# Pricing and Hedging of Credit Derivatives via the Innovations Approach to Nonlinear Filtering

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**Abstract** In this paper we propose a new, information-based approach for modelling the dynamic evolution of a portfolio of credit risky securities. In our setup market prices of traded credit derivatives are given by the solution of a nonlinear filtering problem. The innovations approach to nonlinear filtering is used to solve this problem and to derive the dynamics of market prices. Moreover, the practical application of the model is discussed: we analyse calibration, the pricing of exotic credit derivatives and the computation of risk-minimizing hedging strategies. The paper closes with a few numerical case studies.

**Keywords** Credit derivatives, incomplete information, nonlinear filtering, hedging

## 1 Introduction

Credit derivatives - derivative securities whose payoff is linked to default events in a given portfolio - are an important tool in managing credit risk. However, the subprime crisis and the subsequent turmoil in credit markets highlights the need for a sound methodology for the pricing and the risk management of these securities. Portfolio products pose a particular challenge in this regard: the main difficulty is to capture the dependence structure of the defaults and the dynamic evolution of the credit spreads in a realistic and tractable way.

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In this paper we propose a new, information-based approach to this problem. We consider a reduced-form model driven by an unobservable background factor process $X$. For tractability reasons $X$ is modelled as a finite state Markov chain. We consider a market for defaultable securities related to $m$ firms and assume that the default times are conditionally independent doubly stochastic random times where the default intensity of firm $i$ is given by $\lambda_{t,i} = \lambda_i(X_t)$. This setup is akin to the model of **?**. If $X$ was observable, the Markovian structure of the model would imply that prices of defaultable securities are functions of the past defaults and the current state of $X$.

In our setup $X$ is however not directly observed. Instead, the available information consists of prices of liquidly traded securities. Prices of such securities are given as conditional expectations with respect to a filtration $\mathbb{F}^{\mathbb{M}} = (\mathcal{F}_t^{\mathbb{M}})_{t \geq 0}$ which we call *market information.* We assume that $\mathbb{F}^{\mathbb{M}}$ is generated by the default history of the firms under consideration and by a process $Z$ giving observations of $X$ in additive noise. To compute the prices of the traded securities at $t$ one therefore needs to determine the conditional distribution of $X_t$ given $\mathcal{F}_t^{\mathbb{M}}$. Since $X$ is a finite-state Markov chain this distribution is represented by a vector of probabilities denoted $\boldsymbol{\pi}_t$. Computing the dynamics of the process $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)_{t \geq 0}$ is a nonlinear filtering problem which is solved in Section 3 using martingale representation results and the innovations approach to nonlinear filtering. By the same token we derive the dynamics of the market price of traded credit derivatives.

In Section 4 these results are then applied to the pricing and the hedging of non-traded credit derivatives. It is shown that the price of most credit derivatives common in practice - defined as conditional expectation of the associated payoff given $\mathcal{F}_t^{\mathbb{M}}$ - depends on the realization of $\boldsymbol{\pi}_t$ and on past default information. Here a major issue arises for the application of the model: we view the process $Z$ as abstract source of information which is not directly linked to economic quantities. Hence the process $\boldsymbol{\pi}$ is not directly accessible for typical investors. As we aim at pricing formulas and hedging strategies which can be evaluated in terms of publicly available information, a crucial point is to determine $\boldsymbol{\pi}_t$ from the prices of traded securities (calibration), and we explain how this can be achieved by linear or quadratic programming techniques. Thereafter we derive risk-minimizing hedging strategies. Finally, in Section 5, we illustrate the applicability of the model to practical problems with a few numerical case studies.

The proposed modelling approach has a number of advantages: first, actual computations are done mostly in the context of the hypothetical model where $X$ is fully observable. Since the latter has a simple Markovian structure, computations become relatively straightforward. Second, the fact that prices of traded securities are given by the conditional expectation given the market filtration $\mathbb{F}^{\mathbb{M}}$ leads to rich credit-spread dynamics: the proposed approach accommodates *spread risk* (random fluctuations of credit spreads between defaults) and *default contagion* (the observation that at the default of a company the credit spreads of related companies often react drastically). A prime example for contagion effects is the rise in credit spreads after the default of

Lehman brothers in 2008. Both features are important in the derivation of robust dynamic hedging strategies and for the pricing of certain exotic credit derivatives. Third, the model has a natural factor structure with factor process $\boldsymbol{\pi}$. Finally, the model calibrates reasonably well to observed market data. It is even possible to calibrate the model to single-name CDS spreads and tranche spreads for synthetic CDOs from a *heterogeneous* portfolio, as is discussed in detail in Section 5.2.

Reduced-form credit risk models with incomplete information have been considered previously by Schönbucher (2004), Collin-Dufresne, Goldstein & Helwege (2003), Duffie, Eckner, Horel & Saita (2009) and Frey & Runggaldier (2008). Frey & Runggaldier (2008) concentrate on the mathematical analysis of filtering problems in reduced-form credit risk models. Schönbucher and Collin-Dufresne et. al. were the first to point out that the successive updating of the distribution of an unobservable factor in reaction to incoming default observation has the potential to generate contagion effects. None of these contributions addresses the dynamics of credit-derivative prices under incomplete information or issues related to hedging. The innovations approach to nonlinear filtering has been used previously by Landen (2001) in the context of default-free term-structure models. Moreover, nonlinear filtering problems arise in a natural way in structural credit risk models with incomplete information about the current value of assets or liabilities such as Kusuoka (1999), Duffie & Lando (2001), Jarrow & Protter (2004), Coculescu, Geman, & Jeanblanc (2008) or Frey & Schmidt (2009).

## 2 The Model

Our model is constructed on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions; all processes considered are by assumption $\mathbb{F}$-adapted. $\mathbb{Q}$ is the risk-neutral martingale measure used for pricing. For simplicity we work directly with discounted quantities so that the default-free money market account satisfies $B_t \equiv 1$.

*Defaults and losses.* Consider $m$ firms. The *default time* of firm $i$ is a stopping time denoted by $\tau_i$ and the current *default state* of the portfolio is $Y_t = (Y_{t,1}, \ldots, Y_{t,m})$ with $Y_{t,i} = \mathbb{1}_{\{\tau_i \leq t\}}$. Note that $Y_t \in \{0,1\}^m$. We assume that $Y_0 = 0$. The percentage *loss given default* of firm $i$ is denoted by the random variable $\ell_i \in (0,1]$. We assume that $\ell_1, \ldots, \ell_m$ are independent random variables, independent of all other quantities introduced in the sequel. The *loss state* of the portfolio is given by the process $L = (L_{t,1}, \ldots, L_{t,m})_{t\geq 0}$ where $L_{t,i} = \ell_i Y_{t,i}$.

*Marked-point-process representation.* Denote by $0 = T_0 < T_1 < \cdots < T_m < \infty$ the *ordered default times* and by $\xi_n$ the identity of the firm defaulting at $T_n$. Then the sequence

$$(T_n, (\xi_n, \ell_{\xi_n})) =: (T_n, E_n), \quad 1 \leq n \leq m$$

gives a representation of $L$ as marked point process with mark space $E := \{1,\dots,m\} \times (0,1]$. Let $\mu^L(ds,de)$ be the random measure associated to $L$ with support $[0,\infty) \times E$. Note that any random function $R : \Omega \times [0,\infty) \times E \to \mathbb{R}$ can be written in the form

$$R(s,e) = R(s,(\xi,\ell)) = \sum_{i=1}^{m} \mathbb{1}_{\{\xi=i\}} R_i(s,\ell)$$

with $R_i(s,\ell) := R(s,(i,\ell))$. Hence, integrals with respect to $\mu^L(ds,de)$ can be written in the form

$$\int_0^t \int_E R(s,e)\,\mu^L(ds,de) = \sum_{T_n \le t} R_{\xi_n}(T_n,\ell_{\xi_n}) = \sum_{\tau_i \le t} R_i(\tau_i,\ell_i). \tag{2.1}$$

## 2.1 The underlying Markov model

The default intensities of the firms under consideration are driven by the so-called factor or state process $X$. The process $X$ is modelled as a finite-state Markov chain; in the sequel its state space $S^X$ is identified with the set $\{1,\dots,K\}$. The following assumption states that the default times are conditionally independent, doubly-stochastic random times with default intensity $\lambda_{t,i} := \lambda_i(X_t)$. Set $\mathcal{F}_\infty^X = \sigma(X_s : s \ge 0)$.

**A1** There are functions $\lambda_i : S^X \to (0,\infty)$, $i = 1,\dots,m$, such that for all $t_1,\dots,t_m \ge 0$

$$\mathbb{Q}\big(\tau_1 > t_1,\dots,\tau_m > t_m \mid \mathcal{F}_\infty^X\big) = \prod_{i=1}^{m} \exp\Big(-\int_0^{t_i} \lambda_i(X_s)ds\Big).$$

It is well-known that under **A1** there are no joint defaults, i.e. $\tau_i \ne \tau_j$, for $i \ne j$ almost surely. Moreover, for all $1 \le i \le m$

$$Y_{t,i} - \int_0^{t\wedge\tau_i} \lambda_i(X_s)ds \tag{2.2}$$

is an $\mathbb{F}$-martingale; see for instance Chapter 9 in McNeil, Frey & Embrechts (2005). Furthermore, the process $(X,L)$ is jointly Markov.

Denote by $F_{\ell_i}$ the distribution function of $\ell_i$. A default of firm $i$ occurs with intensity $(1-Y_{t,i})\lambda_i(X_t)$, and the loss given default of firm $i$ has the distribution $F_{\ell_i}$. Hence the $\mathbb{F}$-compensator $\nu^L$ of the random measure $\mu^L$ is given by

$$\nu^L(dt,de) = \nu^L(dt,d\xi,d\ell) = \sum_{i=1}^{m} \delta_{\{i\}}(d\xi)\,F_{\ell_i}(d\ell)\,(1-Y_{t,i})\lambda_i(X_t)dt\,, \tag{2.3}$$

where $\delta_{\{i\}}$ stands for the Dirac-measure in $i$. To illustrate this further, we show how the default intensity of company $j$ can be recovered from (2.3): note that

$$Y_{t,j} = 1\!\!1_{\{\tau_j \le t\}} = \sum_{T_n \le t} 1\!\!1_{\{\xi_n = j\}} = \int_0^t \int_E R^j(s,e)\mu^L(ds,de)$$

with $R^j(s,e) = R^j(s,(\xi,\ell)) := 1\!\!1_{\{\xi=j\}}$. Using (2.1), the compensator of $Y_j$ is given by

$$\begin{aligned}\int_0^t \int_E R^j(s,e)\nu^L(ds,de) &= \int_0^t \int_E 1\!\!1_{\{\xi=j\}} \sum_{i=1}^m \delta_{\{i\}}(d\xi)\, F_{\ell_i}(d\ell)\,(1-Y_{s,i})\lambda_i(X_s)ds \\ &= \int_0^t (1-Y_{s,j})\lambda_j(X_s)ds.\end{aligned}$$

*Example 2.1* In the numerical part we will consider a one-factor model where $X$ represents the global state of the economy. For this we model the default intensities under full information as *increasing* functions $\lambda_i : \{1,\dots,K\} \to (0,\infty)$. Hence, 1 represents the best state (lowest default intensity) and $K$ corresponds to the worst state; moreover, the default intensities are comonotonic. In the special case of a homogeneous model the default intensities of all firms are identical, $\lambda_i(\cdot) \equiv \lambda(\cdot)$.

Furthermore, denote by $(q(i,k))_{1\le i,k\le K}$ the generator matrix of $X$ so that $q(i,k)$, $i \ne k$, gives the intensity of a transition from state $i$ to state $k$. We will consider two possible choices for this matrix. First, let the factor process be constant, $X_t \equiv X$ for all $t$. In that case $q(i,k) \equiv 0$, and filtering reduces to Bayesian analysis. A model of this type is known as *frailty model*, see also Schönbucher (2004). Second, we consider the case where $X$ has *next neighbour dynamics*, that is, the chain jumps from $X_t$ only to the neighbouring points $X_t \pm 1$ (with the obvious modifications for $X_t = 0$ and $X_t = K$).

## 2.2 Market information

In our setting the factor process $X$ is not directly observable. We assume that prices of traded credit derivatives are determined as conditional expectation with respect to some filtration $\mathbb{F}^{\mathbb{M}}$ which we call *market information*. The following assumption states that $\mathbb{F}^{\mathbb{M}}$ is generated by the loss history $\mathbb{F}^L$ and observations of functions of $X$ in additive Gaussian noise.

**A2** $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^L \vee \mathbb{F}^Z$, where the $l$-dimensional process $Z$ is given by

$$Z_t = \int_0^t \mathbf{a}(X_s)\,ds + B_t. \tag{2.4}$$

Here, $B$ is an $l$-dimensional standard $\mathbb{F}$-Brownian motion independent of $X$ and $L$, and $\mathbf{a}(\cdot)$ is a function from $S^X$ to $\mathbb{R}^l$.

In the case of a homogeneous model one could take $l = 1$ and assume that $\mathbf{a}(\cdot) = c \ln \lambda(\cdot)$. Here the constant $c \geq 0$ models the information-content of $Y$: for $c = 0$, $Y$ carries no information, whereas for $c$ large the state $X_t$ can be observed with high precision.

## 3 Dynamics of traded credit derivatives and filtering

In this section we study in detail traded credit derivatives. First, we give a general description of this type of derivatives and discuss the relation between pricing and filtering. In Section 3.2 we then study the dynamics of market prices, using the innovations approach to nonlinear filtering.

### 3.1 Traded securities

We consider a market of $N$ liquidly traded credit derivatives, with - for notational simplicity - common maturity $T$. Most credit derivatives have intermediate cash flows such as payments at default dates and it is convenient to describe the payoff of the $n$th derivative by the cumulative *dividend stream* $D_n$. We assume that $D_n$ takes the form

$$D_{t,n} = \int_0^t d_{1,n}(s, L_s) db(s) + \int_0^t \int_E d_{2,n}(s, L_{s-}, e) \mu^L(ds, de) \tag{3.1}$$

with bounded functions $d_1, d_2$ and an increasing deterministic function $b : [0, T] \to \mathbb{R}$.

Dividend streams of the form (3.1) can be used to model many important credit derivatives, as the following examples show.

*Zero-bond.* A defaultable bond on firm $i$ without coupon payments and with zero recovery pays 1 at $T$ if $\tau_i > T$ and zero otherwise. Hence, we have $b(s) = \mathbb{1}_{\{s \geq T\}}$, $d_1(t, L_t) = \mathbb{1}_{\{L_{t,i}=0\}}$ and $d_2 = 0$.

For CDS and CDO the function $b$ encodes the pre-scheduled payments: for payment dates $t_1 < \cdots < t_{\tilde{n}} < T$ we set $b(s) = |\{i \colon t_i \leq s\}|$.

*Credit default swap (CDS).* A protection seller position in a CDS on firm $i$ offers regular payments of size $S$ at $t_1, \ldots, t_{\tilde{n}}$ until default. In exchange for this, the holder pays the loss $\ell_i$ at $\tau_i$, provided $\tau_i < T$ (accrued premium payments are ignored for simplicity). This can be modelled by taking $d_1(t, L_t) = S\mathbb{1}_{\{L_{t,i}=0\}}$ and $d_2(t, L_{t-}, (\xi, \ell)) = -\mathbb{1}_{\{t \leq T\}} \mathbb{1}_{\{\xi = i\}} \ell$; note that

$$\int_0^t \int_E d_{2,n}(s, L_{s-}, e) \mu^L(ds, de) = -\ell_i \mathbb{1}_{\{L_{t,i}>0\}} = -L_{t,i}.$$

*Collateralized debt obligation (CDO).* A single tranche CDO on the underlying portfolio is specified by an lower and upper detachment point[1] $0 \leq$

[1] In practice, lower and upper detachment points are stated in percentage points, say $0 \leq l < u \leq 1$. Then $x_1 = l \cdot m$ and $x_2 = u \cdot m$.

$x_1 < x_2 \le m$ and a fixed spread $S$. Denote the *cumulative portfolio loss* by $\bar{L}_t = \sum_{i=1}^m L_{t,i}$, and define the function

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ .$$

An investor in a CDO tranche receives at payment date $t_i$ a spread payment proportional to the remaining notional $H(\bar{L}_{t_i})$ of the tranche. Hence, his income stream is given by $\int_0^t SH(\bar{L}_s)db(s)$, so that $d_1(t, L_t) = SH(\bar{L}_t)$. In return the investor pays at the successive default times $T_n$ with $T_n \le T$ the amount

$$-\Delta H(\bar{L}_{T_n}) = -\big(H(\bar{L}_{T_n}) - H(\bar{L}_{T_n-})\big)$$

(the part of the portfolio loss falling in the tranche). This can be modelled by setting

$$d_2(t, L_{t-}, (\xi, \ell)) = \mathbb{1}_{\{t \le T\}} H\left(\ell + \bar{L}_{t-}\right) - H\left(\bar{L}_{t-}\right).$$

Other credit derivatives such as CDS indices or typical basket swaps can be modelled in a similar way.

*Pricing of traded credit derivatives.* Recall that we work with discounted quantities, that $\mathbb{Q}$ represents the underlying pricing measure, and the information available to market participants is the market information $\mathbb{F}^{\mathbb{M}}$. As a consequence we assume that the *current market value* of the traded credit derivatives is given by

$$\widehat{p}_{t,n} := \mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t^{\mathbb{M}}\big), \quad 1 \le n \le N. \tag{3.2}$$

The *gains process* $\widehat{g}_n$ of the $n$-th credit derivative sums the current market value and the dividend payments received so far and is thus given by

$$\widehat{g}_{t,n} := \widehat{p}_{t,n} + D_{t,n} = \mathbb{E}\big(D_{T,n} \mid \mathcal{F}_t^{\mathbb{M}}\big); \tag{3.3}$$

in particular, $\widehat{g}_n$ is a martingale.

Next, we show that the computation of market values leads to a nonlinear filtering problem. We call $\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big)$ the *hypothetical value* of $D_n$. While this quantity will be an important tool in our analysis it does not correspond to market prices as in contrast to $\widehat{p}_n$ it is not $\mathbb{F}^{\mathbb{M}}$-adapted. Observe that by (3.1) $D_{T,n} - D_{t,n}$ is a function of the future path $(L_s)_{s \in (t,T]}$. Hence, the $\mathbb{F}$-Markov property of the pair $(X, L)$ implies that

$$\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big) = p_n(t, X_t, L_t) \tag{3.4}$$

for functions $p_n : [0, T] \times S^X \times [0, 1]^m \to \mathbb{R}$, $n = 1, \dots, N$; see for instance Proposition 2.5.15 in Karatzas & Shreve (1988) for a general version of the Markov property that covers (3.4). By iterated conditional expectations we obtain

$$\widehat{p}_{t,n} = \mathbb{E}\Big(\mathbb{E}\big(D_{T,n} - D_{t,n} | \mathcal{F}_t\big) | \mathcal{F}_t^{\mathbb{M}}\Big) = \mathbb{E}\big(p_n(t, X_t, L_t) | \mathcal{F}_t^{\mathbb{M}}\big). \tag{3.5}$$

In order to compute the market values $\widehat{p}_{t,n}$ we therefore need to determine the conditional distribution of $X_t$ given $\mathcal{F}_t^{\mathbb{M}}$. This a nonlinear filtering problem which we solve in Section 3.3 below.

*Remark 3.1 (Computation of the full-information value)* For bonds and CDSs the evaluation of $p_n$ can be done via the Feynman-Kac formula and related Markov chain techniques; for instance see Elliott & Mamon (2003). In the case of CDOs, the evaluation of $p_n$ via Laplace transforms is discussed in **?**. Alternatively, a two stage method that employs the conditional independence of defaults given $\mathcal{F}^X_\infty$ can be used. For this, one first generates a trajectory of $X$. Given this trajectory, the loss distribution can then be evaluated using one of the known methods for computing the distribution of the sum of independent (but not identically distributed) Bernoulli variables. Finally, the loss distribution is estimated by averaging over the sampled trajectories of $X$. An extensive numerical case study comparing the different approaches is given in Wendler (2010).

## 3.2 Asset price dynamics under the market filtration

In the sequel we use the innovations approach to nonlinear filtering in order to derive a representation of the martingales $\widehat{g}_n$ as a stochastic integral with respect to certain $\mathbb{F}^{\mathbb{M}}$-adapted martingales. For a generic process $U$ we denote by $\widehat{U}_t := \mathbb{E}(U_t|\mathcal{F}^{\mathbb{M}}_t)$ the *optional projection* of $U$ w.r.t. the market filtration $\mathbb{F}^{\mathbb{M}}$ in the rest of the paper. Moreover, for a generic function $f : S^X \to \mathbb{R}$ we use the abbreviation $\widehat{f}$ for the optional projection of the process $(f(X_s))_{s\geq 0}$ with respect to $\mathbb{F}^{\mathbb{M}}$.

We begin by introducing the martingales needed for the representation result. First, define for $i = 1, \dots, l$

$$m^Z_{t,i} := Z_{t,i} - \int_0^t (\widehat{a_i})_s \, ds \,. \tag{3.6}$$

It is well-known that $m^Z$ is an $\mathbb{F}^{\mathbb{M}}$-Brownian motion and thus the martingale part in the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of $Z$. Second, denote by

$$\widehat{\nu}^L(dt, de) := \sum_{i=1}^m \delta_{\{i\}}(d\xi)\, F_{\ell_i}(d\ell)\, (1 - Y_{t,i})(\widehat{\lambda}_i)_t dt \tag{3.7}$$

the compensator of $\mu^L$ w.r.t. $\mathbb{F}^{\mathbb{M}}$ and define the compensated random measure

$$m^L(dt, de) := \mu^L(dt, de) - \widehat{\nu}^L(dt, de) \,. \tag{3.8}$$

Corollary VIII.C4 in Brémaud (1981) yields that for every $\mathbb{F}^{\mathbb{M}}$-predictable random function $f$ such that $\mathbb{E}\big(\int_E \int_0^T |f(s,e)|\, \widehat{\nu}^L(ds, de)\big) < \infty$ the integral $\int_E \int_0^t f(s,e)\, m^L(ds, de)$ is a martingale with respect to $\mathbb{F}^{\mathbb{M}}$.

The following martingale representation result is a key tool in our analysis; its proof is relegated to the appendix.

**Lemma 3.2** *For every $\mathbb{F}^{\mathbb{M}}$-martingale $(U_t)_{0\le t\le T}$ there exists a $\mathbb{F}^{\mathbb{M}}$-predictable function $\gamma : \Omega \times [0,T] \times E \to \mathbb{R}$ and an $\mathbb{R}^l$-valued $\mathbb{F}^{\mathbb{M}}$-adapted process $\boldsymbol{\alpha}$ satisfying $\int_0^T ||\boldsymbol{\alpha}_s||^2 ds < \infty$ $\mathbb{Q}$-a.s. and $\int_0^T \int_E |\gamma(s,e)|\nu^L(ds,de) < \infty$ $\mathbb{Q}$-a.s. such that $U$ has the representation*

$$U_t = U_0 + \int_0^t \int_E \gamma(s,e)\, m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm_s^Z, \quad 0 \le t \le T. \tag{3.9}$$

The next theorem is the basis for the mathematical analysis of the model under the market filtration.

**Theorem 3.3** *Consider a real-valued $\mathbb{F}$-semimartingale*

$$J_t = J_0 + \int_0^t A_s ds + M_t^J, \quad t \le T$$

*such that $[M^J, B] = 0$. Assume that*

*(i)* $\mathbb{E}(|J_0|) < \infty$, $\mathbb{E}(\int_0^T |A_s| ds) < \infty$ *and* $\mathbb{E}(\int_0^T |J_s|\lambda_i(X_s) ds) < \infty$, $1 \le i \le m$.
*(ii)* $\mathbb{E}([M^J]_T) < \infty$.
*(iii)* *For all $1 \le i \le m$ there is some $\mathbb{F}^{\mathbb{M}}$-predictable $R_i : \Omega \times [0,T] \times (0,1] \to \mathbb{R}$ such that*

$$[J, Y_i]_t = \int_0^t \int_E \mathbb{1}_{\{\xi = i\}} R_i(s,\ell)\, \mu^L(ds, d\xi, d\ell). \tag{3.10}$$

*Moreover,* $\mathbb{E}(\int_0^T \int_0^1 |R_i(s,\ell)| F_{\ell_i}(d\ell)(1 - Y_{s,i})\lambda_i(X_s) ds) < \infty$.
*(iv)* $\int_0^t J_s dB_{s,j}$ *and* $\int_0^t Z_{s,j} dM_s^J$, $1 \le j \le l$ *are true $\mathbb{F}$-martingales.*

*Then the optional projection $\widehat{J}$ has the representation*

$$\widehat{J}_t = \widehat{J}_0 + \int_0^t \widehat{A}_s ds + \int_0^t \int_E \gamma(s,e)\, m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm_s^Z, \quad t \le T; \tag{3.11}$$

*here, $\gamma(s,e) = \gamma(s,(\xi,\ell)) = \sum_{i=1}^m \mathbb{1}_{\{\xi = i\}} \gamma_i(s,\ell)$, and $\boldsymbol{\alpha}, \gamma_i$ are given by*

$$\boldsymbol{\alpha}_s = (\widehat{J\mathbf{a}})_s - \widehat{J}_s(\widehat{\mathbf{a}})_s, \tag{3.12}$$

$$\gamma_i(s,\ell) = \frac{1}{(\widehat{\lambda}_i)_{s-}} \Big[ (\widehat{J\lambda_i})_{s-} - \widehat{J}_{s-}(\widehat{\lambda}_i)_{s-} + (\widehat{R_i(\cdot,\ell)\lambda_i})_{s-} \Big]. \tag{3.13}$$

*Proof* The proof uses the following two well-known facts.

1. For every true $\mathbb{F}$-martingale $N$, the projection $\widehat{N}$ is an $\mathbb{F}^{\mathbb{M}}$-martingale.
2. For any progressively measurable process $\phi$ with $\mathbb{E}\big(\int_0^T |\phi_s| ds\big) < \infty$ the process $\widehat{\int_0^t \phi_s ds} - \int_0^t \widehat{\phi}_s\, ds$, $t \le T$, is an $\mathbb{F}^{\mathbb{M}}$-martingale.

The first fact is simply a consequence of iterated expectations, while the second follows from the Fubini theorem, see for instance Davis & Marcus (1981).

As $M^J$ is a true martingale by $(ii)$, Fact 1 and 2 immediately yield that $\widehat{J}_t - \widehat{J}_0 - \int_0^t \widehat{A}_s ds$ is an $\mathbb{F}^{\mathbb{M}}$-martingale. Lemma 3.2 thus gives the existence of the representation (3.11).

It remains to identify $\gamma$ and $\boldsymbol{\alpha}$. The idea is to use the elementary identity

$$\widehat{J\phi} = \widehat{J}\,\phi$$

for any $\mathbb{F}^{\mathbb{M}}$-adapted $\phi$. Each side of this equation gives rise to a different semimartingale decomposition of $\widehat{J\phi}$; comparing those for suitably chosen $\phi$ one obtains $\gamma$ and $\boldsymbol{\alpha}$.

In order to identify $\gamma$, fix $i$ and let

$$\phi_t^i = \int_0^t \int_E \varphi(s,\ell) \mathbb{1}_{\{\xi = i\}}\, \mu^L(ds, d\xi, d\ell)$$

for a bounded and $\mathbb{F}^{\mathbb{M}}$-predictable $\varphi$. Note that $\phi^i$ is $\mathbb{F}^{\mathbb{M}}$-adapted. We first determine the $\mathbb{F}$-semimartingale decomposition of $J\phi^i$. Itô's formula gives

$$d(J_t \phi_t^i) = \phi_{t-}^i dJ_t + J_{t-} d\phi_t^i + d[J, \phi^i]_t. \tag{3.14}$$

With (3.10),

$$[J, \phi^i]_t = \sum_{s \le t} \Delta J_s \Delta \phi_s^i = \int_0^t \int_E R_i(s,\ell) \varphi(s,\ell) \mathbb{1}_{\{\xi = i\}} \mu^L(ds, d\xi, d\ell).$$

Hence, using (2.3), the predictable compensator of $[J, \phi^i]$ is

$$\langle J, \phi^i \rangle_t = \int_0^t \int_0^1 R_i(s,\ell) \varphi(s,\ell) F_{\ell_i}(d\ell)(1 - Y_{s,i}) \lambda_i(X_s) ds. \tag{3.15}$$

Moreover, $[J, \phi^i] - \langle J, \phi^i \rangle$ is a true martingale by (iii), as $\varphi$ is bounded. Using (3.14) and (3.15) the finite variation part in the $\mathbb{F}$-semimartingale decomposition of $J\phi^i =: \tilde{A} + \tilde{M}$ computes to

$$\begin{aligned} \tilde{A}_t = \int_0^t \Bigg( \phi_s^i A_s + J_s (1 - Y_{s,i}) \lambda_i(X_s) \int_0^1 \varphi(s,\ell) F_{\ell_i}(d\ell) \\ + \int_0^1 R_i(s,\ell) \varphi(s,\ell) (1 - Y_{s,i}) \lambda_i(X_s) F_{\ell_i}(d\ell) \Bigg) ds. \end{aligned}$$

Moreover, $\tilde{M}$ is a true $\mathbb{F}$-martingale by (i) - (iii). Using Fact 1 and 2 the finite variation part in the $\mathbb{F}^{\mathbb{M}}$-semimartingale decomposition of $\widehat{J\phi^i}$ turns out to be

$$\int_0^t \Bigg( \phi_s^i \widehat{A}_s + (1 - Y_{s,i})(\widehat{J\lambda_i})_s \int_0^1 \varphi(s,\ell) F_{\ell_i}(d\ell) + \int_0^1 \varphi(s,\ell)(1 - Y_{s,i})(\widehat{R_i(\cdot,\ell)\lambda_i})_s \, F_{\ell_i}(d\ell) \Bigg) ds. \tag{3.16}$$

On the other hand, we get from Lemma 3.2 that

$$\widehat{J}_t = \int_0^t \widehat{A}_s ds + \int_0^t \int_E \gamma(s,e) m^L(ds,de) + \int_0^t \boldsymbol{\alpha}_s^\top dm_s^Z.$$

Hence, Itô's formula gives

$$\widehat{J}_t \phi_t^i = M_t + \int_0^t \Bigg( \phi_s^i \widehat{A}_s + \widehat{J}_s \int_0^1 \varphi(s,\ell) F_{\ell_i}(d\ell)(1 - Y_{s,i})(\widehat{\lambda}_i)_s + \int_0^1 \gamma_i(s,\ell)\varphi(s,\ell) F_{\ell_i}(d\ell)(\widehat{\lambda}_i)_s (1 - Y_{s,i}) \Bigg) ds \tag{3.17}$$

where $M$ is a local $\mathbb{F}^{\mathbb{M}}$-martingale. Recall that $\widehat{J\phi} = \widehat{J}\,\phi$. By the uniqueness of the semimartingale decomposition, (3.16) must equal the finite variation part in (3.17) which leads to

$$0 = \int_0^t \int_0^1 \varphi(s,\ell)(1 - Y_{s,i}) \Bigg( (\widehat{J\lambda_i})_s - \widehat{J}_s(\widehat{\lambda}_i)_s + (\widehat{R_i(\cdot,\ell)\lambda_i})_s - \gamma_i(s,\ell)(\widehat{\lambda}_i)_s \Bigg) F_{\ell_i}(d\ell) ds$$

for all $0 \le t \le T$. Since $\varphi$ was arbitrary and $\gamma$ is predictable, we get (3.13).

In order to establish (3.12) we use a similar argument with $\phi = Z_i$. For this, note that the arising local martingales in the semimartingale decomposition of $JZ_i$ are true martingales by (iv). □

The following theorem describes the dynamics of the gains processes of the traded credit derivatives and gives their instantaneous quadratic covariation.

**Theorem 3.4** *Under* **A1** *and* **A2** *the gains processes $\widehat{g}_1, \ldots, \widehat{g}_N$ of the traded securities have the martingale representation*

$$\widehat{g}_{t,n} = \widehat{g}_{0,n} + \sum_{i=1}^m \int_0^t \int_E \mathbb{1}_{\{\xi = i\}} \gamma_i^{\widehat{g}_n}(s,\ell) \, m^L(ds, d\xi, d\ell) + \int_0^t (\boldsymbol{\alpha}_s^{\widehat{g}_n})^\top dm_s^Z; \tag{3.18}$$

*here the integrands are given by*

$$\boldsymbol{\alpha}_t^{\widehat{g}_n} = \widehat{p_{t,n} \cdot \mathbf{a}}_t - \widehat{p}_{t,n}\, \widehat{\mathbf{a}}_t\,, \tag{3.19}$$

$$\gamma_i^{\widehat{g}_n}(s,\ell) = \frac{1}{(\widehat{\lambda}_i)_{s-}} \Big[ (\widehat{p_n \lambda_i})_{s-} - (\widehat{p}_n)_{s-} (\widehat{\lambda}_i)_{s-} + (\widehat{R_{i,n}(\cdot,\ell)\lambda_i})_{s-} \Big] \text{ with} \tag{3.20}$$

$$R_{i,n}(s,\ell) = p_n\,(s, X_s, L_s + \ell \boldsymbol{e}_i) - p_n(s, X_s, L_s) + d_{2,n}\,(s, X_s, L_s + \ell \boldsymbol{e}_i) \tag{3.21}$$

*and* $\boldsymbol{e}_i$ *the ith unit vector in* $\mathbb{R}^m$. *The predictable quadratic variation of the gains processes* $\widehat{g}_1, \ldots, \widehat{g}_N$ *with respect to* $\mathbb{F}^{\mathbb{M}}$ *satisfies* $d\langle \widehat{g}_i, \widehat{g}_j \rangle_t^{\mathbb{M}} = v_t^{ij}\, dt$ *with*

$$v_t^{ij} := \sum_{k=1}^m \int_0^1 \gamma_k^{\widehat{g}_i}(t,\ell)\, \gamma_k^{\widehat{g}_j}(t,\ell)\, F_{\ell_k}(d\ell)\, \widehat{\lambda}_{t,k}(1 - Y_{t,k}) + \sum_{k=1}^l \alpha_{t,k}^{\widehat{g}_i} \alpha_{t,k}^{\widehat{g}_j}\,. \tag{3.22}$$

*Proof* We apply Theorem 3.3 to the $\mathbb{F}$-martingale $J_t = \mathbb{E}(D_{T,n}|\mathcal{F}_t)$ and verify the conditions therein: first, $[J,B] = 0$ as $B$ is independent of $X$ and $L$. As $d_{1,n}$ and $d_{2,n}$ from (3.1) are bounded, so is $J$. By **A1** $\lambda_i$ is bounded and hence (i) holds. Second, $M^J = J$ is bounded and hence a square-integrable true martingale which gives (ii). Next, note that $J_t = p_n(t, X_t, L_t) + D_{t,n}$. Hence

$$\begin{aligned}
[J, Y_i]_t &= (\Delta J_{\tau_i} \Delta Y_{\tau_i,i}) \mathbb{1}_{\{\tau_i \le t\}} \\
&= \mathbb{1}_{\{\tau_i \le t\}} \Big( p_n(\tau_i, X_{\tau_i}, L_{\tau_i}) - p_n(\tau_i, X_{\tau_i -}, L_{\tau_i -}) + \Delta D_{\tau_i,n} \Big) \\
&= \int_0^t \int_E \mathbb{1}_{\{\xi = i\}} R_{i,n}(s-, \ell) \mu^L(ds, d\xi, d\ell)
\end{aligned}$$

with $R_{i,n}$ as in (3.21). Here we have implicitly used, that $p_n$ is the solution of a backward equation for the Markov process $(X,L)$ and therefore continuous in $t$, and that $X$ and $L$ have no joint jumps. As $R$ is bounded, (iii) follows. Next, as $J$ is bounded, $\int J dB_j$ is a true martingale. Moreover,

$$\int_0^t Z_{s,j} dJ_s = \int_0^t \int_0^s a_j(X_u) du\, dJ_s + \int_0^t B_{s,j} dJ_s.$$

As $\mathbf{a}(\cdot)$ is bounded, the first term has integrable quadratic variation and is thus a true martingale. Since $B$ and $J$ are independent, we get

$$\mathbb{E}\Big( \int_0^t (B_{s,j})^2 d[J]_s \Big) = \mathbb{E}\Big( \int_0^t \mathbb{E}(B_{s,j}^2) d[J]_s \Big) \le T\mathbb{E}([J]_T) < \infty.$$

This together yields (iv) and hence (3.18) with $p_{t,n}$ instead of $J$ in (3.19) and (3.20). Recall that $\widehat{g}_{t,n} = \widehat{p}_{t,n} + D_{t,n}$ where $D_{t,n}$ is $\mathbb{F}_t^{\mathbb{M}}$-measurable. This allows us to replace $J$ by $p_{t,n}$ and yields the first part of the theorem.

The second part (the statement regarding the predictable quadratic variations) follows immediately from (3.18) and (3.7). □

*Remark 3.5* The assumption that $X$ is a finite state Markov chain was only used to insure integrability conditions in Theorem 3.3 and in Theorem 3.4 so that these results are easily extended to a more general setting. The filtering results in Section 3.3 below on the other hand do exploit the specific structure of $X$.

## 3.3 Filtering and factor representation of market prices

Since $X$ is a finite state Markov chain, the conditional distribution of $X_t$ given $\mathcal{F}_t^{\mathbb{M}}$ is given by the vector $\boldsymbol{\pi}_t = (\pi_t^1, \ldots, \pi_t^K)^\top$ with $\pi_t^k := \mathbb{Q}(X_t = k|\mathcal{F}_t^{\mathbb{M}})$. The following proposition shows that the process $\boldsymbol{\pi}$ is the solution of a $K$-dimensional SDE system driven by $m^Z$ and the $\mathbb{F}^{\mathbb{M}}$-martingale $M$ given by

$$M_{t,j} := Y_{t,j} - \int_0^t (1 - Y_{s,j})\,(\widehat{\lambda}_j)_s ds = \int_0^t \int_E \mathbb{1}_{\{\xi=j\}} m^L(ds, d\xi, d\ell),\ 1 \le j \le m.$$

**Proposition 3.6** *Denote the generator matrix of $X$ by $(q(i,k))_{1\le i,k\le K}$. Then, for $k = 1, \ldots, K$,*

$$d\pi_t^k = \sum_{i\in S^X} q(i,k)\pi_t^i dt + (\boldsymbol{\gamma}^k(\boldsymbol{\pi}_{t-}))^\top dM_t + (\boldsymbol{\alpha}^k(\boldsymbol{\pi}_t))^\top dm_t^Z\,, \tag{3.23}$$

*with coefficients given by*

$$\gamma_j^k(\boldsymbol{\pi}_t) = \pi_t^k \Big( \frac{\lambda_j(k)}{\sum_{i\in S^X} \lambda_j(i)\pi_t^i} - 1 \Big), \quad 1 \le j \le m, \tag{3.24}$$

$$\boldsymbol{\alpha}^k(\boldsymbol{\pi}_t) = \pi_t^k \Big( \mathbf{a}(k) - \sum_{i\in S^X} \pi_t^i \mathbf{a}(i) \Big)\,. \tag{3.25}$$

*Proof* Denote the generator of $X$ by $\mathscr{L}$ and set $f_k(x) = \mathbb{1}_{\{x=k\}}$. Then the $\mathbb{F}$-semimartingale decomposition of $(f_k(X_t))_{t\ge 0}$ is

$$f_k(X_t) = f_k(X_0) + \int_0^t \mathscr{L} f_k(X_s)\,ds + \Big( f_k(X_t) - f_k(X_0) - \int_0^t \mathscr{L} f_k(X_s)\,ds \Big)\,.$$

Note that $\pi^k = \widehat{f}_k$ and that $\mathscr{L} f_k(X_t) = q(X_t, k)$. We apply Theorem 3.3 with $J = f_k(X_t) = \mathbb{1}_{\{X_t=k\}}$. First, $[f_k(\cdot), B] = [M^J, B] \equiv 0$, as $B$ is continuous and $f_k(\cdot)$ is of finite variation. Moreover, $[f_k(\cdot), Y_i] = 0$ for all $i$ as $X$ and $Y$ have a.s. no common jumps, so that the random function $R_i$ in Condition (iii) of Theorem 3.3 vanishes for all $i$. Boundedness of $J$ implies Conditions (i)-(iv) from that theorem by a similar argument as in the proof of Theorem 3.4. Hence

$$d\pi_t^k = \widehat{q(X_t, k)} dt + \int_E \sum_{i=1}^m \gamma_i(t,\ell) \mathbb{1}_{\{\xi=i\}}\, m^L(dt, d\xi, d\ell) + \boldsymbol{\alpha}_t^\top dm_t^Z$$

with $\gamma_i$ given by

$$\gamma_i(t,\ell) = \frac{1}{(\widehat{\lambda}_i)_{t-}} \left( (\widehat{\lambda_i(k)J})_{t-} - (\widehat{\lambda}_i)_{t-} \widehat{J}_{t-} \right) = \frac{1}{(\widehat{\lambda}_i)_{t-}} \left( \lambda_i(k)\pi^k_{t-} - (\widehat{\lambda}_i)_{t-}\, \pi^k_{t-} \right).$$

Note that $(\widehat{\lambda}_i)_{t-} = \sum_{k \in S^X} \lambda_i(k)\pi^k_{t-}$. As $\gamma_i(t,\ell)$ does not depend on $\ell$,

$$\int_0^t \int_E \gamma_i(s,\ell) 1\!\!1_{\{\xi = i\}} m^L(ds, d\xi, d\ell) = \int_0^t \gamma_i^k(\boldsymbol{\pi}_{s-}) dM_{s,i}\,,$$

and (3.24) follows. For (3.25), note finally that

$$\boldsymbol{\alpha}^k_t = f_k(\widehat{X_t)\mathbf{a}(}X_t) - \widehat{f_k(X_t)}\widehat{\mathbf{a}(X_t)} = \pi^k_t\,\mathbf{a}(k) - \pi^k_t \sum_{i \in S^X} \pi^i_t \mathbf{a}(i)\,.$$

*Remark 3.7* Related results have previously appeared in the filtering literature. For the case of diffusion observations, (3.23) is given in Liptser & Shiryaev (2000) and Wonham (1965). For the case of marked-point-process observations we refer to Brémaud (1981) and further references therein.

*Contagion.* The previous results permit us to give an explicit expression for the contagion effects induced in our model. For $i \neq j$ we get from (3.24) that

$$\begin{aligned} \widehat{\lambda}_{\tau_j,i} - \widehat{\lambda}_{\tau_j-,i} &= \sum_{k=1}^K \lambda_i(k) \cdot \pi^k_{\tau_j-} \left( \frac{\lambda_j(k)}{\sum_{l=1}^K \lambda_j(l) \pi^l_{\tau_j-}} - 1 \right) \\ &= \frac{\operatorname{cov}^{\boldsymbol{\pi}_{\tau_j-}}\left(\lambda_i, \lambda_j\right)}{\mathbb{E}^{\boldsymbol{\pi}_{\tau_j-}}(\lambda_j)}\,. \end{aligned} \tag{3.26}$$

Moreover, $\boldsymbol{\pi}_{\tau_j-}$ gives the conditional distribution of $X$ immediately prior to the default event. According to (3.26), default contagion is proportional to the covariance of the random variables $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under $\boldsymbol{\pi}_{\tau_j-}$. This implies that contagion is largest for firms with similar characteristics and hence a high correlation of $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$. This effect is very intuitive.

The process $(L, \boldsymbol{\pi})$ is a natural state variable process for the model: first, $(L, \boldsymbol{\pi})$ is a Markov process (see Proposition 3.8 below). Second, all quantities of interest at time $t$ can be represented in terms of $L_t$ and $\boldsymbol{\pi}_t$. In particular, the market values from (3.5) can be expressed as follows

$$\widehat{p}_{t,n} = \sum_{k \in S^X} p_n(t,k,L_t)\,\pi^k_t,$$

and a similar representation can be obtained for the integrands $\boldsymbol{\alpha}^{\widehat{g}_n}_t$ and $\gamma_i^{\widehat{g}_n}(t,\ell)$ from Theorem 3.4. Motivated by these two observations we call $(L, \boldsymbol{\pi})$ the *market state process*. The next result characterizes its probabilistic properties.

**Proposition 3.8** *The market state process $(L, \boldsymbol{\pi})$ is the unique solution of the martingale process associated with the generator $\mathscr{L}$ given by formula* (A.1) *in the appendix. In particular, $(L, \boldsymbol{\pi})$ is an $\mathbb{F}^{\mathbb{M}}$-Markov process of jump-diffusion type.*

To prove this claim we use Itô's formula to identify the generator of $(L, \boldsymbol{\pi})$ and show uniqueness of the related martingale problem; see Appendix A.2 for details.

## 4 Practical issues: pricing, calibration and hedging

In this section we discuss the pricing, the calibration, and the hedging of credit derivatives. Consider a non-traded credit derivative. In accordance with (3.2), we define the price at time $t$ of the credit derivative as conditional expectation of the associated payoff given $\mathcal{F}_t^{\mathbb{M}}$. For the credit derivatives common in practice this conditional expectation is given by a function of the current market state $(L_t, \boldsymbol{\pi}_t)$, as we show in Section 4.1. Here a major issue arises for the application of the model: as explained in the introduction, we view the process $Z$ generating the market filtration $\mathbb{F}^{\mathbb{M}}$ as abstract source of information so that the process $\boldsymbol{\pi}$ is not directly observable for investors. On the other hand, pricing formulas and hedging strategies need to be evaluated using only publicly available information. Section 4.2 is therefore devoted to model calibration. In particular we explain how to determine $\boldsymbol{\pi}_t$ from prices of traded securities observed at time $t$. In Section 4.3 we finally consider dynamic hedging strategies in our framework.

### 4.1 Pricing

Basically all credit derivatives common in practice fall in one of the following two classes:

*Options on the loss state.* This class comprises derivatives with payoff given by an $\mathbb{F}^L$-adapted dividend stream $D$ of the form (3.1); examples are typical basket derivatives or (bespoke) CDOs. As in (3.4), the hypothetical value of an option on the loss state in the underlying Markov model, $\mathbb{E}\big(D_T - D_t | \mathcal{F}_t\big)$, is equal to $p(t, X_t, L_t)$ for some function $p : [0, T] \times S^X \times [0, 1]^m \to \mathbb{R}$.[2] Hence, the price of the option at time $t$ is given by

$$\widehat{p}_t := \mathbb{E}(D_T - D_t | \mathcal{F}_t^{\mathbb{M}}) = \sum_{k \in S^X} p(t, k, L_t) \pi_t^k \, . \tag{4.1}$$

Note that for an option on the loss state the price $\widehat{p}_t$ depends only on the current market state $(L_t, \boldsymbol{\pi}_t)$ and on the function $p(\cdot)$ that gives the hypothetical value of the option in the underlying Markov model. Hence the precise

[2] The evaluation of $p(\cdot)$ can be done with similar methods as in Remark 3.1.

form of the function $\mathbf{a}(\cdot)$ from **A2** and thus of the dynamics of $\boldsymbol{\pi}$ is irrelevant for the pricing of these claims; the dynamics of $\boldsymbol{\pi}$ do however matter in the computation of hedging strategies as will be shown below.

*Options on traded assets.* This class contains derivatives whose payoff depends on the future *market value* of traded securities: the payoff is of the form $\tilde{H}(L_U, \widehat{p}_{U,1}, \ldots, \widehat{p}_{U,N})$, to be paid at maturity $U \leq T$. Examples include options on corporate bonds, options on CDS indices or options on synthetic CDO tranches.

Denote by $\mathcal{M} = \{\boldsymbol{\pi} \geq 0\colon \sum_{k \in S^X} \pi_k = 1\}$ the unit simplex in $\mathbb{R}^K$. Using (4.1), the payoff of the option can be written in the form $H\big(U, L_U, \boldsymbol{\pi}_U\big)$, where

$$H\left(t, L, \boldsymbol{\pi}\right) = \tilde{H}\Big(L, \sum_{k \in S^X} \pi^k p_1(t,k,L), \ldots, \sum_{k \in S^X} \pi^k p_N(t,k,L)\Big).$$

Since the market state $(L, \boldsymbol{\pi})$ is a $\mathbb{F}^{\mathbb{M}}$-Markov process, the price of the option at time $t$ is of the form

$$\mathbb{E}\left(H(U, L_U, \boldsymbol{\pi}_U) | \mathcal{F}_t^{\mathbb{M}}\right) = h(t, L_t, \boldsymbol{\pi}_t), \tag{4.2}$$

for some $h\colon [0,U] \times [0,1]^m \times \mathcal{M} \to \mathbb{R}$, where $\mathcal{M} = \{\boldsymbol{\pi} \geq 0\colon \sum_{k \in S^X} \pi_k = 1\}$ denotes the unit simplex in $\mathbb{R}^K$. By standard results the function $h$ is a solution of the backward equation

$$\partial_t h(\cdot) + \mathscr{L} h(\cdot) = 0.$$

However, the market state is usually a high-dimensional process so that the practical computation of $h(\cdot)$ will typically be based on Monte Carlo methods. Note that for an option on traded assets the function $h(\cdot)$ and hence its price depends on the entire generator $\mathscr{L}$ of $(L, \boldsymbol{\pi})$ and therefore also on the form of $\mathbf{a}(\cdot)$.

*Example 4.1 (Options on a CDS index)* Index options are a typical example for an option on a traded asset. Upon exercise the owner of the option holds a protection-buyer position on the underlying index with a pre-specified spread $S$ (the exercise spread of the option); moreover, he obtains the cumulative portfolio loss up to time $U$. Denote by $V^{\text{def}}(t, X_t, L_t)$ and $V^{\text{prem}}(t, X_t, L_t)$ the full-information value of the default and the premium payment leg of the CDS index. In our setup the value of the option at maturity $U$ is then given by the following function of the market state at $U$:

$$h(L_U, \boldsymbol{\pi}_U) = \Big(\bar{L}_U + \sum_{k \in S^X} \pi_U^k \big(V^{\text{def}}(U,k,L_U) - S V^{\text{prem}}(U,k,L_U)\big)\Big)^+, \tag{4.3}$$

with $\bar{L}_t = \sum_{i=1}^m L_{t,i}$. Numerical results on the pricing of credit index options in our setup can be found in (Frey & Schmidt 2010).

4.2 Calibration

Model calibration involves two separate tasks: on the one hand, at fixed current time $t$ one needs to determine $\boldsymbol{\pi}_t$, the current value of the process $\boldsymbol{\pi}$. On the other hand, the model parameters (the generator matrix of $X$ and parameters of the functions $\mathbf{a}(\cdot)$ and $\lambda_i(\cdot), i = 1, \ldots, m$) need to be estimated. The latter task depends on the specific parametrization of the model and on the available data. We discuss parameter estimation for the frailty model in Section 5.

Here we concentrate on the determination of $\boldsymbol{\pi}_t$. The key point is the observation that the set of all probability vectors consistent with the price information at $t$ can be described in terms of a set of linear inequalities. Details depend on the way the traded credit derivatives are quoted in practice, and we discuss zero coupon bonds and CDSs as representative examples.

*Zero-bond.* Consider a zero coupon bond on firm $i$. Its hypothetical value prior to default in the underlying Markov model is given by

$$\mathbb{E}\Big(e^{-\int_t^T \lambda_i(X_s)\,ds} \,\big|\, X_t = k\Big) =: p_i(t,k).$$

The precise form of $p_i(\cdot)$ is irrelevant here. Suppose that at $t$ we observe bid and ask quotes $\underline{p} \leq \overline{p}$ for the bond. In order to be consistent with this information, a solution $\boldsymbol{\pi} \in \mathcal{M}$ of the calibration problem at $t$ needs to satisfy the linear inequalities

$$\underline{p} \leq \sum_{k \in S^X} p_i(t,k)\pi_k \leq \overline{p}\,.$$

*Credit default swap.* A CDS on firm $i$ is quoted by its spread $S_t$. The spread is chosen in such a way that the market value of the contract is zero. In our setup this translates as follows. Let

$$\begin{aligned} V_i^{\mathrm{def}}(t,k) &:= \mathbb{E}\Big(\int_t^T dL_{s,i} \,\big|\, X_t = k, L_{t,i} = 0\Big), \\ V_i^{\mathrm{prem}}(t,k) &:= \sum_{t_j \in (t,T]} \mathbb{Q}\big(L_{t_j,i} = 0 | X_t = k, L_{t,i} = 0\big). \end{aligned} \tag{4.4}$$

Then the quoted CDS spread solves $\sum_{k \in S^X} \pi_t^k \left(S_t\, V_i^{\mathrm{prem}}(t,k) - V_i^{\mathrm{def}}(t,k)\right) = 0$, given $\tau_i > t$. Suppose now that at time $t$ we observe bid and ask spreads $\underline{S} \leq \overline{S}$ for the contract. Then $\boldsymbol{\pi}$ must satisfy the following two inequalities:

$$\begin{aligned} &\sum_{k \in S^X} \pi_k \big(\underline{S} V_i^{\mathrm{prem}}(t,k) - V_i^{\mathrm{def}}(t,k)\big) \leq 0\,, \\ &\sum_{k \in S^X} \pi_k \big(\overline{S} V_i^{\mathrm{prem}}(t,k) - V_i^{\mathrm{def}}(t,k)\big) \geq 0\,. \end{aligned} \tag{4.5}$$

Standard linear programming techniques can be used to detect if the system of linear inequalities corresponding to the available market quotes is nonempty and to determine a solution $\boldsymbol{\pi} \in \mathcal{M}$. [3] In case that there is more than one probability vector $\boldsymbol{\pi} \in \mathcal{M}$ consistent with the given price information at time $t$, a unique solution $\boldsymbol{\pi}^*$ of the calibration problem can be determined by a suitable *regularization procedure.* More precisely, given a reference measure $\boldsymbol{\nu}$ on $S^X$ and a distance $d$, $\boldsymbol{\pi}^*$ is given by

$$\boldsymbol{\pi}^* = \operatorname{argmin}\big\{d(\boldsymbol{\pi}, \boldsymbol{\nu}) : \boldsymbol{\pi} \text{ is consistent with the price information in } t\big\}. \tag{4.6}$$

A possible choice is to minimize relative entropy to the uniform distribution; in that case $d(\boldsymbol{\pi}, \boldsymbol{\nu}) \propto \sum_{k \in S^X} \pi_k \ln \pi_k$ and the optimization problem that defines $\boldsymbol{\pi}^*$ is convex.

## 4.3 Hedging

Hedging is a key issue in the management of portfolios of credit derivatives. The standard market practice is to use sensitivity-based hedging strategies computed by ad-hoc rules within the static base-correlation framework; see for instance Neugebauer (2006). Clearly, it is desirable to work with hedging strategies which are based on a methodologically sound approach instead. In this section we therefore use our results from Section 3 to derive model-based dynamic hedging strategies. We expect the market to be incomplete, as the prices of the traded credit derivatives follow a jump-diffusion process. In order to deal with this problem we use the concept of risk minimization as introduced by Föllmer & Sondermann (1986). The hedging of credit derivatives via risk minimization is also studied in Frey & Backhaus (2010) and Cont & Kan (2008), albeit in a different setup; other relevant contributions include the papers Laurent, Cousin & Fermanian (2007) or Bielecki, Jeanblanc & Rutkowski (2007).

We begin by recalling the notion of a risk-minimizing hedging strategy. Consider traded assets with prices $\widehat{\mathbf{p}}$ and associated filtration $\mathbb{F}^{\widehat{\mathbf{p}}}$. Denote by $\widehat{\mathbf{g}} = (\widehat{g}_1, \ldots, \widehat{g}_N)^\top$ the vector of gains processes of the traded securities and by $\mathbf{v}_t = (v_t^{ij})_{1 \le i,j \le N}$ their instantaneous quadratic variation as given in Theorem 3.4, and let $L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ be the space of all $N$-dimensional $\mathbb{F}^{\mathbb{M}}$-predictable processes $\boldsymbol{\theta}$ such that $\mathbb{E}(\int_0^T \boldsymbol{\theta}_s^\top \mathbf{v}_s \boldsymbol{\theta}_s ds) < \infty$. An *admissible trading strategy* is given by a pair $\varphi = (\boldsymbol{\theta}, \eta)$ where $\boldsymbol{\theta} \in L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ and $\eta$ is $\mathbb{F}^{\mathbb{M}}$-adapted. Moreover the value process $V_t = V_t(\varphi) = \boldsymbol{\theta}_t^\top \widehat{\mathbf{p}}_t + \eta_t$ is RCLL and $\mathbb{E}(\sup_{0 \le t \le T} V_t^2) < \infty$. The *cost* process $C = C(\varphi)$ and the *remaining risk* process $R = R(\varphi)$ of the trading strategy $\varphi$ are finally defined by

$$C_t = V_t - \int_0^t \boldsymbol{\theta}_s^\top d\widehat{\mathbf{g}}_s \quad \text{and} \quad R_t = \mathbb{E}\big((C_T - C_t)^2 | \mathcal{F}_t^{\mathbb{M}}\big), \quad t \le T.$$

[3] In abstract terms the set of linear inequalities corresponding to the calibration problem can be written in the form $A\boldsymbol{\pi} \le b$. Consider the auxiliary problem $\min c^\top \mathbf{y}$ subject to $A\boldsymbol{\pi} \le b + \mathbf{y}$, $\mathbf{y} \ge 0$ for a suitable vector of weights $c > 0$. Consider a solution $(\mathbf{y}, \boldsymbol{\pi})$ of the auxiliary problem. If $\mathbf{y} = 0$, $\boldsymbol{\pi}$ is a solution to the original calibration problem.

Consider now a claim $H$ with square integrable, $(\mathbb{F}^L \vee \mathbb{F}^{\widehat{\mathbf{p}}})$-adapted cumulative dividend stream $D$ such as the credit derivatives considered in Section 4.1. An admissible strategy $\varphi$ is called a *risk-minimizing hedging strategy* for $H$ if $V_T(\varphi) = D_T$ and if moreover for any $t \in [0,T]$ and any admissible strategy $\tilde{\varphi}$ satisfying $V_T(\tilde{\varphi}) = D_T$ we have $R_t(\varphi) \le R_t(\tilde{\varphi})$.

Risk-minimization is well-suited for our setup as the ensuing hedging strategies are relatively easy to compute and as it suffices to know the risk-neutral dynamics of credit derivative prices. From a methodological point of view it might however be more natural to minimize the remaining risk under the historical probability measure. This would lead to alternative quadratic-hedging approaches; see for instance Schweizer (2001). However, the computation of the corresponding strategies becomes a very challenging problem. Moreover, it is quite hard to determine the dynamics of CDS and CDO spreads under the historical measure as this requires the estimation of historical default intensities.

**Proposition 4.2** *Consider a claim $H$ with cumulative dividend stream $D_T \in L^2(\Omega, \mathcal{F}_T^L \vee \mathcal{F}_T^{\widehat{\mathbf{p}}}, \mathbb{Q})$ and gains process $\widehat{g}_t^H = \mathbb{E}(D_T | \mathcal{F}_t^{\mathbb{M}})$. A risk-minimizing hedging strategy $\varphi = (\boldsymbol{\theta}, \eta)$ for $H$ is given by*

$$\boldsymbol{\theta}_t = \mathbf{v}_{t-}^{\mathrm{inv}} \frac{d}{dt} \langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}} \quad \text{and } \eta_t = \widehat{g}_t^H - \theta_t^\top \widehat{\mathbf{p}}_t, \quad t \le T \tag{4.7}$$

*where $\mathbf{v}_t^{\mathrm{inv}}$ denotes the pseudo inverse of the instantaneous quadratic variation $\mathbf{v}_t$ and where $\frac{d}{dt}\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}$ is the predictable Lebesgue-density of $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}$.*

*Proof* It is well-known that risk-minimizing hedging strategies relate to the *Galtchouk-Kunita-Watanabe* decomposition of the martingale $g^H$ with respect to the gains processes of traded securities:

$$\widehat{g}_t^H = \widehat{g}_0^H + \sum_{n=1}^{N} \int_0^t \xi_{s,n}^H \, d\widehat{g}_{s,n} + H_t^{\perp}, \quad t \le T \tag{4.8}$$

with $\xi_i^H \in L^2(\widehat{\mathbf{g}}, \mathbb{F}^{\mathbb{M}})$ and $\langle H^\perp, \widehat{\mathbf{g}} \rangle^{\mathbb{M}} \equiv 0$: one has that $\boldsymbol{\theta} = \xi^H$, $V_t(\varphi) = \widehat{g}_t^H$ and $C = H^\perp$. From $\langle H^\perp, \widehat{\mathbf{g}} \rangle^{\mathbb{M}} \equiv 0$ we get the following equation for $\xi^H$:

$$\frac{d}{dt} \langle \widehat{g}^H, \widehat{g}_j \rangle_t^{\mathbb{M}} = \sum_{n=1}^{N} \xi_{t,j}^H v_t^{n,j}, \quad t \le T; \tag{4.9}$$

by definition of $\mathbf{v}_t^{\mathrm{inv}}$ a solution of (4.9) is given by $\mathbf{v}_t^{\mathrm{inv}} \frac{d}{dt} \langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle_t^{\mathbb{M}}$. $\square$

The crucial step in applying Proposition 4.2 is to compute the quadratic variation $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle^{\mathbb{M}}$, and we now explain how this can be achieved for the claims considered in Section 4.1. First, if $H$ represents an option on the loss state, by an argument analogous to the proof of Theorem 3.4 one obtains that $\widehat{g}_t^H$ has a representation of the form (3.18) with integrands $\boldsymbol{\alpha}^H$ and $\boldsymbol{\gamma}^H$ given by the analogous expressions to (3.19) and (3.20). Then, $\langle \widehat{g}^H, \widehat{\mathbf{g}} \rangle^{\mathbb{M}}$ is given by

$$d\langle \widehat{g}^H, \widehat{g}_i \rangle_t^{\mathbb{M}} = \Big( \sum_{j=1}^{m} \int_0^1 \Big( \gamma_j^H(t,l) \gamma_j^{\widehat{g}_i}(t,l) \Big) F_{l_j}(dl) \widehat{\lambda}_{t,j} (1 - Y_{t,j}) + \sum_{j=1}^{l} \alpha_{t,j}^H \alpha_{t,j}^{\widehat{g}_i} \Big) dt \,, \quad 1 \le i \le N. \tag{4.10}$$

The main step in computing $\boldsymbol{\alpha}^H$ and $\boldsymbol{\gamma}^H$ is to compute the function $p(t,k,L)$ that gives the hypothetical value of the derivative in the underlying Markov model.

Second, if $H$ is an option on traded assets with payoff $\tilde{H}(\widehat{\mathbf{p}}_U, L_U)$, we have $\widehat{g}_t^H = h(t, L_t, \boldsymbol{\pi}_t)$, compare (4.2). Applying Itô's formula to $h(t, L_t, \boldsymbol{\pi}_t)$ gives a martingale representation of $\widehat{g}^H$, see the proof of Proposition 3.8 in the appendix and the related comment A.1. From this $[\widehat{g}^H, \widehat{g}_n]$ and its compensator $\langle \widehat{g}^H, \widehat{g}_n \rangle^{\mathbb{M}}$ can be computed via standard arguments. Note finally that in both cases $\boldsymbol{\theta}_t$ depends only on the current market state $(L_t, \boldsymbol{\pi}_t)$. A numerical example is presented in Section 5.3.

## 5 Numerical case studies

In this section we present results from a number of small numerical and empirical case studies that serve to further illustrate the application of the model to practical problems.

### 5.1 Dynamics of credit spreads

As remarked earlier, the fact that in our model prices of traded securities are given by the conditional expectation given the market filtration leads to rich credit-spread dynamics with spread risk (random fluctuations of credit spreads between defaults) and default contagion. This is illustrated in Figure 5.1 where we plot a simulated credit-spread trajectory. The random fluctuation of the credit spreads between defaults as well contagions effects at default times (e.g. around $t = 600$) can be spotted clearly.

### 5.2 Calibration to CDO spreads

We work in a frailty model where the generator matrix of $X$ is identically zero, see Example 2.1. In that model default times are independent, exponentially distributed random variables given $X$, and dependence is created by mixing over the states of $X$. Moreover, the computation of full-information values is particularly easy. A static model of this form (no dynamics of $\boldsymbol{\pi}$) has been

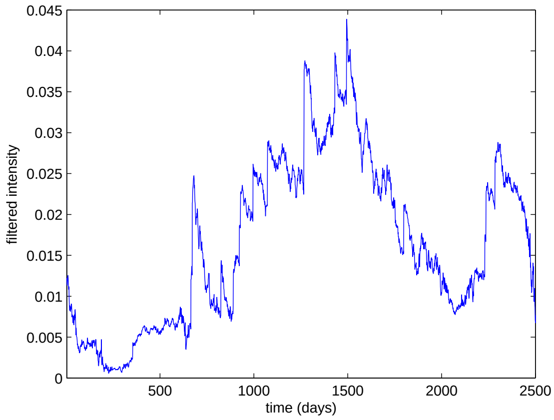


**Fig. 5.1** A simulated path of credit spreads under zero recovery. The graph has been created for the case where $X$ is a Markov chain with next-neighbour dynamics (Example 2.1).

proposed by Hull & White (2006) under the label *implied copula model*; see also Rosen & Saunders (2009). Since prices of CDS-indices and CDO tranches are independent of the form of the dynamics of $\boldsymbol{\pi}$, pricing and calibration techniques for these products in the frailty model are similar to those in the implied copula models. However, our framework permits the pricing of tranche- and index options and the derivation of model-based hedging strategies, issues which cannot be addressed in the static implied copula models.

We choose a parametrization which is motivated by the popular one-factor Gauss or double-$t$ copula models. Assume $X$ takes values in $\{x_1, \dots, x_K\} \subset \mathbb{R}$ and that firm $i$ defaults in a given year, if

$$\sqrt{\rho}X + \sqrt{1-\rho}\,\epsilon_i > d_i;$$

here $\epsilon_1, \dots, \epsilon_m$ are i.i.d. standard normal random variables, $\rho \in (-1, 1)$ and $d_1, \dots, d_m \in \mathbb{R}$ are given default thresholds. Hence, given $X = x_k$, the one-year default probability of firm $i$ is given by

$$p_i(x_k) := \Phi\left(\frac{\sqrt{\rho}}{\sqrt{1-\rho}}x_k - \frac{d_i}{\sqrt{1-\rho}}\right) \tag{5.1}$$

and the corresponding default intensity is $\lambda_i(x_k) = -\ln(1 - p_i(x_k))$. In the homogeneous version of this model all thresholds are identical, that is $d_1 = \cdots = d_m$.

In order to obtain calibration and pricing results which are robust with respect to the precise location of the grid points, it is advisable to choose the number of states $K$ relatively large. We work with $K = 100$, and we choose [4] the levels $x_1, \dots, x_K$ as quantiles of a $t_6$-distribution and set $\rho = 0.5$.

[4] Experiments with different values of these parameters yielded similar results.

The following algorithm determines the thresholds $\mathbf{d} = (d_1, \ldots, d_m)$ and probabilities $\boldsymbol{\pi} = (\pi^1, \ldots, \pi^K)$ from $m$ individual CDS spreads and CDO tranche spreads.

**Algorithm 5.1** 1. Choose initial values for $\boldsymbol{\pi}^{(0)}$, for instance the uniform distribution on $x_1, \ldots, x_K$.
2. Given $\boldsymbol{\pi}^{(0)}$, compute the thresholds $\mathbf{d}^{(1)}$ such that CDS spreads are matched exactly, using that the CDS-spread of firm $i$ is decreasing in $d_i$.
3. Given $\mathbf{d}^{(1)}$, determine $\boldsymbol{\pi}^{(1)}$ from CDO and CDS spreads via linear programming as outlined in Section 4.2.
4. Iterate Steps 2. and 3. until a desired precision level is reached.

*Comments.* In Step 3. one could alternatively minimize the squared distance between market- and model value via quadratic programming.

To obtain smoother results, a regularization procedure such as entropy minimization can be applied to the outcome of Step 4 (see (4.6)).

In the homogeneous case (i.e. $d_1 = \cdots = d_m$) the parameter $d_1$ can be kept fixed during the calibration.

*Calibration results.* We present results from two types of numerical experiments[5]. First, we calibrated the homogeneous version of the model to tranche and index spreads from the iTraxx Europe in the years 2006 (before the credit crisis) and 2009. The calibration precision (Step 4) was chosen as 1% relative error and regularization was used to obtain a smooth distribution.

The outcome is plotted in Figure 5.2. We clearly see that with the emergence of the credit crisis the calibration procedure puts more mass on states where the default intensity is high (2009-data). This reflects the increased awareness of future defaults and the increasing risk aversion in the market after the arrival of the crisis. This effect can also be observed in other model types; see for instance (Brigo, Pallavicini & Torresetti 2009).

Second, we calibrated the inhomogeneous version of the model jointly to CDS spreads and CDO tranche spreads, with quite satisfactory results. The data consists of iTraxx Europe tranche spreads and CDS spreads from the corresponding constituents on the same day in 2009 as in the first experiment. This is a challenging calibration exercise, such that we choose the calibration precision (Step 4) as 4% relative error for the tranche data and up to 8% relative error for the single-name CDSs; see the right graph in Figure 5.3. In the left graph in Figure 5.3 we plot the distribution of the average default probability: $\frac{1}{m}\sum_{i=1}^{m} p_i(\cdot)$. The outcome is qualitatively similar to the homogeneous case, however, the distribution is less smooth due to the additional constraints in the calibration problem.

[5] All calibrations run on a Pentium-III in about 1 minute.

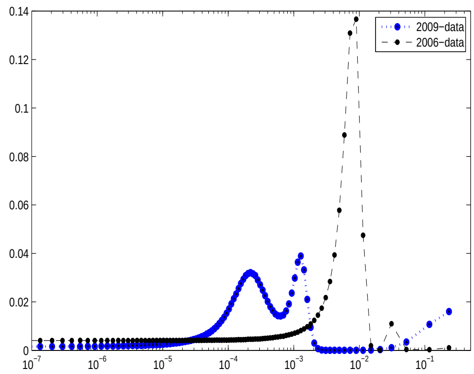


**Fig. 5.2** One-year default probabilities $(p(x_1), \ldots, p(x_{100})$, as in (5.1)) obtained via calibration in a homogeneous one-factor frailty model for data from 2006 and 2009. Note that logarithmic scaling is used on the x-axis.

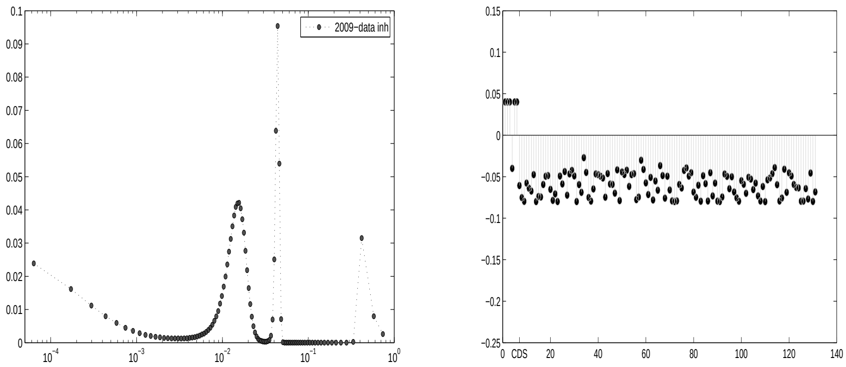


**Fig. 5.3** Left: Average one-year default probabilities $(p(x_1), \ldots, p(x_{100})$, as in (5.1)) obtained via calibration in a one-factor frailty model for CDS and CDO data from 2009. Right: Corresponding relative calibration errors for tranches (first 6 data points) and CDSs in Step 4.

## 5.3 Hedging of CDO tranches

Finally we consider the hedging of synthetic CDO tranches on the iTraxx Europe, using the underlying CDS index as hedging instrument. This choice is motivated by tractability reasons: it is much easier to manage a hedge portfolio in the CDS index than in the 125 single-name CDSs on the constituents of the index. Moreover, the empirical study in Cont & Kan (2008) shows that the use of single-name CDS as additional hedging instruments does not lead to a significant performance improvement in the hedging of CDOs. Given, that we use the CDS index as hedging instrument, it is most natural to work in a homogeneous model and we use the homogeneous version of the frailty model introduced in Section 5.2. Moreover, we take $Z$ to be one-dimensional and assume that $a(x) = c\ln(\lambda(x))$ for $c \geq 0$.

Recall from Section 4.3 that the function $a(\cdot)$ from **A2** does have an impact on the hedge ratios generated within the model. Hence we need to estimate the parameter $c$. For this we use a simple method-of-moment type procedure. First, we computed the empirical quadratic variation of index spreads. Since there were no defaults within the iTraxx Europe in the observation period, this quantity is an estimate of the continuous part of the quadratic variation of index spreads. Second, we computed the model-implied instantaneous continuous quadratic variation (the quadratic variation of the diffusion part of the spread dynamics) as a function of $c$.[6] Matching these two expressions gives an estimate for $c$. We obtain $c = 0.42$ ($c = 0.71$) for the 2009 data (2006 data).

The hedge ratio $\theta_t$ giving the number of CDS index contracts to be held in the portfolio was computed from Proposition 4.2 using relation (4.10); numerical results are given in Table 5.1. For comparison, we additionally state the hedge ratios for $c = 0$. With $c = 0$ the dynamics of credit derivatives are not affected by fluctuations in $Z$ (no spread risk), and it is easily seen that the risk-minimizing hedging strategy is a perfect replication strategy in that case, see also Frey & Backhaus (2010). We see that $\theta$ is affected by $c$ and that higher values of $c$ mostly lead to larger hedge ratios. Moreover, due to the relatively high probability attributed to extreme states where default probabilities are very high, in 2009 model-implied contagion effects were more pronounced than in 2006. Hence a default leads to a huge increase in the value of a protection-buyer position in the CDS index and in turn to a relatively low hedge ratio for the equity tranche (labeled [0-3]). This is in line with observations in Frey & Backhaus (2010).

| Tranche | [0-3] | [3-6] | [6-9] | [9-12] | [12-22] |
|---|---|---|---|---|---|
| 2006-data, $c$ estimated | 0.469 | 0.091 | 0.053 | 0.036 | 0.096 |
| 2006-data, $c = 0$ | 0.346 | 0.091 | 0.056 | 0.041 | 0.110 |
| 2009-data, $c$ estimated | 0.068 | 0.0392 | 0.0369 | 0.0349 | 0.105 |
| 2009-data, $c = 0$ | 0.066 | 0.0390 | 0.0366 | 0.0346 | 0.105 |

**Table 5.1** Risk-minimizing hedge ratio $\theta$ for hedging a CDO tranche with the underlying CDS index in the homogeneous version of the frailty model. The numbers were computed using the probability vector $\boldsymbol{\pi}^*$ obtained via calibration to the iTraxx data from 2006 and 2009.

[6] For this we used (3.19) to determine the diffusion part in the dynamics of the default leg $\widehat{V}_t^{\mathrm{def}}$ and the premium leg $\widehat{V}_t^{\mathrm{prem}}$. The diffusion part of the spread $S_t = \widehat{V}_t^{\mathrm{def}} / \widehat{V}_t^{\mathrm{prem}}$ can then be obtained via Ito's formula; it is given by $c/\widehat{V}_t^{\mathrm{prem}} \big(\widehat{V_t^{\mathrm{def}} \ln \lambda} - S_t\, \widehat{V_t^{\mathrm{def}} \ln \lambda}\big)$. The model-implied instantaneous continuous quadratic variation is then equal to the square of this expression.

# A Proofs

## A.1 Proof of Lemma 3.2

The proof goes in three steps. First, we introduce a new measure $\mathbb{Q}^*$, so that under $\mathbb{Q}^*$ the $\mathbb{F}^{\mathbb{M}}$-compensator of $\mu^L$ is independent of $X$, and $Z$ is a $\mathbb{Q}^*$-Brownian motion. Next, we use available martingale representation results under $\mathbb{Q}^*$ and finally we change back to the original measure $\mathbb{Q}$.

In the following, we simply write $\mathbf{a}_s := \mathbf{a}(X_s)$. Define the density martingale

$$\eta_t := \prod_{T_n \le t} (\widehat{\lambda}_{T_n-,\xi_n})^{-1} \exp\Bigg( \int_0^t \sum_{i=1}^m (1 - Y_{s,i})(\widehat{\lambda}_{s-,i} - 1) ds - \int_0^t \widehat{\mathbf{a}}_s^\top dm_s^Z - \frac{1}{2} \int_0^t \|\widehat{\mathbf{a}}_s\|^2 \, ds \Bigg), \ t \le T,$$

and note that the dynamics of $\eta$ is

$$d\eta_t = \eta_{t-} \left( \sum_{i=1}^m ((\widehat{\lambda}_{t-,i})^{-1} - 1)\big(dY_{t,i} - \widehat{\lambda}_{t,i}(1 - Y_{t,i})dt\big) - \widehat{\mathbf{a}}_t^\top dm_t^Z \right).$$

By **A1**, $\lambda_j > 0$. As $S^X$ is finite, the functions $\lambda, \widehat{\lambda}, \widehat{\lambda}^{-1}$, and $\widehat{\mathbf{a}}$ are bounded, hence $\eta$ is a true martingale; see for instance Protter (2004), Exercise V.14. Define a measure $\mathbb{Q}^*$ by ${}^{d\mathbb{Q}^*}/{}_{d\mathbb{Q}}{}_{|\mathcal{F}_T^{\mathbb{M}}} = \eta_T$. Then, by the Girsanov theorem, $Z$ is a $\mathbb{Q}^*$-Brownian motion and the $\mathbb{F}^{\mathbb{M}}$-compensator of $\mu^L$ under $\mathbb{Q}^*$ is

$$\nu^*(dt, de) := \sum_{i=1}^m \delta_{\{i\}}(d\xi) \, F_{\ell_i}(d\ell) \, (1 - Y_{t,i})dt.$$

Consider now the $(\mathbb{Q}, \mathbb{F}^{\mathbb{M}})$-martingale $U$ and define the $\mathbb{Q}^*$-integrable random variable $N_T := U_T \eta_T^{-1}$ and the associated martingale $N_t = \mathbb{E}^{\mathbb{Q}^*}(N_T \mid \mathcal{F}_t^{\mathbb{M}})$. Note that by the Bayes formula,

$$N_t = \frac{1}{\eta_t} \mathbb{E}^{\mathbb{Q}}(N_T \eta_T \mid \mathcal{F}_t^{\mathbb{M}}) = \frac{1}{\eta_t} \mathbb{E}^{\mathbb{Q}}(U_T \mid \mathcal{F}_t^{\mathbb{M}}) = \frac{U_t}{\eta_t}.$$

The next step is to establish a martingale representation. For this we rely on representation results for jump diffusions from Jacod & Shiryaev (2003). Therefore we need to rewrite $L$ in a suitable form: consider the semimartingale $S := (Z, Y, L)^\top$, such that the jumps of $S$ take values in $\tilde{E} := \mathbb{R}^l \times \{\mathbf{e}_1, \dots, \mathbf{e}_m\} \times (0,1]^m$ where $\mathbf{e}_i$ stands for the $i$-th unit vector in $\mathbb{R}^m$. The $\mathbb{F}^{\mathbb{M}}$-compensator of the random measure $\mu^S$ associated with the jumps of $S$ under $\mathbb{Q}^*$ is

$$\nu^S(dt, d\tilde{e}) := \sum_{i=1}^m \mathbb{1}_{\{\tilde{e} = (0, \mathbf{e}_i, \ell \mathbf{e}_i)\}} \, F_{\ell_i}(d\ell) \, (1 - Y_{t-,i})dt.$$

Theorem III.2.34 Jacod & Shiryaev (2003) now shows that the martingale problem associated with the characteristics of $S$ has a unique solution; Theorem III.4.29 of the same source then gives that the $\mathbb{Q}^*$-martingale $(N_t)_{0 \le t \le T}$ has a representation of the form

$$N_t = \mathbb{E}(U_T) + \int_0^t \tilde{\boldsymbol{\alpha}}_s^\top dZ_s + \int_0^t \int_{\tilde{E}} \tilde{\tilde{\gamma}}(s, \tilde{e}) m^S(ds, d\tilde{e}),$$

where $m^S = \mu^S - \nu^S$. Moreover, $\int_0^T \| \tilde{\boldsymbol{\alpha}}_s \|^2 \, ds < \infty$ $\mathbb{Q}^*$-a.s. as well as $\int_0^T \int_{\tilde{E}} \mid \tilde{\tilde{\gamma}}(s, \tilde{e}) \mid \nu^S(ds, de) < \infty$ $\mathbb{Q}^*$-a.s. Let $\tilde{\gamma}(s, (i, \ell)) := \tilde{\tilde{\gamma}}(s, (0, \mathbf{e}_i, \ell \mathbf{e}_i))$. Then

$$\int_0^t \int_{\tilde{E}} \tilde{\tilde{\gamma}}(s, \tilde{e}) m^S(ds, d\tilde{e}) = \int_0^t \int_E \tilde{\gamma}(s, e) m^*(ds, de),$$

where $m^* := \mu^L - \nu^*$. Hence we established the existence of a martingale representation of $N$ with respect to $(Z, m^*)$.

In order to change back to the original measure, we compute the differential of $U_t = \eta_t N_t$:

$$\begin{aligned}
dU_t &= d(\eta_t N_t) = \eta_{t-} dN_t + N_{t-} d\eta_t + d[\eta, N]_t \\
&= \eta_{t-} \tilde{\boldsymbol{\alpha}}_t^\top \big(dm_t^Z + \widehat{\mathbf{a}}_t dt\big) + \int\limits_E \eta_{t-} \tilde{\gamma}(t,e) m^*(dt, de) \\
&+ \sum_{i=1}^m N_t \eta_t ((\widehat{\lambda}_{t,i})^{-1} - 1)\big(dY_{t,i} - \widehat{\lambda}_{t,i}(1 - Y_{t,i})dt\big) - \eta_t N_t \widehat{\mathbf{a}}_t^\top dm_t^Z - \eta_t \widehat{\mathbf{a}}_t^\top \tilde{\boldsymbol{\alpha}}_t dt \\
&+ \int\limits_E \tilde{\gamma}(t, \xi, \ell) \eta_{t-} (\widehat{\lambda}_{t-,\xi}^{-1} - 1)\, \mu^L(dt, d\xi, d\ell).
\end{aligned}$$

Rearranging terms gives

$$\begin{aligned}
dU_t &= \eta_{t-} \left(\tilde{\boldsymbol{\alpha}}_t^\top - N_{t-} \widehat{\mathbf{a}}_t^\top\right) dm_t^Z \\
&+ \int\limits_E \eta_{t-} \left(N_{t-}((\widehat{\lambda}_{t-,\xi})^{-1} - 1) + (\widehat{\lambda}_{t-,\xi})^{-1} \tilde{\gamma}(t, \xi, \ell)\right) m^L(dt, d\xi, d\ell),
\end{aligned}$$

which is the desired martingale representation for $U$. Moreover, since $\lambda$ is bounded away from zero and since $N$ and $\eta$ have cadlag-paths, we also have $\int_0^T \parallel \boldsymbol{\alpha}_s \parallel^2 ds < \infty$ $\mathbb{Q}$-a.s. as well as $\int_0^T \int_E \mid \gamma(s,e) \mid \nu^L(ds, de) < \infty$ $\mathbb{Q}$-a.s. □

## A.2 Proof of Proposition 3.8

First, we identify the generator of the process $(L, \boldsymbol{\pi})$. Denote by $\mathcal{M} = \{\boldsymbol{\pi} \geq 0 \colon \sum_{k \in S^X} \pi_k = 1\}$ the unit simplex in $\mathbb{R}^K$. Consider $f : [0,T] \times [0,1]^m \times \mathcal{M} \to \mathbb{R}$, sufficiently regular. The Itô formula gives

$$\begin{aligned}
f(t, L_t, \boldsymbol{\pi}_t) &= f(0, L_0, \boldsymbol{\pi}_0) + \int\limits_0^t \partial_t f(\cdot) ds + \sum_{k \in S^X} \int\limits_0^t \partial_{\pi^k} f(\cdot) d\pi_s^k \\
&+ \frac{1}{2} \sum_{k,l \in S^X} \int\limits_0^t \partial_{\pi^k} \partial_{\pi^l} f(\cdot) d[\pi^k, \pi^l]_s^c \\
&+ \sum_{T_n \leq t} \Big( f(T_n, L_{T_n}, \boldsymbol{\pi}_{T_n}) - f(T_n, L_{T_n-}, \boldsymbol{\pi}_{T_n-}) - \sum_{k \in S^X} \partial_{\pi^k} f(\cdot) \Delta\pi_{T_n}^k \Big),
\end{aligned}$$

where $f(\cdot)$ stands for $f(s, L_{s-}, \boldsymbol{\pi}_{s-})$. With (3.23) we obtain

$$\begin{aligned}
\sum_{k \in S^X} \int\limits_0^t \partial_{\pi^k} f(\cdot) d\pi_s^k &= \sum_{k \in S^X} \int\limits_0^t \partial_{\pi^k} f(\cdot) \boldsymbol{\alpha}^k(\boldsymbol{\pi}_s)^\top dm_s^Z \\
&+ \sum_{k \in S^X} \int\limits_0^t \partial_{\pi^k} f(\cdot) \boldsymbol{\gamma}^k(\boldsymbol{\pi}_{s-})^\top dM_s \\
&+ \sum_{k \in S^X} \int\limits_0^t \partial_{\pi^k} f(\cdot) \Big( \sum_{i \in S^X} q(i,k) \pi_s^i \Big) ds.
\end{aligned}$$

Letting $c_{lk}(\boldsymbol{\pi}) := \boldsymbol{\alpha}^k(\boldsymbol{\pi})^\top \boldsymbol{\alpha}^l(\boldsymbol{\pi})$, we have that $d[\pi^k, \pi^l]^c_s = c_{lk}(\boldsymbol{\pi}_s)ds$. Finally, with $\mathbf{e}_i$ being the $i$-th unit vector in $\mathbb{R}^m$,

$$\begin{aligned}
&\sum_{T_n \le t} \Big( f(T_n, L_{T_n}, \boldsymbol{\pi}_{T_n}) - f(T_n, L_{T_n-}, \boldsymbol{\pi}_{T_n-}) - \sum_{k \in S^X} \partial_{\pi^k} f(\cdot) \Delta \pi^k_{T_n} \Big) \\
&= \int_0^t \int_E \sum_{i=1}^m \mathbb{1}_{\{\xi = i\}} \Big( f\Big(s, L_{s-} + \ell \mathbf{e}_i, \pi^1_{s-} \frac{\lambda_i(1)}{(\widehat{\lambda}_i)_{s-}}, \dots, \pi^K_{s-} \frac{\lambda_i(K)}{(\widehat{\lambda}_i)_{s-}}\Big) - f(s, L_{s-}, \boldsymbol{\pi}_{s-}) \\
&\qquad - \sum_{k \in S^X} \partial_{\pi^k} f(\cdot) \pi^k_{s-} \big( \frac{\lambda_i(k)}{(\widehat{\lambda}_i)_{s-}} - 1 \big) \Big) \mu^L(ds, d\xi, d\ell).
\end{aligned}$$

Let $\widehat{\lambda}_i(\boldsymbol{\pi}) := \sum_{l \in S^X} \lambda_i(k) \pi^l$. The above shows that $f(t, L_t, \boldsymbol{\pi}_t) - \int_0^t \mathscr{L} f(s, L_s, \boldsymbol{\pi}_s) ds$ is a (local) martingale where

$$\begin{aligned}
\mathscr{L} f(t, L, \boldsymbol{\pi}) &= \partial_t f(t, L, \boldsymbol{\pi}) + \sum_{k \in S^X} \partial_{\pi^k} f(t, L, \boldsymbol{\pi}) \Big( \sum_{i \in S^X} q(i,k) \pi^i \Big) \\
&+ \frac{1}{2} \sum_{k,l \in S^X} \partial_{\pi^k} \partial_{\pi^l} f(t, L, \boldsymbol{\pi}) c_{lk}(\boldsymbol{\pi}) \\
&- \sum_{i=1}^m \mathbb{1}_{\{L_i = 0\}} \sum_{k \in S^X} \partial_{\pi^k} f(t, L, \boldsymbol{\pi}) \pi^k \big( \lambda_i(k) - \widehat{\lambda}_i(\boldsymbol{\pi}) \big) \\
&+ \sum_{i=1}^m \mathbb{1}_{\{L_i = 0\}} \widehat{\lambda}_i(\boldsymbol{\pi}) \int_0^1 \Big( f\Big(t, L + \ell \mathbf{e}_i, \pi^1 \frac{\lambda_i(1)}{\widehat{\lambda}_i(\boldsymbol{\pi})}, \dots, \pi^K \frac{\lambda_i(K)}{\widehat{\lambda}_i(\boldsymbol{\pi})}\Big) - f(t, L, \boldsymbol{\pi}) \Big) F_{\ell_i}(d\ell).
\end{aligned} \tag{A.1}$$

Theorem 4.4.2 a) in (Ethier & Kurtz 1986) shows that $(L, \boldsymbol{\pi})$ is a Markov process w.r.t. filtration $\mathbb{F}^{\mathbb{M}}$, if any two solutions of the martingale problem have the same one-dimensional marginals. This holds in particular, if the associated martingale problem has a unique solution measure, which we now show. Consider the following SDE system:

$$L_{t,i} = \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{L_{t-,i} = 0\}} K_i(L_{s-}, \boldsymbol{\pi}_{s-}; s, u) \mathcal{N}(ds, du), \tag{A.2}$$

$$\begin{aligned}
d\pi^k_t = &\sum_{i \in S^X} q(i,k) \pi^i_t dt + \sum_{j=1}^m \mathbb{1}_{\{L_{t-,j} = 0\}} \int_{\mathbb{R}} \mathbb{1}_{[\widehat{\Lambda}_{j-1,s}, \widehat{\Lambda}_{j,s}]}(u) \gamma^k_j(\boldsymbol{\pi}_{t-}) \left( \mathcal{N}(dt, du) - dt du \right) \\
&+ \boldsymbol{\alpha}^k(\boldsymbol{\pi}_t) dW_t;
\end{aligned} \tag{A.3}$$

where $\mathcal{N}$ is a Poisson random measure on $\mathbb{R}$ with compensator $dsdu$, $W$ is a Brownian motion, and

$$K_i(L_{s-}, \boldsymbol{\pi}_{s-}; s, u) = \mathbb{1}_{[\widehat{\Lambda}_{i-1,s}, \widehat{\Lambda}_{i,s}]}(u) F^{-1}_{\ell_i} \left( \frac{u - \widehat{\Lambda}_{i-1,s}}{(\widehat{\lambda}_i)_{s-}} \right)$$

with $\widehat{\Lambda}_{i,s} := \widehat{\Lambda}_i(\boldsymbol{\pi}_{s-}) = \sum_{j=1}^i (\widehat{\lambda}_j)_{s-}$. A similar computation as in the first part of the proof shows that $\mathscr{L}$ is the associated generator.

The system (A.2), (A.3) has a unique strong solution by Theorem 2.2 in Kliemann, Koch & Marchetti (1990): conditions 1) and 2) are straightforward to verify; the associated diffusion ((2.4) in the paper) is given by:

$$\begin{aligned}
d\tilde{L}_{t,i} &= \mathbb{1}_{\{\tilde{L}_{t-,i} = 0\}} \mathbb{E}(\ell_i) \widehat{\lambda}_i(\tilde{\boldsymbol{\pi}}_t) dt, \\
d\tilde{\pi}^k_t &= \sum_{i \in S^X} q(i,k) \tilde{\pi}^i_t dt + \boldsymbol{\alpha}^k(\tilde{\boldsymbol{\pi}}_t) dW_t.
\end{aligned}$$

Given $\tilde{\pi}$, $\tilde{L}$ can be computed by ordinary integration. Moreover, the SDE for $\tilde{\boldsymbol{\pi}}$ is the standard filter for Markov chains observed in additive Gaussian noise. Existence and uniqueness of this SDE is well-known, see for example Exercise 3.27 from Bain & Crisan (2009).

By Theorem III.2.26 from Jacod & Shiryaev (2003) uniqueness in law of the system (A.2), (A.3) gives uniqueness in law of the martingale problem for the generator $\mathscr{L}$ and we conclude.

□

*Remark A.1* The above proof shows additionally, that the martingale part of $f(t, L_t, \boldsymbol{\pi}_t)$ can be written as

$$\sum_{k \in S^X} \int_0^t \partial_{\pi^k} f(\cdot) \boldsymbol{\alpha}^k(\boldsymbol{\pi}_s)^\top dm_s^Z + \int_0^t \int_E \sum_{i=1}^m \mathbb{1}_{\{\xi=i\}}$$
$$\left( f\Big(s, L_{s-} + \ell \mathbf{e}_i, \pi_{s-}^1 \frac{\lambda_i(1)}{(\widehat{\lambda}_i)_{s-}}, \ldots, \pi_{s-}^K \frac{\lambda_i(K)}{(\widehat{\lambda}_i)_{s-}}\Big) - f(s, L_{s-}, \boldsymbol{\pi}_{s-}) \right) m^L(ds, d\xi, d\ell).$$

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