-exponentially distributed random variables $E_1, E_2, \dots$, independent of $\mathscr{L}$. Set $T_0 = 0$ and define

$$T_i := \inf\{t \geq 0 : \mathscr{L}_t \geq E_1 + \dots + E_i\}, \quad i \geq 1.$$

Then $\mathscr{L}$ takes the rôle of a cumulated intensity process. Note that in this model it is possible that $T_i = T_{i-1}$, if $\mathscr{L}$ has jumps. We will call this effect *joint jumps* in the claims arrival process.

On the other side, if $\mathscr{L}$ is absolutely continuous, i.e.

$$\mathscr{L}_t = \int_0^t \ell_s ds,$$

the probability of joint jumps vanishes. The process ℓ is then the *intensity* of the point process $(T_i)_{i\geq 1}$.

Without further assumptions, given $\mathscr{L}$, the point process $(T_1, T_2, \dots)$ always exists, but can be explosive. Uniqueness of the distribution of the point process $(T_1, T_2, \dots)$ requires some further assumptions, in particular on the considered filtration, see Jacod (1975).

In the following we generalize the definitions of intensity to that of cumulated intensities. First, for a point process $(T_n)_{n\geq 1}$ with associated counting process $N_t := \sum_{n\geq 1} \mathbb{1}_{\{T_n \leq t\}}$, $t \geq 0$ we call a predictable random measure $\mathscr{L}$ *cumulated intensity measure* if

$$\mathbb{E}\bigg[\int_0^\infty Y_s dN_s\bigg] = \mathbb{E}\bigg[\int_0^\infty Y_s \mathscr{L}(ds)\bigg],$$

for all non-negative $\mathbb{F}$-predictable processes Y. The non-decreasing, predictable process $\mathscr{L}_t := \mathscr{L}([0,t])$ will be called *cumulated intensity process*.

The model we will study falls into this general class. In our applications, if $\mathscr{L}_t = \int_0^t \ell_s ds$, then ℓ will also be called *arrival rate*. If $\ell_t = c$ with $c > 0$ then we are back in the Poisson-process case as in Example 2.1.

Intuitively, this set-up can also be viewed as a random time change of a Poisson process: the number of arrival times before t can be represented as $\tilde{M}(\mathscr{L}(t)),\ t \geq 0$ with an independent Poisson process $\tilde{M}$ with intensity 1.

Definition 2.2. Consider a marked point process with associated random measure M. Suppose that for each $A \in \mathcal{B}(\mathbb{R})$, $M([0,t], A)$ has the cumulated intensity measure $\mathscr{L}(dt, A)$. Then $\mathscr{L}(dt, dz)$ is called $\mathbb{F}$-cumulated intensity measure of M.

The cumulated intensity measure determines the compensator in the Doob-Meyer decomposition, such that $\mathscr{L}$ is also called *compensator* of M: if Y is predictable, such that

$$\mathbb{E}\left[\int_0^t \int |Y(s,z)| \mathscr{L}_s(ds, dz)\right] < \infty,$$

for all $t \geq 0$, the following process

$$\int_0^t \int Y(s,z)(M(ds,dz) - \mathscr{L}_s(ds,dz)), \quad t \geq 0$$

is a $\mathbb{F}$-martingale. The compensator in the Doob-Meyer decomposition is unique, and so is the cumulated intensity measure of M. For further details see Jacod and Shiryaev (2003) Section II.1.

In the following paragraph we will introduce specific processes which we will later use to construct cumulated intensity measures.

Shot-noise processes. Consider an inhomogeneous Poisson process N with intensity function λ and denote by $0 < \tau_1 < \tau_2 < \cdots$ its jump times. Let $\xi_n,\ n \geq 1$ be random variables with values in $\mathbb{R}^d$, i.i.d. and independent of N. Then the driving process $(\tau_n, \xi_n)_{n\geq 1}$ is a inhomogeneous compound Poisson process. Finally, consider a measurable function $h : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ and define the process S by

$$S_t := \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} h(t, \tau_n, \xi_n), \qquad t \geq 0. \tag{2}$$

Then we call S a *shot-noise process.* The function h is called *noise-function.* This definition is general enough for our purposes, but could be extended at the cost of more complicated results. For example, it is possible to include general random compensators for N or even infinity activity for the driving process. We refer to Parzen (1962) or Schmidt and Stute (2007) for references and further literature on shot-noise processes.

If μ is the random measure associated with the marked point process $(\tau_n, \xi_n)_{n\geq 1}$, then

$$S_t = \int_0^t \int_{\mathbb{R}^d} h(t, s, x) \mu(ds, dx), \qquad t \geq 0. \tag{3}$$

This representation shows that in general, S will not be a semi-martingale. In most applications, however, we will consider $h(t,s,x) = g(t-s, x)$ and the semi-martingale property in this case is simpler to study.

Example 2.4. If G is not of finite absolute variation, S is no longer a semi-martingale. For example, consider a Brownian motion W such that $(W_t)_{t\geq 0}$ is $\mathcal{F}_0$-measurable. Letting

$$g(t,x) = xW_t$$

gives that $dS_t = Z_{t-}dW_t + dZ_t$ which is not a semi-martingale (W is $\mathcal{F}_0$-measurable!).

For the following result, we denote by ν the compensator of μ and consider shot-noise processes of the form

$$S_t := \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} g(t-\tau_n, \xi_n), \qquad t \geq 0. \tag{4}$$

Lemma 2.1. *Fix $T>0$ and assume that $g(t,x) = g(0,x) + \int_0^t g'(s,x)ds$ for all $0 \leq t \leq T$ and all $x \in \mathbb{R}^d$. If*

$$\int_0^T \int_{\mathbb{R}^d} (g'(s,x))^2 \nu(ds,dx) < \infty, \tag{5}$$

$\mathbb{P}$-a.s., then $(S_t)_{0\leq t\leq T}$ as in (4) is a semi-martingale.

Proof. Under condition (5), we can apply the stochastic Fubini theorem in the general version given in Theorem IV.65 in Protter (2004). Observe that

$$\begin{aligned} S_t &= \int_0^t \int_{\mathbb{R}^d} \int_s^t g'(u-s,x)du\, \mu(ds,dx) + \int_0^t \int_{\mathbb{R}^d} g(0,x)\mu(ds,dx) \\ &= \int_0^t \int_0^s \int_{\mathbb{R}^d} g'(u-s,x)\mu(ds,dx)\, du + \int_0^t \int_{\mathbb{R}^d} g(0,x)\nu(ds,dx) + M_t \end{aligned}$$

with a local martingale M. This is the semi-martingale representation of S and we conclude. □

In the exponential case, i.e. when $g(t,x) = xe^{-bt}$, we obtain $g(t,x) = -bg(t,x)$ and $g(0,x) = x$, such that

$$S_t = \int_0^t -bS_u du + Z_t.$$

In this case, S is also a Markov process. This is, under quite weak assumptions, the only specification where a shot-noise process is Markovian.

For applications it is important to have a repertory of parametric families which can be used to estimate the shot-noise process from data. We give some specifications in the following example which lead to highly tractable models. These examples will partly be taken up in Example 2.10 in an integrated form.

Example 2.5 (Parametric families)**.** In this example we concentrate on the multiplicative structure

$$g(t,x) = g(t)\, x$$

and give a number of useful specifications for the noise function g.

(1) *Regime switching*: The shot at T_i has a constant impact for a specified time length β and after β the impact jumps to a new level (regime) which could even be zero. For $\alpha \in \mathbb{R}$, $\beta > 0$, let

$$g(t) = \mathbb{1}_{\{t\leq\beta\}} + \alpha \mathbb{1}_{\{t>\beta\}}.$$

For $\alpha = 0$, the effect of the shot vanishes totally after a time period of length β.

(2) *Exponential structure*: for $\beta > 0$, let

$$g(t) = e^{-\beta t}.$$

Here, the effect of a shot decreases exponentially over time. As already mentioned, in this case S is Markovian.

We close this section with an example where claims are discounted with respect to deterministic, but non-constant interest rates.

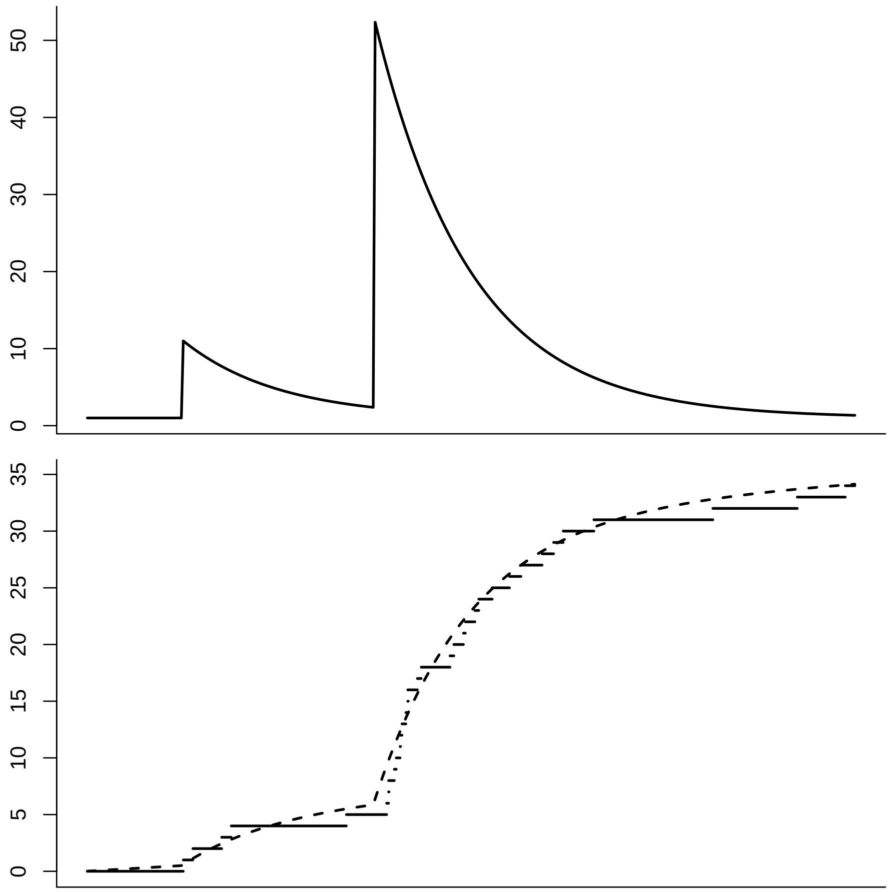

FIGURE 1. Illustration of a shot-noise process (top) with exponential structure. The graph on the bottom shows a counting process whose jump times have the shot-noise process as intensity ℓ. The dashed line is the cumulated intensity process $\mathscr{L}(t)$.

Example 2.6 (Discounting claims)**.** Following Example 2.2 we consider claims arriving according to a Poisson process with constant intensity ℓ. The risk-free rate of interest r is a deterministic, measurable function such that $\int_0^T r(s)ds < \infty$. The value of claims of size 1 arriving before T, discounted to current time 0 is given by

$$C_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_i \le t\}} e^{-\int_0^{T_n} r(s)ds},$$

which is a shot-noise process with noise function $h(t,T) = e^{-\int_t^T r(s)ds}$. Proposition 2.2 will enable us to compute the expectation of the discounted claims. This approach can be extended to incorporate stochastic interest rates as well.

Example 2.7 (Delayed claims)**.** Often, when a claim is announced to the insurer, the size of the claim is not known immediately. In this case, there is a delay of the claim. We could incorporate this in our set-up by letting $\xi \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ where ξ_2 denotes the delay. The noise function

$$h(t, T, x) = x_1 g(t - (T + x_2)) \mathbb{1}_{\{t \geq T + x_2\}},$$

$x = (x_1, x_2)^\top$, allows to include such effects in multiplicative model as in Example 2.5.

For the description of the statistical properties of the model, the Fourier transform of the shot-noise process is a central quantity which is given in the following result. For convenience of the reader we give a proof of this classical result in our general set-up. We denote by $\Lambda(t)$ the cumulated intensity function of the time-inhomogeneous Poisson process N.

Proposition 2.2. *Fix $t \geq 0$ and assume that $\Lambda(t) < \infty$ for all $s \in [0, t]$. Let η be $U[0, \Lambda(t)]$-distributed, independent of ξ_1 and*

$$\varphi(t, \theta) := \mathbb{E}[\exp(i\theta h(t, \Lambda^{-1}(\eta), \xi_1))] - 1.$$

Then, for a shot-noise process S as in (2) it holds for all $\theta \in \mathbb{R}$ that

$$\mathbb{E}\left(e^{i\theta S_t}\right) = \exp\Big(\Lambda(t)\varphi(t, \theta)\Big). \tag{6}$$

The independence of ξ_1 and η allows to compute φ by simple integration:

$$\mathbb{E}[\exp(i\theta h(t, \eta, \xi_1))] = \int_0^t \frac{1}{t} \mathbb{E}[\exp(i\theta h(t, s, \xi_1))] ds.$$

In a model with multiplicative structure, i.e. $h(t, s, x) = h(t, s)x$ we have that

$$\mathbb{E}[\exp(i\theta h(t, s, \xi_1))] = \mathbb{E}[\exp(i\theta h(t, s)\xi_1)],$$

such that φ can be computed from the Fourier transform of ξ_1. We illustrate this in Example 2.8 below

Central to the proof is the following lemma which gives a relation of the jump times of the Poisson process to order statistics of i.i.d., uniformly distributed random variables. The order statistic of $\eta_1, \ldots, \eta_k$ is obtained through ordering the sample by size, $\eta_{1:k} < \eta_{2:k} < \cdots < \eta_{k:k}$ (in our case there are no ties, i.e. all values are different).

Lemma 2.3. *Consider a (homogeneous) Poisson process N with jump times $\sigma_1, \sigma_2, \ldots$, $t > 0$ and $k \in \mathbb{N}$. Conditional on $N_t = k$ it holds that*

$$(\sigma_1, \ldots, \sigma_k) \stackrel{\mathscr{L}}{=} (\eta_{1:k}, \ldots, \eta_{k:k}) \tag{7}$$

where $\eta_1, \eta_2, \ldots, \eta_k$ are i.i.d., and uniformly distributed on $[0, t]$.

For a proof we refer to p.502 in Rolski et al. (1999).

Proof. We first consider the case when $\lambda_t \equiv 1$. Then N is a standard Poisson process and we denote its jump times by $\sigma_1, \sigma_2, \ldots$. By Lemma 2.3, independence of $\boldsymbol{\xi} := (\xi_1, \xi_2, \ldots)$ and N, and the i.i.d. property of $\boldsymbol{\xi}$ and measurability of h we obtain that, conditionally on $N_T = k$

$$\sum_{n=1}^{k} h(t, \sigma_n, \xi_n) \stackrel{\mathscr{L}}{=} \sum_{n=1}^{k} h(t, \eta_{n:k}, \xi_n) \stackrel{\mathscr{L}}{=} \sum_{n=1}^{k} h(t, \eta_n, \xi_n). \tag{8}$$

Hence, as k was arbitrary it follows that

$$\sum_{\sigma_n \leq t} h(t, \sigma_n, \xi_n) \stackrel{\mathscr{L}}{=} \sum_{\sigma_n \leq t} h(t, \eta_n, \xi_n)$$

where $(\eta_1, \eta_2, \dots)$ are i.i.d., $U[0,t]$-distributed, *and* independent of N and $\boldsymbol{\xi}$. Hence,

$$\mathbb{E}\Big[\prod_{\sigma_n \le t} e^{i\theta h(t,\eta_n,V_n)}\Big] = \sum_{k\ge 0} e^{-t}\frac{t^k}{k!}\prod_{n=1}^{k}\mathbb{E}\Big[e^{i\theta h(t,\eta_1,V_1)}\Big]$$

$$= \exp\bigg(-t + t\,\mathbb{E}\Big[e^{i\theta h(t,\eta_1,\xi_1)}\Big]\bigg). \tag{9}$$

Now we utilize the representation of an inhomogeneous Poisson process as time-transformation of a standard Poisson process: the process $(N(\Lambda(s)))_{s\ge 0}$ with $\Lambda(s) := \int_0^s \lambda_v dv$ is a time-inhomogeneous Poisson process with intensity function λ. The jump times of $N(\Lambda)$ are given by $\tau_n := \Lambda^{-1}(\sigma_n)$ because

$$\sum_{n\ge 1}\mathbb{1}_{\{\sigma_n \le \Lambda(t)\}} = \sum_{n\ge 1}\mathbb{1}_{\{\Lambda^{-1}(\sigma_n)\le t\}}$$

where $\Lambda^{-1}(t) := \inf\{s \ge 0 : \Lambda(s) \ge t\}$ denotes the generalized inverse of Λ. We obtain that

$$S_t \overset{\mathscr{L}}{=} \sum_{\tau_n \le t} h(t,\tau_n,\xi_n)$$

$$= \sum_{\sigma_n \le \Lambda(t)} h(t,\Lambda^{-1}(\sigma_n),\xi_n)$$

and, by (9),

$$\mathbb{E}\big[\exp(i\theta S_t)\big] = \mathbb{E}\Big[\exp\Big(\sum_{\sigma_n\le\Lambda(t)} h(t,\Lambda^{-1}(\sigma_n),\xi_n)\Big)\Big]$$

$$= \exp\bigg(-\Lambda(t) + \Lambda(t)\,\mathbb{E}\Big[e^{i\theta h(t,\Lambda^{-1}(\eta_1),\xi_1)}\Big]\bigg).$$

Note that $\sigma_1, \dots$ now take values in $[0,\Lambda(t)]$, such that $\eta_1 \sim U[0,\Lambda(t)]$. The expectation in the last equation equals

$$\mathbb{E}\Big[e^{i\theta h(t,\Lambda^{-1}(\eta_1),\xi_1)}\Big] = \varphi(\theta) + 1$$

and we conclude. □

Corollary 2.4. *Assume that $\Lambda(t) = \lambda t$, such that N is a Poisson process with intensity $\lambda > 0$.*

(i) *If $h(t,T,x) = 1$ we obtain that $N(t)$ is Poisson(λt)-distributed:*

$$\mathbb{E}\big[e^{i\theta S_t}\big] = \mathbb{E}\big[e^{i\theta N_t}\big] = \exp\big(\lambda t(e^{i\theta} - 1)\big).$$

(ii) *If $h(t,T,x) = x$ then S is a compound Poisson process. We denote by $\varphi_\xi(\theta) := \mathbb{E}[e^{i\theta\xi_1}]$ the Fourier transform of ξ_1 and obtain*

$$\mathbb{E}\big[e^{i\theta S_t}\big] = \exp\big(\lambda t(\varphi_\xi(\theta) - 1)\big).$$

(iii) *If $h(t,T,x) = xe^{-b(t-T)}$ we obtain the classical Markovian shot-noise process and*

$$\mathbb{E}\big[e^{i\theta S_t}\big] = \exp\big(\lambda t\varphi(t,\theta)\big) \tag{10}$$

with

$$\varphi(t,\theta) = \mathbb{E}\Big[e^{i\theta e^{-b\eta_1}\xi_1}\Big] - 1 = \frac{1}{t}\int_0^t \varphi_\xi(\theta e^{-bx})dx - 1.$$

Proof. The first two results follow immediately. Regarding the third claim, note that $\Lambda^{-1}(t) = \frac{t}{\lambda}$. Together with $\eta \sim U[0, \lambda t]$ we obtain that

$$\Lambda^{-1}(\eta) = \frac{1}{\lambda}\eta \sim U[0,t].$$

Then also $t - \eta \sim U[0,t]$ and we obtain that

$$\begin{aligned}\varphi(t,\theta) &= \mathbb{E}\Big[e^{i\theta e^{-b\eta_1}\xi_1}\Big] - 1 \\ &= \frac{1}{t}\int_0^t \mathbb{E}\Big[e^{i\theta e^{-bx}\xi_1}\Big]dx - 1 \\ &= \frac{1}{t}\int_0^t \varphi_\xi(\theta e^{-bx})dx - 1\end{aligned}$$

by Fubini's theorem. □

Related results may be found in Rice (1977). The semi-Markov case is considered in Smith (1973).

Example 2.8 (A parametric example for the jump distribution)**.** The following example illustrates the applicability of Proposition 2.2. Consider a Poisson process with intensity λ as driver and ξ_i which have an Erlang distribution. This is a flexible class of positive random variables which contains the exponential and χ^2_n-distribution as special cases: consider $\xi_1 \sim \Gamma(n,\nu)$ with $n \in \mathbb{N}$ and $\nu > 0$. Then

$$\varphi_\xi\,(\theta) = \mathbb{E}(e^{i\theta V_1}) = \left(1 - \frac{i\theta}{\nu}\right)^{-n}.$$

The tractability of the Erlang-distribution mainly attributes to the following result:

$$\int \frac{a^n}{x(a+bx)^n}dx = \ln\Big(\frac{x}{a+bx}\Big) + \sum_{j=1}^{n-1}\frac{a^j}{j(a+bx)^j}. \tag{11}$$

We choose $h(t,T,x) = e^{-b(t-t)}x$ and compute

$$\begin{aligned}\int_0^t \varphi_\xi(\theta e^{-bx})dx &= \int_0^t \frac{\nu^n}{\big(\nu - i\theta e^{-bs}\big)^n}ds \\ &= \frac{1}{b}\int_{e^{-bt}}^1 \frac{\nu^n}{x\,(\nu - i\theta x)^n}dx \\ &\overset{(11)}{=} \ln\frac{i\nu + \theta e^{-bt}}{i\nu + \theta} + \sum_{j=1}^{n-1}\left(\frac{\nu^j}{j(\nu - i\theta)^j} - \frac{\nu^j}{j(\nu - i\theta e^{-bt})^j}\right)\end{aligned}$$

and we obtain the characteristic function of S_t from (10). For $n = 1$ we obtain an exponential distribution with parameter $\nu > 0$ and the obvious simplification.

2.1. **Claims driven by shot-noise processes.** Now we are in the position to put our ingredients together for the modelling of insurance claims. Let $A : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing function denoting the cumulated claim arrival intensity when there is no shot-noise process present. As previously, we consider an inhomogeneous Poisson process N with jump times $(\tau_n)_{n\geq 1}$ and intensity function λ. The shots are given by the i.i.d. sequence $\xi_1, \xi_2, \ldots$. The considered shot-noise process S is

$$S_t := \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} G(t - \tau_n, \xi_n), \qquad t \geq 0,$$

similar to Equation (4).

As before, claims arrive at times $T_1, T_2, \dots$ where the associated point process has cumlated intensity measure (compensator) $\mathscr{L}$. In this section, the shot-noise process will be used as basis for $\mathscr{L}$, such that we assume that the function $G : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}$ is non-decreasing in its first coordinate, time. Moreover, we assume that

$$\mathscr{L}_t = A(t) + \sum_{n \geq 1} \mathbb{1}_{\{\tau_n \leq t\}} G(t - \tau_n, \xi_n), \quad t \geq 0. \tag{12}$$

Example 2.9 (Shot-noise arrival rate)**.** If the claim arrival rate ℓ is given by a shot-noise process with noise function g, then $\mathscr{L}$ falls into the above class: note that

$$\mathscr{L}_t = \int_0^t \ell_s ds = \sum_{\tau_n \leq t} \int_{\tau_i}^t g(s - \tau_n, \xi_n) ds = \sum_{\tau_i \leq t} G(t - \tau_n, \xi_n) \tag{13}$$

with $G(t,x) = \int_0^t g(s,x)ds$. In this case, $G(0,x) = 0$ reflecting the continuity of $\mathscr{L}$.

As indicated in the above example we will consider integrals over shot-noise processes as cumulated intensity processes. In view of classical applications this class of processes is quite unusual as the noise function is increasing. We distinguish these two cases in our notation by always using g and G for the noise function in the original shot-noise process and the integrated shot-noise process, respectively.

For concrete implementations it is important to have a repertory of non-decreasing shot-noise processes which can be used to estimate the shot-noise process from data. We give some specifications in the following example which lead to highly tractable models.

Example 2.10 (Parametric families)**.** In the following examples we consider the multiplicative structure

$$G(t,x) = G(t)x$$

where $G : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is non-negative and increasing in its first coordinate, and the random variables $\xi_n,\ n \geq 1$ have values in $\mathbb{R}_{\geq 0}$.

(1) *Linear structure*: for $\alpha \in [0,1], \beta > 0$, let

$$G(t) = \alpha + (1-\alpha)\frac{t}{\beta}\mathbb{1}_{\{t \leq \beta\}} + (1-\alpha)\mathbb{1}_{\{t > \beta\}}.$$

This response function starts at α and increases linearly over the interval $[0,\beta]$ until it reaches 1. For $\alpha = 0$, this function is absolutely continuous.

(2) *Exponential structure*: for $\alpha \in [0,1], \beta > 0$, let

$$G(t) = \alpha + (1-\alpha)\left(1 - e^{-\beta t}\right).$$

Here, G starts at α and increases exponentially to 1. The parameter α controls the impact of the jump size on S. If $\alpha = 0$, G is differentiable. The parameter β controls the speed of the growth.

(3) *Rational structure*: for $\alpha \in [0,1], \beta > 0$, let

$$G(t) = \alpha + (1-\alpha)\frac{t}{t+\beta}.$$

This provides an alternative specification to the exponential structure.

An illustration of the last example may be found in Figure 2.

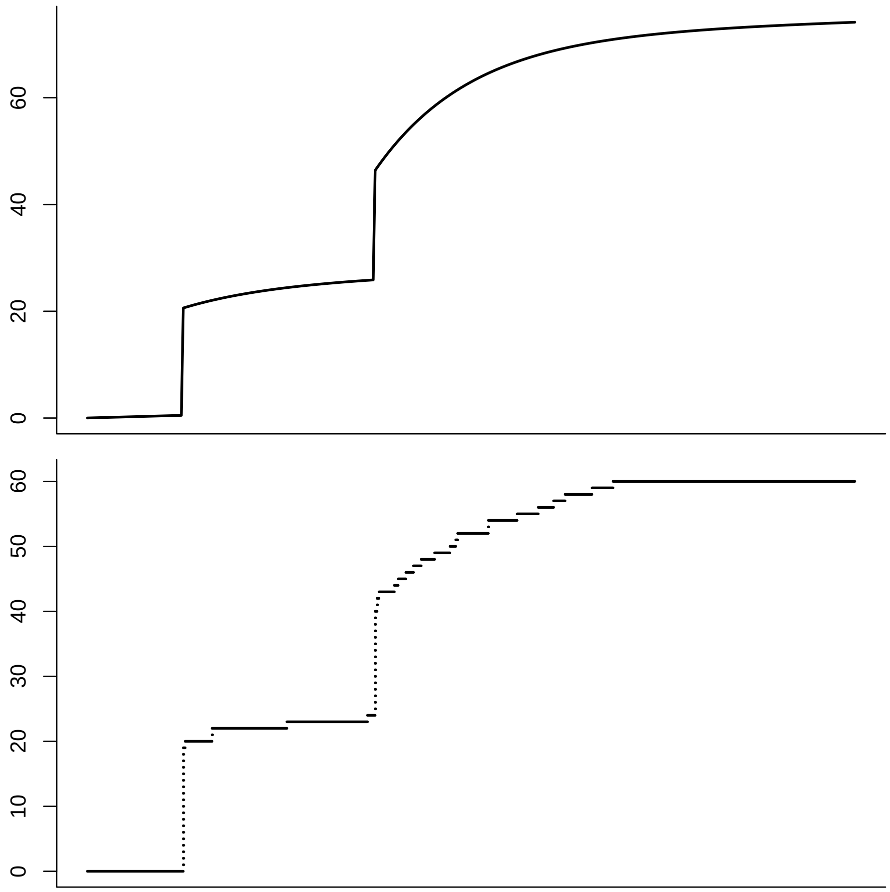

FIGURE 2. Illustration of the cumulated shot-noise intensity $\mathscr{L}$ with exponential structure and jumps ($\alpha \neq 0$. The graph on the bottom shows a counting process whose jump times have cumulated intensity process $\mathscr{L}$. Multiple claim arrivals occur when $\mathscr{L}$ jumps.

3. Catastrophe bonds

Catastrophe bonds (CAT bonds) are risk-linked securities which allow to transfer insurance risks to investors. While the valuation of car insurance can effectively be done using the law of large numbers, catastrophe risks pose a large challenge due to highly dependent claim arrivals. Our shot-noise approach sets a framework which is ideally suited to model such risks.

The size of CAT bond markets has been increasing continuously over the last decade and has reached an outstanding volume of $19 billion dollars in October 2013.

We consider the following stylized version: a CAT bond offers a coupon payment c at payment dates $t_1, \dots, t_K$ and the repayment of the principal 1 at t_K if no trigger event happened. In the case of a trigger event happened, the coupons are ceased and a fraction δ of the principal is paid back.

As an example we consider as trigger event if the claims process $C = \sum_{T_n \le t} Z_n$ crosses a barrier B and assume zero interest rates. In this case the payment at t_k would be

$$f_k(C_{t_k}) = \begin{cases} c + \mathbb{1}_{\{k=K\}}, & \text{if } C_{t_k} \le B, \\ \delta \mathbb{1}_{\{k=K\}}, & \text{if } C_{t_k} > B. \end{cases}$$

for $k = 1, \dots, K$.

For the pricing of the CAT bond we need to choose a risk-neutral measure Q and obtain that the value of the CAT bond computes to the expectation (under Q) of discounted payoffs, i.e.

$$\sum_{k=1}^{T} \mathbb{E}^Q \Big[\beta(t_k)\, f_k(C_{t_k}) \Big].$$

Here $\beta(t)$ is the discounting function for the time period $[0,t]$, so for example $\beta(t) = \exp(-\int_0^t r_u du)$ with risk-free rate of interest r. The expectations can of course always be computed by means of a Monte-Carlo simulation. In the following, we show how to obtain a more explicit result.

First, we assume that β is deterministic. This is reasonable in insurance applications as the risks due to claims are huge in comparison to the effect of stochastic interest rates. This assumption can easily be relaxed to interest rates which are independent of the claim sizes. More general interest rate models, however, require a change of numéraire which comes at the cost of more complicated results.

If interest rates are deterministic, we obtain that

$$\mathbb{E}^Q \Big[\beta(t_k) \mathbb{1}_{\{C_{t_k} \le B\}} \Big] = \beta(t_k) Q(C_{t_k} \le B)$$

and it remains to compute the boundary crossing probabilities of the claims process in the following.

For more information on CAT bonds we refer to Cox and Pederson (2000), Louberge et al. (1999) or Lewis (2007). Our model also extends the approach in Dassios and Jang (2003) where shot-noise Cox processes in an exponential structure with $\alpha = 0$ (see Example 2.5) have been applied to derivatives on a catastrophe index.

3.1. **Equivalent measure changes.** Following the results in Schmidt (2013) we study measure changes for shot-noise processes. This is an important tool for pricing, filtering as well as for importance sampling of shot-noise processes.

The basic driver of the shot-noise process S as given in (2) is the marked point process $\Phi = (\tau_n, \xi_n)_{n \ge 0}$. It is thus sufficient to study changes of measure for Φ. Already in Brémaud (1981) it was shown how to change measure as in the Girsanov theorem for marked point processes. We will present this results in the following. In Schmidt (2013) it was shown that these measure changes include *all* equivalent measure changes.

We consider an initial filtration $\mathcal{H} \subset \mathcal{F}_0$ and denote by $\mathcal{P}$ the predictable σ-field. Denote by μ the random measure associated to Φ as in (3). As above we assume that the compensator of the process $\Phi_t = \sum_{\tau_n \le t} \xi_n,\ t \ge 0$ is given by $\nu(t, dx)dt$, i.e.

$$\Phi_t - \int_0^t \int_{\mathbb{R}^d} u \nu(s, dx) ds$$

is a local martingale and the kernel $\nu(\omega, t, dx)$ is $\mathcal{P} \otimes \mathbb{R}^d$-measurable. Then $\nu(t, \mathbb{R}^d)$ is the intensity at t for a jump and, if the intensity is positive, $\frac{\nu(t,dx)}{\nu(t,\mathbb{R}^d)}$ is the respective jump-size distribution.

Consider a $\mathcal{P}\otimes\mathbb{R}^d$-measurable positive function Y such that

$$\int_0^t \int_{\mathbb{R}^d} Y(s,u)\nu(s,du)ds < \infty \tag{14}$$

$\mathbb{P}$-alomst surely and let the likelihood-process Z be given by

$$Z_t = e^{-\int_0^t \int_{\mathbb{R}^d}(Y(s,u)-1)G(s,du)ds} \prod_{\tau_n \le t} Y(\tau_n,\xi_n), \quad t \ge 0. \tag{15}$$

Fix a time horizon $T>0$ and assume that $\mathbb{E}[Z_T]=1$. Then $d\mathbb{P}' := Z_T\, d\mathbb{P}$ defines a probability measure which is equivalent (as Y is positive and so Z) to $\mathbb{P}$. Under $\mathbb{P}'$, Φ is a (possibly explosive) marked point process and its compensator w.r.t. $\mathbb{P}'$ is given by

$$Y(t,u)\nu(t,du)dt.$$

3.1.1. *Preserving independent increments.* For tractability reasons one often considers shot-noise processes driven by a marked point process which has independent increments. If the increments are moreover stationary, then Φ is a Lévy process. We cover both cases in this section.

Theorem 3.1. *Assume that $\mathbb{P}\sim\mathbb{P}'$. Let the density process of $\mathbb{P}'$ relative to $\mathbb{P}$ be of the form (15).*

(1) *If Φ has independent increments under $\mathbb{P}$ and $\mathbb{P}'$, then Y is deterministic.*
(2) *If Φ has independent and stationary increments under $\mathbb{P}$ and $\mathbb{P}'$, then Y is deterministic and does not depend on time.*

For a proof, see Schmidt (2013).

Example 3.1 (The Esscher measure)**.** Consider a generic n-dimensional stochastic process X. Then the Esscher measure (Esscher (1932)) is given by the density

$$Z_t = \frac{e^{aX_t}}{\mathbb{E}(e^{aX_t})}$$

where $a\in\mathbb{R}^d$ is chosen in such a way that Z is a martingale. Esche and Schweizer (2005) showed that the Esscher measure preserves the Lévy property. Dassios and Jang (2003) applied the Esscher measure to obtain an arbitrage-free pricing methodology for catastrophe bonds under shot-noise processes.

Example 3.2 (The minimal martingale measure)**.** The minimal martingale measure as proposed in Föllmer and Schweizer (1990) for a certain class of shot-noise processes has been analysed in Schmidt and Stute (2007). It can be described as follows: consider the semi-martingale $X=A+M$ where A is an increasing process of bounded variation and M is a local martingale. Assume that there exists a process ℓ which satisfies

$$A_t = \int_0^t \ell_s d\langle M\rangle_s.$$

Then the density of the minimal martingale measure with respect to $\mathbb{P}$ is given by

$$Z_t = \mathcal{E}\left(\int_0^\cdot \ell_{s-}dM_s\right)_t.$$

Here $\mathcal{E}$ denotes the Doleans-Dade stochastic exponential, i.e. Z is the solution of $dZ_t = Z_{t-}\ell_{t-}dM_t$.

In Altmann et al. (2008) the minimal martingale measure was obtained by a considering discrete time first and then taking limits.

3.2. **Pricing.** In Dassios and Jang (2003) the authors choose the Esscher measure to obtain a pricing measure in the context of CAT bonds. Choosing the pricing measure in the case of a CAT bond is simpler than in many other cases because the underlying (the catastrophe index) is not a traded asset. In this case any equivalent measure is a martingale measure.

We take a more general approach here and only assume that certain properties of the shot-noise process hold under Q. Given this properties, we derive general pricing rules. A calibration to market data gives access to the risk-neutral measure Q. Possible ways to do this are to proceed as in Dassios and Jang (2005) via Kalman filtering, or to use a minimal-distance estimation as in Section 4.

According to Theorem 3.1 we assume a simple structure of Φ under Q. This is in spirit with many applied results in mathematical finance, see for example Elliott and Madan (1998).

(A1): We assume that under Q the marked point process Φ has i.i.d. marks $(\xi_n)_{n\geq 1}$ and the point process $(\tau_n)_{n\geq 1}$ is a inhomogeneous Poisson process.

This assumption will be satisfied under an Esscher change of measure, which is an important class for insurance applications. If we have a deterministic interest rate, $\beta(t)$ is constant and so for the pricing it is sufficient to compute the expectation of $f_k(L_{t_k})$ only.

The key to efficient pricing methodologies is to obtain the Fourier transform of the claims arrival process. In this regard, we consider the set-up as in Section 2.1: claims arrive at times $T_1, T_2, \dots$ where the associated point process $M_t = \sum_{n\geq 1} \mathbb{1}_{\{T_i \leq t\}}$ has cumulated intensity measure (compensator) $\mathscr{L}$. We assume that

$$\mathscr{L}_t = A(t) + \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} G(t-\tau_n, \xi_n), \quad t \geq 0$$

where $A : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a non-decreasing measurable function. The driver of the shot-noise process is an inhomogeneous Poisson process with jump times $(\tau_n)_{n\geq 1}$ and intensity function λ. The shots are given by the i.i.d. sequence $\xi_1, \xi_2, \dots$.

Proposition 3.2. *Consider the point process $M_t = \sum_{n\geq 1} \mathbb{1}_{\{T_i\leq t\}}$, $t \geq 0$, the independent sequence $Z_1, Z_2, \dots$ and the cumulated claims at time t,*

$$C_t = \sum_{T_n \leq t} Z_n.$$

Then, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}^Q\left[e^{i\theta C_t}\right] = \sum_{n\geq 1} Q(M_t = n)(\varphi_Z(\theta))^n.$$

Proof. The result follows immediately by independence as

$$\begin{aligned}\mathbb{E}^Q\left[e^{i\theta C_t}\right] &= \sum_{n\geq 1} \mathbb{E}^Q\left[\mathbb{1}_{\{M_t=n\}} e^{i\theta(Z_1+\cdots+Z_n)}\right] \\ &= \sum_{n\geq 1} Q(M_t = n)(\varphi_Z(\theta))^n.\end{aligned}$$

□

Of course, if Z_n stems from a family of distributions which is stable under convolution, $(\varphi_Z(\theta))^n$ will be easy to compute. In the following result we compute the remaining probabilities.

In the doubly-stochastic case as in Example 2.3 we have the following, important result: recall that this setting can be viewed as a stochastic time change: $M(t) = \tilde{M}(\mathscr{L}(t))$, with an Poisson process $\tilde{M}$ with intensity 1, independent of $\mathscr{L}$. Then

$$\begin{aligned}\mathbb{Q}(M(t)=n) &= \mathbb{E}^{\mathbb{Q}}\big[\mathbb{Q}(\tilde{M}(\mathscr{L}(t))=n|\mathscr{L})\big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{n!}\exp(-\mathscr{L}(t))(\mathscr{L}(t))^n\Big].\end{aligned}$$

Proposition 3.3. *Assume that*

$$Q(M_t=n)=\frac{1}{n!}\mathbb{E}^Q\Big[\exp(-\mathscr{L}(t))(\mathscr{L}(t))^n\Big]. \tag{16}$$

Set

$$\varphi(t,\theta):=\mathbb{E}\Big[\exp\big(-\theta G(t-\Lambda^{-1}(\eta),\xi_1)\big)\Big]-1;$$

here η is $U(0,\Lambda(t))$-distributed, independent of ξ_1. Then

$$Q(M_t=n)=\frac{1}{n!}e^{-\int_0^t A(s)ds}\cdot\Big((-\partial_\theta)^n e^{\Lambda(t)\varphi(t,\theta)}\Big).$$

Proof. We compute the right hand side of (16). Consider an integrable, non-negative random variable X. Then, $\mathbb{E}[\exp(-\theta X)]<\infty$ for all $\theta\geq 0$. Moreover, by monotone convergence,

$$\mathbb{E}[X\exp(-\theta X)]=-\mathbb{E}[\partial_\theta\exp(-\theta X)]=-\partial_\theta\mathbb{E}[\exp(-\theta X)]$$

and, proceeding iteratively,

$$\mathbb{E}[X^n\exp(-\theta X)]=(-\partial_\theta)^n\mathbb{E}[\exp(-\theta X)].$$

Then, analogously to Proposition 2.2, we obtain that

$$\mathbb{E}^Q\Big[\exp(-\theta\mathscr{L}(t))\Big]=e^{-\theta A(t)}\cdot\exp\big(\Lambda(t)\varphi(t,-i\theta)\big)$$

and the conclusion follows. □

In Example 2.8 the n-th derivative can be computed. Otherwise one has to resort to numerical methods.

Now the way to pricing of the CAT-bond is clear: one can either invert the Fourier transform by Fast-Fourier methods or, alternatively compute

$$q_n:=Q(Z_1+\cdots+Z_n\leq B)$$

which can sometimes be obtained explicitly, such that

$$\mathbb{E}^Q[\mathbb{1}_{\{C_t\leq B\}}]=\sum_{n\geq 1}Q(M_t=n)q_n.$$

4. Estimating shot-noise processes

The estimation of shot-noise processes is an important part in the application of these models. A possible approach in this direction uses filtering methods and has been started in Dassios and Jang (2005). Further approaches for point process estimation may be found in Jacobsen (1982) or Karr (1986). A recent account which especially treats shot-noise processes may be found in Kopperschmidt and Stute (2013) which we will present now.

The key assumption in the approach of Kopperschmidt and Stute (2013) is that i.i.d. observations of the shot-noise process are at hand. The key tool to estimation is to use a parametric compensator of the point process and estimate the unknown parameter in terms of a minimum-distance estimator. In the insurance context it is often the case that i.i.d. observations are available: if used for modelling the claims arrivals after catastrophes, each catastrophe with associated claims process constitutes such a single observation.

We will consider the following case: observations consist of data of $i = 1, \dots, n$ catastrophes. For each catastrophe i the claims arrive at times $T_1^i, T_2^i, \dots$ and we observe the point processes

$$N_t^i = \sum_{n \geq 1} \mathbb{1}_{\{T_n^i \leq t\}}, \qquad t \in [0, \bar{T}]$$

on a fixed time interval $[0, \bar{T}]$. Typically $\bar{T}$ will be quite large such that all claims are included in the study.

We assume that N^i are independent and identically distributed such that each N^i has a compensator of the same type. Each claims arrival process N^i is driven by an individual shot-noise process in spirit of Equation (12). We assume that the time points of the catastrophes are observable: more generally, to each N^i we associate the catastrophe arrivals $\tau_1^i, \tau_2^i, \dots$ which are observable. Moreover, to each τ_n^i there is an associated ξ_n^i which is also assumed to be observable. It denotes a proxy for the overall size of the catastrophe. This could be obtained from expert opinions, the area of land reached by the catastrophe, or the cumulated claim sizes. It refers to the size of the shot in the compensator of N^i.

Choosing a parametric approach, we follow Equation (12) and consider a parametric shot-noise form. More precisely, given the parameter $\theta \in \Theta \subset \mathbb{R}^K$ we assume that compensator of N^i is given by

$$\mathscr{L}_t^i(\theta_0) = A(t, \theta_0) + \sum_{\tau_n^i \leq t} G(\theta_0, t - \tau_n^i, \xi_n^i), \quad t \in [0, \bar{T}]$$

for some $\theta_0 \in \Theta$.

The first step towards the estimation is the introduction of the aggregated point process and the aggregated compensator:

$$\bar{N}_n = \frac{1}{n} \sum_{i=1}^n N^i, \qquad \bar{\mathscr{L}}_n(\theta) = \frac{1}{n} \sum_{i=1}^n \mathscr{L}^i.$$

The second step is to define a suitable distance. For the finite measure μ we consider the semi-norm

$$\parallel f \parallel_\mu := \left[\int_{[0, \bar{T}]} f^2(t) \mu(dt) \right]^{1/2}.$$

The measure μ induced by $\bar{N}_n$ leads to the following semi-norm

$$\parallel f \parallel_{\bar{N}_n} = \frac{1}{n} \sum_{i=1}^n \sum_{j \geq 1} f^2(\tau_j^i) \mathbb{1}_{\{\tau_j^i \leq \bar{T}\}}.$$

Then, the quantity

$$\parallel \bar{N}_n - \bar{\mathscr{L}}_n(\theta) \parallel_{\bar{N}_n}$$

represents an overall measure of fit for the observed data $\bar{N}_n$ to the compensator $\bar{\mathscr{L}}_n(\theta)$. The final estimator of θ_0 is the parameter which maximizes this fit:

$$\theta_n := \arg \inf_{\theta \in \Theta} \parallel \bar{N}_n - \bar{\mathscr{L}}_n(\theta) \parallel_{\bar{N}_n} . \tag{17}$$

The following weak identifiability assumption will be needed for consistency. By Θ^c we denote the closure of Θ. First, we assume that for all $i = 1, \dots, n$

$$\mathbb{E}[N^i(\bar{T})] < \infty \quad \text{and} \quad \mathbb{E}[\mathscr{L}_{\bar{T}}^i(\theta)] < \infty.$$

(A2): Let $\Theta \subset \mathbb{R}^K$ be a bounded open set and suppose that for each $\epsilon > 0$

$$\inf_{\|\theta-\theta_0\|\geq\epsilon} \| \, \mathbb{E}[\mathscr{L}(\theta_0) - \mathscr{L}(\theta)] \, \|_{\mathbb{E}[\mathscr{L}(\theta_0)]} > 0.$$

Moreover, the process $(t, \theta) \to \mathscr{L}_t(\theta)$ is continuous with probability one and admits a continuous extension to $[0, \bar{T} \times \Theta^c$.

The following result, given in Theorem 1 in Kopperschmidt and Stute (2013), shows consistency of the minimum-distance estimator.

Theorem 4.1. *Assume that (A2) holds. Then*

$$\lim_{n\to\infty} \theta_n = \theta_0 \quad \textit{with probability one.}$$

The proof of the theorem may be found in Kopperschmidt and Stute (2013). It relies on generalized U-Statistics and an appropriate version of the Hewitt-Savage 0-1 law. Under further assumptions, they also obtain asymptotic normality of the estimator θ_n and we refer to Theorem 2 in their paper for a precise statement.

This estimation procedure seems a very promising approach compared to existing methodologies and will be taken up in a future article for an estimation on insurance catastrophe data.

5. Simulation

Efficient simulation algorithms are often the key to widespread application of a model. In particular, when closed-form results are expensive or not at hand, Monte-Carlo simulation always provides an alternative which is nowadays often feasible due to available computer power. Similar to Filipović et al. (2011) we can give general simulation routines for counting processes in a doubly-stochastic setting following the methodology in Jacod (1975).

Consider a fixed time horizon T. We will use the fact (see Lemma 2.3), that conditional on the number of jumps of a Poisson process its jump times are equal in distribution to the order statistics of i.i.d. uniform random variables on $[0, T]$. The second key ingredient will be the time-transform: $\tilde{N}(\Lambda(t))$ is a inhomogeneous Poisson process if $\tilde{N}$ is a standard Poisson process.

We shortly recall our model: $N = \tilde{N} \circ \Lambda$ is a time-inhomogeneous Poisson process with intensity function λ and jump times $\tau_1, \tau_2, \dots$. The shots $\xi_1, \xi_2, \dots$ are i.i.d. and $\mathbb{R}^d$-valued. We denote the distribution of ξ_1 by F_ξ. Then our shot-noise process is given by

$$S_t := \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} g(t - \tau_n, \xi_n), \qquad t \geq 0,$$

following (4). The insurance claims arrive at times $T_1, T_2, \dots$ which are doubly-stochastic random times with cumulated intensity process

$$\mathscr{L}_t = A(t) + \sum_{n\geq 1} \mathbb{1}_{\{\tau_n \leq t\}} G(t - \tau_n, \xi_n).$$

The claim sizes $Z_1, Z_2, \dots$ itself are i.i.d. with distribution function F_Z.

Algorithm 5.1. Simulate one path of the shot-noise process S and, afterwards, a vector of claim arrivals together with associated claim sizes. A realized path may be found in Figure 3.

(1) Draw the number of jumps N from a Poisson$(\Lambda(T))$-distribution.
(2) Simulate N i.i.d. U$[0, \Lambda(T)]$ random variables $\eta_1, \eta_2, \dots$ and set $tau_i := \Lambda^{-1}(\eta_{i:N})$, $i = 1, \dots, N$, $\eta_{i:N}$ being the i-th order statistic.

(3) Simulate N i.i.d. random variables $\xi_1, \dots, \xi_N$ (jump heights) according to the chosen distribution F_ξ.
(4) Compute the path $\mathscr{L}(t) = A(t) + \sum_{i=1}^N V_i h(t - T_i)$.
(5) Simulate the claim arrival times by taking i.i.d. exponential(1)-random variables $E_1, E_2, \dots$ and calculating

$$T_i = \inf\{t \geq 0 : \mathscr{L}(t) \geq E_1 + \cdots + E_i\}, \qquad i \geq 1.$$

(6) Simulate the claim sizes $Z_1, Z_2, \dots$ from the distribution F_Z.

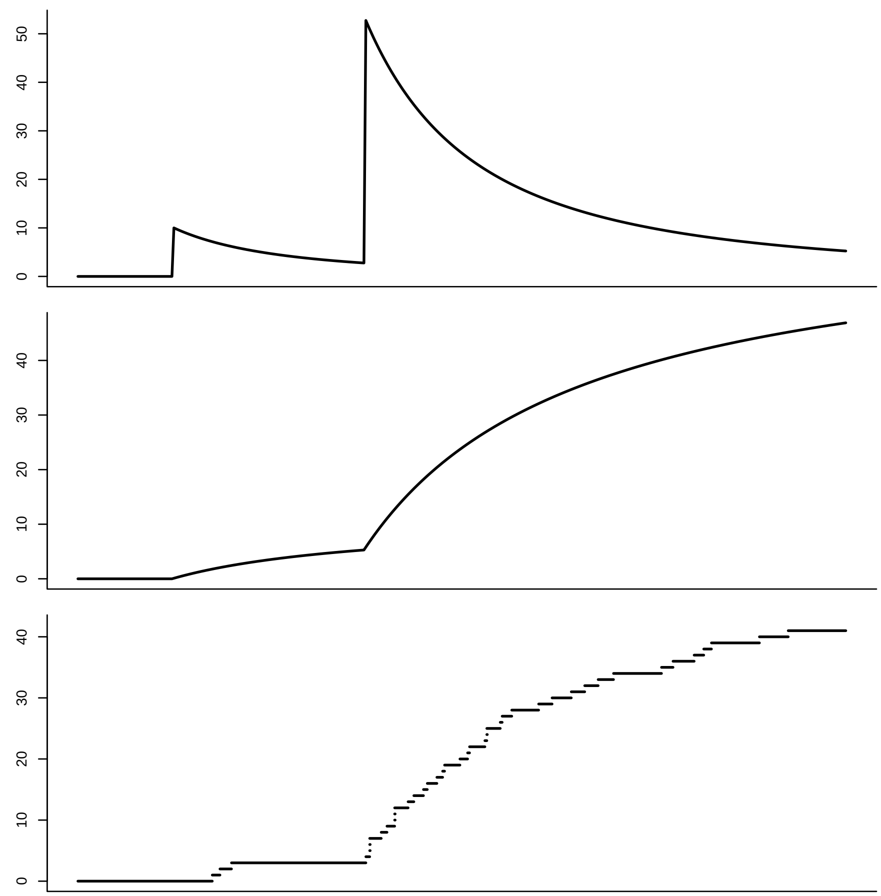

Figure 3. Simulation of a claims process driven by a shot-noise process with rational structure. The graph shows the intensity process ℓ (top), the cumulated intensity process $\mathscr{L}(t) = \int_0^t \ell(s)ds$ (middle) and the simulated claims process $\sum \mathbb{1}_{\{T_n \leq t\}}$ (bottom).

References

Altmann, T., Schmidt, T. and Stute, W. (2008), 'A shot noise model for financial assets', *International Journal of Theoretical and Applied Finance* **11**(1), 87 – 106.

Artemis (2013), 'Catastrophe bond market hits $19 billion outstanding for first time'.
URL: *http://www.artemis.bm/blog/2013/10/08/catastrophe-bond-market-hits-19-billion-outstanding-for-first-time/*

Bielecki, T. and Rutkowski, M. (2002), *Credit Risk: Modeling, Valuation and Hedging*, Springer Verlag. Berlin Heidelberg New York.

Brémaud, P. (1981), *Point Processes and Queues*, Springer Verlag. Berlin Heidelberg New York.

Bremaud, P. (2000), 'An insensitivity property of Lundberg's estimate for delayed claims.', *Journal of Applied Probability* **37**(3), 914.

Cox, S. and Pederson, H. (2000), 'Catastrophe risk bonds', *North American Actuarial Journal* **4**, 56 – 82.

Czado, C., Kastenmeier, R., Brechmann, E. C. and Min, A. (2012), 'A mixed copula model for insurance claims and claim sizes', *Scand. Actuar. J.* (4), 278–305.

Dassios, A. and Jang, J. (2003), 'Pricing of catastrophe reinsurance & derivatives using the cox process with shot noise intensity', *Finance and Stochastics* **7**(1), 73–95.

Dassios, A. and Jang, J. (2005), 'Kalman-Bucy filtering for linear systems driven by the Cox process with shot noise intensity and its application to the pricing of reinsurance contracts', *J. Appl. Probab.* **42**(1), 93–107.

Elliott, R. J. and Madan, D. B. (1998), 'A discrete time extended Girsanov prinicple', *Mathematical Finance* **8**, 127–152.

Esche, F. and Schweizer, M. (2005), 'Minimal entropy preserves the Lévy property: how and why', *Stochastic Process. Appl.* **115**(2), 299–327.

Esscher, F. (1932), 'On the probability function in the collective theory of risk', *Skandinavisk Aktuarietidskrift* **15**, 175 – 95.

Filipović, D. (2009), *Term Structure Models: A Graduate Course*, Springer Verlag. Berlin Heidelberg New York.

Filipović, D., Overbeck, L. and Schmidt, T. (2011), 'Dynamic CDO term structure modelling', *Mathematical Finance* **21**, 53 – 71.

Föllmer, H. and Schweizer, M. (1990), Hedging of contingent claims under incomplete information, *in* M. H. A. Davis and R. J. Elliott, eds, 'Applied Stochastic Analysis', Vol. 5, Gordon and Breach, London/New York, pp. 389 – 414.

Insurance Insider (2013), 'Cat bond market reaches record $18bn size'.
URL: *http://www.insuranceinsider.com/-1244580/8*

Jacobsen, M. (1982), *Statistical analysis of counting processes*, Vol. 12 of *Lecture Notes in Statistics*, Springer-Verlag, New York.

Jacod, J. (1975), 'Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales', *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **31**, 235 – 253.

Jacod, J. and Shiryaev, A. (2003), *Limit Theorems for Stochastic Processes*, 2nd edn, Springer Verlag, Berlin.

Jang, J., Herbertsson, A. and Schmidt, T. (2011), 'Pricing basket default swaps in a tractable shot noise model', *Statistics and Probability Letters* **8**, 1196 – 1207.

Karr, A. F. (1986), *Point processes and their statistical inference*, Vol. 2 of *Probability: Pure and Applied*, Marcel Dekker Inc., New York.

Kopperschmidt, K. and Stute, W. (2013), 'The statistical analysis of self-exciting point processes', *Working paper* .

Kühn, C. (2004), 'Shot-noise processes', *Encyclopedia of Actuarial Science* pp. 1556 – 1558.

Lewis, M. (2007), 'In natures' casino', *New York Times, 26th of August* .

Louberge, H., Kellezi, E. and Gilli, M. (1999), 'Using catastrophe-linked securities to diversify insurance risk: A financial analysis of CAT bonds.', *Journal of Insurance Issues* **22**, 125 – 146.

Lund, R. B., Butler, R. W. and Paige, R. L. (1999), 'Prediction of shot noise', *Journal of Applied Probability* pp. 374 – 388.

Marshall, A. W. and Olkin, I. (1967), 'A multivariate exponential distribution', *Journal of the American Statistical Association* **62**, 30–44.

Mikosch, T. (2009), *Non-life insurance mathematics*, Universitext, second edn, Springer-Verlag, Berlin. An introduction with the Poisson process.

Parzen, E. (1962), *Stochastic processes*, Holden-Day, Inc., San Francisco.

Protter, P. (2004), *Stochastic Integration and Differential Equations*, 2nd edn, Springer Verlag. Berlin Heidelberg New York.

Rice, J. (1977), *Advances in Applied Probability* **9**, 553 – 565.

Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999), *Stochastic Processes for Insurance and Finance*, John Wiley & Sons. New York.

Scherer, M., Schmid, L. and Schmidt, T. (2012), 'Shot-noise multivariate default models', *European Actuarial Journal* **2**, 161 – 186.

Schmidt, T. (2013), 'Shot-noise processes: equivalent measure changes and applications in finance and filtering', *Working paper* .

Schmidt, T. and Stute, W. (2007), 'General shot-noise processes and the minimal martingale measure', *Statistics & Probability Letters* **77**, 1332 – 1338.

Schottky, W. (1918), 'Über spontane Stromschwankungen in verschiedenen Elektrizitätsleitern', *Annalen der Physik* **362**(23), 541 – 567.

Smith, W. (1973), 'Shot noise generated by a semi-Markov process', *Journal of Applied Probability* **10**, 685 – 690.

Chemnitz University of Technology, Reichenhainer Str. 41, 09126 Chemnitz, Germany. Email: thorsten.schmidt@mathematik.tu-chemnitz.de.