# Doubly Stochastic CDO Term Structures*

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**Abstract**

This paper provides a general framework for doubly stochastic term structure models for portfolio of credits, such as collateralized debt obligations (CDOs). We introduce the defaultable $(T, x)$-bonds, which pay one if the aggregated loss process in the underlying pool of the CDO has not exceeded $x$ at maturity $T$, and zero else. Necessary and sufficient conditions on the stochastic term structure movements for the absence of arbitrage are given. Moreover, we show that any exogenous specification of the forward rates and spreads volatility curve actually yields a consistent loss process and thus an arbitrage-free family of $(T, x)$-bond prices. For the sake of analytical and computational efficiency we then develop a tractable class of affine term structure models.

**Key words**: affine term structure, collateralized debt obligations, loss process, single tranche CDO, term structure of forward spreads

## 1 Introduction

*Collateralized debt obligations (CDOs)* are securities backed by a pool of reference entities such as bonds, loans or credit default swaps. The reference entities form the *asset side* of a CDO-structure. Traded products are notes on the CDO *tranches*. They have different seniorities and build the *liability side* of the CDO.

The most liquidly traded CDOs are those based on so-called indices, such as the CDX in the U.S. and the Itraxx in Europe. Both indices consist of the most liquidly traded and quoted credit default swaps in the given market. The standard instrument for investing in a CDO pool is a so-called single tranche

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CDO, which will be formally defined below. For more background and references on CDOs we refer, e.g., to the respective chapters in [17].

Recently, there have emerged several new attempts on CDO valuation based on the aggregate loss function (the so-called top-down approach). Bennani [1] models the evolution of the conditional expectation of the aggregate loss at some fixed maturity. However, this approach focuses on one maturity date only, and neither market interest rate and nor spread risk is explicitly considered. Schönbucher [18] introduces the forward loss distributions and finds a Markov chain with the same marginal distribution as the loss process. Some corresponding efficient calibration algorithms have recently been developed in Cont et al. [6, 7]. Ehlers and Schönbucher [9] extend [18] by considering non-constant interest rates for pricing. They introduce conditional forward interest-rates $f_n(t,T)$ and forward protection rates (spreads) $F_n(t,T)$ given a particular realization of the loss process $L(t) = n$. An HJM-type specification of the loss-contingent forward interest and loss rates $f_n$ and $F_n$ is then proposed and no-arbitrage conditions are given. Ehlers and Schönbucher [10] analyze the interplay of the background (i.e. forward interest and protection rates) and the loss process conditional on an increasing sequence of filtrations. The technical analysis in [18, 9, 10] relies on the assumption that the loss process lives on a finite grid, and their extension to multi-step increments (loss given default risk) becomes notationally demanding. The paper of Sidenius et al. (SPA) [19] is closest to our framework. However, SPA assume zero risk-free rates. Moreover, some crucial problems, e.g. regarding the construction of a consistent loss process, have remained open in [19].

The aim of our paper is to provide a unifying approach for the modelling of the forward rate and spread curve in a doubly stochastic setup (see Remark 3.5 below for a formal definition of the doubly stochastic property). This approach encompasses the above mentioned under a doubly stochastic regime. We therefore introduce the defaultable $(T,x)$-bonds, which pay one if the aggregated CDO loss process has not exceeded $x$ at maturity $T$, and zero else. It turns out that essentially all contingent claims on the CDO-pool, such as STCDOs, can be written—and thus priced—as linear combinations of $(T,x)$-bonds.

We then model the term structure of risk free $T$-forward rates and $(T,x)$-spreads as system of Itô processes driven by some Brownian motion. First, we provide necessary and sufficient conditions for the absence of arbitrage on these dynamics. Most important from a modelling point of view, we then formulate sufficient conditions on the stochastic basis such that any exogenous specification of the forward rates and spreads volatility curve actually yields a consistent loss process and thus an arbitrage-free family of $(T,x)$-bond prices. This is very much in the spirit of the Heath–Jarrow–Morton [15] approach to the modelling of the term structure of risk free interest rates. Moreover, we obtain efficient pricing formulas for STCDOs. For the sake of analytical and computational efficiency we then develop a tractable class of affine term structure models.

The novelty of our approach is its focus on the $(T,x)$-bonds and their exogenous stochastic specification. This perspective facilitates the mathematical analysis and it should also facilitate the empirical estimation for dynamic CDO term structure modelling, as it is the case for Heath–Jarrow–Morton [15] type forward rate models. Moreover, to our knowledge, the integrated affine specification of the $(T, x)$-term structure developed below is new in the literature.

The structure of the paper is as follows. In Section 2, we formally introduce the $(T, x)$-bonds. In Section 3, we first provide necessary and sufficient conditions for the absence of arbitrage. We then give sufficient conditions on the stochastic basis such that any given specification of the volatility parameters implies an arbitrage-free $(T, x)$-bond market. In Section 4, we derive STCDO price formulas. In Section 5, we provide an affine specification.

# 2 $(T, x)$-Bonds

As stochastic basis, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$. We assume that $\mathbb{Q}$ is a risk-neutral pricing measure. An equivalent measure change will be discussed below in Remark 3.7.

Consider a pool of credits (the CDO-pool) with an overall nominal normalized to 1, and let $\mathcal{I} = [0, 1]$ denote the set of loss fractions, i.e. $x \in \mathcal{I}$ represents the state where $100x\%$ of the overall nominal has defaulted. We denote by $L$ the $\mathcal{I}$-valued increasing aggregate CDO-*loss process*. That is, $L_t$ represents the ratio of CDO-losses occurred by time $t$.

The basic instrument that we consider is a $(T, x)$-*bond* which pays $1_{\{L_T \leq x\}}$ at maturity $T$, for $x \in \mathcal{I}$. Its price at time $t \leq T$ is denoted by $P(t, T, x)$. Obviously, $P(t, T, x)$ is increasing in $x$ and decreasing in $T$. Since $L_t \leq 1$ for all $t$, the risk free $T$-*bond* price $P(t, T)$ at time $t \leq T$ equals

$$P(t, T) = P(t, T, 1). \tag{1}$$

$(T, x)$-bonds are the fundamental components for the hedging and pricing of CDO-derivatives. Indeed, any European type contingent claim on the loss process with (regular enough) payoff function $F(L_T)$ at maturity $T$ can be decomposed into a linear combination of $(T, x)$-bonds

$$F(L_T) = F(1) - \int_{\mathcal{I}} F'(x) 1_{\{L_T \leq x\}} \, dx.$$

Hence the static portfolio

$$F(1)P(t, T) - \int_{\mathcal{I}} F'(x) P(t, T, x) \, dx$$

replicates, and thus prices the claim at any time $t \leq T$, model independently. For example, the basic components of the payment leg of the STCDO in Section 4 below are put options with payoff $(K - L_T)^+ = \int_{(0,K]} 1_{\{L_T \leq x\}} \, dx$.

**Remark 2.1.** *Note that this setup contains the finite case $\mathcal{I} = \{\frac{i}{n} \mid i = 0, \ldots, n\}$ in particular. Indeed, if $L$ can only assume fractions $\frac{i}{n}$, $i = 0, \ldots, n$, then $1_{\{L_T \leq x\}} = 1_{\{L_T \leq \frac{i}{n}\}}$, and hence $P(t, T, x) = P(t, T, \frac{i}{n})$, for all $x \in [\frac{i}{n}, \frac{i+1}{n})$.*

# 3 Arbitrage-free Term Structure Movements

Our aim is to describe the $(T, x)$-bond price term structure movements explicitly in the form

$$P(t, T, x) = 1_{\{L_t \le x\}} e^{-\int_t^T (f(t,u)+\phi(t,u,x))du}, \tag{2}$$

where $f(t, T)$ denotes the risk free $T$-*forward rate* and $\phi(t, T, x)$ the $(T, x)$-*forward spread* prevailing at date $t$, respectively. That is, $f(t, T) + \phi(t, T, x)$ is the rate that one can contract for at time $t$, given that $L_t \le x$, on a defaultable forward investment of one euro that begins at date $T$ and is returned an instant $dT$ later conditional on $L_{T+dT} \le x$.

Let us reflect for a moment why (2) is a well defined concept. From arbitrage theory, we know that the $(T, x)$-bond price can be written as conditional expectation of its payoff with respect to the $T$-forward measure $\mathbb{Q}^T \sim \mathbb{Q}$: $P(t, T, x) = \mathbb{Q}^T[L_T \le x \mid \mathcal{F}_t]$, see e.g. [12]. That is, $x \mapsto P(t, T, x)$ is the $\mathcal{F}_t$-conditional $\mathbb{Q}^T$-distribution function of $L_T$. As $t$ tends to $T$, this distribution converges to a Dirac measure at $L_T$. Since $L$ is increasing, this singularity is captured by the indicator function $1_{\{L_t \le x\}}$, which becomes dominant for $t \uparrow T$ while the smooth part $e^{-\int_t^T (f(t,u)+\phi(t,u,x))du}$ converges smoothly to 1, see Figure 1. Note that (2) would not make sense if $L$ were not increasing but diffusive, such as a stock price process.

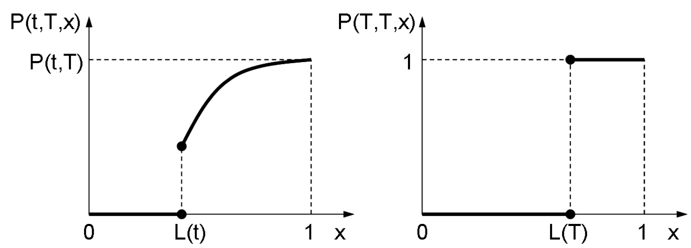


Figure 1: $P(t, T, x) = \mathbb{Q}^T[L_T \le x \mid \mathcal{F}_t]$ for $t < T$ and $t = T$.

We now assume that

**(A1)** $L_t = \sum_{s \le t} \Delta L_s$ is an $\mathcal{I}$-valued increasing marked point process[1] which admits an absolutely continuous compensator $\nu(t, dx)\, dt$.

This setup implies totally inaccessible default times of the $(T, x)$-bonds:

**Lemma 3.1.** *Assume that* **(A1)** *holds. Then, for any $x \in \mathcal{I}$, the indicator process $1_{\{L_t \le x\}}$ is càdlàg with intensity process*

$$\lambda(t, x) = \nu(t, (x - L_t, 1] \cap \mathcal{I}). \tag{3}$$

[1] Also called multivariate point process. For a definition see e.g. [2].

*That is,*

$$M_t^x = 1_{\{L_t \le x\}} + \int_0^t 1_{\{L_s \le x\}} \lambda(s,x)\, ds \tag{4}$$

*is a martingale. Moreover, $\lambda(t,x)$ is progressive, decreasing and càdlàg in $x \in \mathcal{I}$ with $\lambda(t,1) = 0$.*

*Proof.* Right-continuity of $1_{\{L_t \le x\}}$ follows from the structure **(A1)** of $L_t$. By the very definition of $\nu(t,dx)$,

$$F(L_t) - \int_0^t \int_{\mathcal{I}} (F(L_s + y) - F(L_s))\nu(s,dy)\, ds$$

is a martingale, for any bounded measurable function $F$. Moreover, for $F(L_t) = 1_{\{L_t \le x\}}$ we have

$$F(L_s + y) - F(L_s) = -1_{\{L_s + y > x\}} 1_{\{L_s \le x\}}. \tag{5}$$

This proves (4). The other properties of $\lambda(t,x)$ hold by inspection. $\square$

We now assume that, for any $(T,x)$, the forward rates and spreads follow Itô processes of the form

$$f(t,T) = f(0,T) + \int_0^t a(s,T)ds + \int_0^t b(s,T)^\top \cdot dW_s, \tag{6}$$

$$\phi(t,T,x) = \phi(0,T,x) + \int_0^t \alpha(s,T,x)ds + \int_0^t \beta(s,T,x)^\top \cdot dW_s, \tag{7}$$

where $W$ is some $d$-dimensional Brownian motion. To assert that the subsequent analysis and formal manipulations be meaningful, we make the following sufficient technical assumptions:

**(A2)** the initial forward curves $f(0,T)$ and $\phi(0,T,x)$ are continuous in $(T,x)$,

**(A3)** $a(t,T)$ and $\alpha(t,T,x)$ are $\mathbb{R}$-valued adapted processes, jointly continuous in $(t,T,x)$ with $\alpha(t,T,1) = 0$,

**(A4)** $b(t,T)$ and $\beta(t,T,x)$ are $\mathbb{R}^d$-valued adapted processes, jointly continuous in $(t,T,x)$ with $\beta(t,T,1) = 0$.

Conditions **(A2)**–**(A4)** assert that the risk free *short rate* $r_t = f(t,t)$ has a progressive version and satisfies $\int_0^T |r_t|\, dt < \infty$ for all $T$ (see [12]). Hence the *savings account* $e^{\int_0^t r_s ds}$ is well defined.

It is well known that there exists no arbitrage in the $(T,x)$-bond market if the discounted price processes

$$e^{-\int_0^t r_s ds} P(t,T,x) \text{ are local martingales for all } (T,x). \tag{8}$$

We now give necessary and sufficient conditions for (8) to hold.

**Theorem 3.2.** *Assume* **(A1)–(A4)** *hold. Then the no-arbitrage condition* (8) *is equivalent to*

$$a(t,T) = b(t,T)^\top \cdot \int_t^T b(t,u)du, \tag{9}$$

$$\alpha(t,T,x) = b(t,T)^\top \cdot \int_t^T \beta(t,u,x)du + \beta(t,T,x)^\top \cdot \int_t^T (b(t,u)+\beta(t,u,x))du, \tag{10}$$

$$\lambda(t,x) = \phi(t,t,x) \tag{11}$$

*where* (10) *and* (11) *hold on* $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$*-a.s. for all* $(T,x)$.

*Proof.* We denote

$$p(t,T,x) = e^{-\int_t^T (f(t,u)+\phi(t,u,x))du} \tag{12}$$

so that $P(t,T,x) = 1_{\{L_t \le x\}} p(t,T,x)$. Using a stochastic Fubini argument proposed by Heath et al. [15], see also [12], we derive

$$\begin{aligned}\frac{dp(t,T,x)}{p(t,T,x)} = \Bigg( f(t,t) + \phi(t,t,x) - \int_t^T (a(t,u)+\alpha(t,u,x))du \\ + \frac{1}{2}\left\| \int_t^T (b(t,u)+\beta(t,u,x))du \right\|^2 \Bigg) dt \\ - \left( \int_t^T (b(t,u)+\beta(t,u,x))du \right)^\top \cdot dW_t.\end{aligned} \tag{13}$$

Denote $Z(t,T,x) = e^{-\int_0^t r_s ds} P(t,T,x)$. Integrating by parts and using (4) yields

$$dZ(t,T,x) = Z(t,T,x)\left( -r_t dt + dM_t^x - \lambda(t,x)dt + \frac{dp(t,T,x)}{p(t,T,x)} \right). \tag{14}$$

Combining (13) and (14) shows that (8) holds if and only if

$$\begin{aligned}-\lambda(t,x) + \phi(t,t,x) - \int_t^T (a(t,u)+\alpha(t,u,x))du \\ + \frac{1}{2}\left\| \int_t^T (b(t,u)+\beta(t,u,x))du \right\|^2 = 0\end{aligned} \tag{15}$$

on $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$-a.s. for all $(T,x)$.

Since $L_t \le 1$ for all $t$ and by differentiating in $T$, we obtain that (15) is equivalent to (9)–(11). □

Theorem 3.2 states that, under the no-arbitrage condition (8), the *drift* parameters $a(t,T)$ and $\alpha(t,T,x)$ are determined by the *volatility* parameters $b(t,T)$ and $\beta(t,T,x)$ . However, there is still an implicit relation between the exogenously given loss process $L_t$ and $\phi(t,t,x)$ in (11). From a modelling point of view, this is unsatisfactory. It would be desirable if the sole exogenous specification of the volatility structure $b(t,T)$ and $\beta(t,T,x)$ already fully determines an arbitrage-free $(T,x)$-bond model. The main problem consists in constructing a consistent loss process $L_t$ which satisfies (8) for some given $f(t,T)$ and $\phi(t,T,x)$. This is best illustrated if we assume, for the moment, that

$$dL_t = \int_E \delta(t,\xi)\, m(dt,d\xi) \tag{16}$$

is driven by a Poisson random measure $m(dt,d\xi)$ with compensator $F(d\xi)dt$ on some mark space $E$, for some appropriate process $\delta(t,\xi)$. The compensator of $L$ then satisfies

$$\nu(t,dx) = \int_E 1_{\{\delta(t,\xi)\in dx\}} F(d\xi).$$

In view of (3), the no-arbitrage condition (11) thus reads

$$\int_E 1_{\{\delta(t,\xi)\in dx\}} F(d\xi) = -\phi(t,t,L_t+dx).$$

It is genuinely difficult to solve this last equation for $\delta(t,\xi)$ under the premise that (6), (7), (16) forms a strongly solvable stochastic dynamic system.

Such "non-classical" stochastic differential equations where the characteristics of certain driving semimartingales depend on the solution-process appear first in [13]. We will follow here a similar path as in [13] and find (the law of) $L$ as solution of a martingale problem. This is achieved under some additional assumptions on the stochastic basis:

**(A5)** $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}$, $\mathbb{Q}(d\omega_1, d\omega_2) = \mathbb{Q}_1(d\omega_1)\mathbb{Q}_2(\omega_1, d\omega_2)$, and $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$, where

- (i) $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ is some filtered probability space carrying the Brownian motion $W$,
- (ii) $(\Omega_2, \mathcal{H})$ is the canonical space of càdlàg paths from $\mathbb{R}_+$ to $\mathcal{I}$, and
- (iii) $\mathbb{Q}_2$ is a probability kernel from $\Omega_1$ to $\mathcal{H}$ to be determined below.

The next theorem is the constructive counterpart to Theorem 3.2 and contains a useful formula for CDO derivatives pricing.

**Theorem 3.3.** *Assume* **(A5)** *holds. Let $f(0,T)$ and $\phi(0,T,x)$ be some initial forward curves satisfying* **(A2)**, *and $b(t,T)$ and $\beta(t,T,x)$ some $(\mathcal{G}_t)$-adapted processes satisfying* **(A4)**. *Define $a(t,T)$ and $\alpha(t,T,x)$ by (9)–(10), $f(t,T)$ and $\phi(t,T,x)$ by (6)–(7), and $\lambda(t,x)$ by (11), for all $(t,T,x)$. If $\lambda(t,x)$ is jointly continuous in $(t,x)$ and decreasing in $x$ with $\lambda(t,1)=0$, then there exists a loss process $L_t$ satisfying* **(A1)** *and such that the no-arbitrage condition (8) holds. Moreover, for any positive $\mathcal{G}$-measurable random variable $X$ and all $x \in \mathcal{I}$,*

$$\mathbb{E}[X 1_{\{L_T \le x\}} \mid \mathcal{G} \otimes \mathcal{H}_t] = X 1_{\{L_t \le x\}} e^{-\int_t^T \lambda(s,x)ds}. \tag{17}$$

*Proof.* Denote $\lambda(t,x) := 0$ for $x \geq 1$. Fix $\omega_1 \in \Omega_1$ and define, reversely to (3), the Borel measure on $\mathcal{I}$

$$\nu(t,\omega_1,x;(y,z]) = \lambda(t,\omega_1,x+y) - \lambda(t,\omega_1,x+z),$$

for $y < z$ in $\mathcal{I}$. Since $\sup_{y<z\in\mathcal{I}} |\nu(t,\omega_1,x_n;(y,z]) - \nu(t,\omega_1,x;(y,z])| \to 0$ for $x_n \to x$ in $\mathcal{I}$, uniformly in $t$ on compacts, we infer that

$$\mathcal{A}(\omega_1)f(t,x) = \int_{\mathcal{I}} (f(t,x+y) - f(t,x))\nu(t,\omega_1,x;dy)$$

defines a bounded linear operator on the Banach space of continuous functions with compact support on $\mathbb{R}_+ \times \mathcal{I}$. Moreover, $\mathcal{A}(\omega_1)$ satisfies the positive maximum principle, that is, $\sup_{(t,x)\in\mathbb{R}_+\times\mathcal{I}} f(t,x) = f(t_0,x_0) \geq 0$ implies $\mathcal{A}(\omega_1)f(t_0,x_0) \leq 0$. Hence there exists a probability measure $\mathbb{Q}_2(\omega_1,\cdot)$ such that the coordinate process $L_t(\omega_2) = \omega_2(t)$ becomes a Markov process on $(\Omega_2,\mathcal{H},(\mathcal{H}_t),\mathbb{Q}_2(\omega_1,\cdot))$ with compensator $\nu(t,\omega_1,x;dy)$ (see Theorem 5.4 in Chapter 4 of Ethier and Kurtz [11]). As a consequence, arguing as in (5),

$$M_t^x(\omega_1,\cdot) = 1_{\{L_t\leq x\}} + \int_0^t 1_{\{L_s\leq x\}}\lambda(s,\omega_1,x)ds$$

is a $(\mathcal{H}_t,\mathbb{Q}_2(\omega_1,\cdot))$-martingale. Moreover, by the Markov property of $L$, the probabilities

$$\mathbb{Q}_2(\omega_1,\{L_{t_1}\in A_1,\dots,L_{t_n}\in A_n\})$$

depend measurable on $\lambda(s,\omega_1,x)$, $s \leq t$, for all $t_1 < \cdots < t_n \leq t$ and $A_i \in \mathcal{B}(\mathcal{I})$ for $n \geq 1$. By a monotone class argument, we deduce that

$$\mathbb{Q}_2 \text{ is a probability kernel from } (\Omega_1,\mathcal{G}_t) \text{ to } \mathcal{H}_t, \text{ for all } t. \tag{18}$$

By construction, the loss process $L_t(\omega_1,\omega_2) = \omega_2(t)$ satisfies **(A1)** on the stochastic basis $(\Omega,\mathcal{F},\mathbb{Q})$. The no-arbitrage property (8) now follows by Theorem 3.2.

Finally, let $X \geq 0$ be $\mathcal{G}$-measurable. In view of the above, we calculate

$$\begin{aligned}
\Phi(T) &:= \mathbb{E}\left[X1_{\{L_T\leq x\}} \mid \mathcal{G}\otimes\mathcal{H}_t\right] \\
&= \mathbb{E}\left[XM_T^x - \int_0^T X1_{\{L_s\leq x\}}\lambda(s,x)ds \mid \mathcal{G}\otimes\mathcal{H}_t\right] \\
&= XM_t^x - \int_0^T \lambda(s,x)E\left[X1_{\{L_s\leq x\}} \mid \mathcal{G}\otimes\mathcal{H}_t\right]ds \\
&= X1_{\{L_t\leq x\}} - \int_t^T \lambda(s,x)\Phi(s)ds.
\end{aligned}$$

We infer that

$$\Phi(T) = X1_{\{L_t\leq x\}}e^{-\int_t^T \lambda(s,x)ds},$$

which yields (17). □

By conditioning both sides in (17) on $\mathcal{F}_t$, we immediately obtain the following

**Corollary 3.4.** *Under the assumptions of Theorem 3.3, for any positive $\mathcal{G}$-measurable random variable $X$ and all $x \in \mathcal{I}$,*

$$\mathbb{E}[X 1_{\{L_T \leq x\}} \mid \mathcal{F}_t] = 1_{\{L_t \leq x\}} \mathbb{E}\left[X e^{-\int_t^T \lambda(s,x) ds} \mid \mathcal{G}_t\right]. \tag{19}$$

**Remark 3.5.** *From* (17)*, we see that the individual default times*

$$\tau_x := \inf\{t \mid L_t > x\}, \quad x \in \mathcal{I},$$

*of the* $(T, x)$*-bonds are* doubly stochastic*, in the sense that each $\tau_x$ can be considered as the first jump time of a $\mathcal{G}$-conditional time-inhomogeneous Poisson process with intensity $\lambda(t, x)$. That is, for $t \leq T$,*

$$\mathbb{Q}[\tau_x > T \mid \mathcal{G} \otimes \mathcal{H}_t] = 1_{\{\tau_x > t\}} \mathrm{e}^{-\int_t^T \lambda(s,x)\, ds}.$$

*See e.g. [2, Section II.1] or [12].*

**Remark 3.6.** *In view of* (18)*, we infer that all $(\mathcal{G}_t)$-martingales are also martingales with respect to the larger filtration $(\mathcal{F}_t)$. Hence the so called "(H)-hypothesis" (see [3]) is simultaneously satisfied for all default times $\tau_x$.*

**Remark 3.7.** *We present our approach under the assumption that $\mathbb{Q}$ is a risk-neutral measure, i.e. the no-arbitrage condition* (8) *is supposed to hold under $\mathbb{Q}$. It is of course possible to consider the above characteristics $a$, $\alpha$, $\nu(t, dx)$ and $\lambda$ with respect to some objective probability measure $\mathbb{P} \sim \mathbb{Q}$. The measure change from $\mathbb{P}$ to $\mathbb{Q}$ will have the following impact:*

$$\begin{aligned} a(t,T) &\rightsquigarrow a(t,T) + b(t,T)^\top \cdot \Phi(t) \\ \alpha(t,T,x) &\rightsquigarrow \alpha(t,T,x) + \beta(t,T,x)^\top \cdot \Phi(t) \\ \nu(t,dx) &\rightsquigarrow \Psi(t,x)\nu(t,dx) \end{aligned}$$

*for some appropriate stochastic processes $\Phi(t, \omega)$ and $\Psi(t, \omega, x)$ with values in $\mathbb{R}^d$ and* $(0, \infty)$*, respectively. We do not intend to provide further general results on this, as it is rather standard and regularity conditions have to be checked from case to case. For a general reference see Theorem III.3.24 in [14], for Markovian models see also [5].*

# 4 Single Tranche CDOs (STCDOs)

Throughout this section, we assume that the requirements in Theorem 3.3, in particular **(A5)**, are satisfied.

The standard instrument for investing in a CDO-pool is a *single tranche CDO (STCDO)*, also called *tranche credit default swap*. A STCDO is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$,
- a *tranche* with lower and upper detachment points $x_1 < x_2$ in $\mathcal{I}$,
- a fixed spread $S$.

We write

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1,x_2]} 1_{\{x \le y\}} dy.$$

An investor in this STCDO

- receives $SH(L_{T_i})$ at $T_i$, $i = 1, \dots, n$ (payment leg),
- pays $-dH(L_t) = H(L_{t-}) - H(L_t)$ at any time $t \in (T_0, T_n]$ where $\Delta L_t \neq 0$ (default leg).

As in (12), we denote the $(\mathcal{G}_t)$-adapted part of the $(T,x)$-bond price by $p(t,T,x)$, so that $P(t,T,x) = 1_{\{L_t \le x\}} p(t,T,x)$.

**Lemma 4.1.** *The value of the STCDO at time $t \le T_0$ is*

$$\begin{aligned} &\Gamma(t,S) \\ &= \int_{(x_1,x_2]} 1_{\{L_t \le y\}} \left( S \sum_{i=1}^n p(t,T_i,y) - p(t,T_0,y) + p(t,T_n,y) + \gamma(t,y) \right) dy \end{aligned} \tag{20}$$

*where $\gamma(t,y) = \int_{T_0}^{T_n} \mathbb{E}\left[ r_u e^{-\int_t^u (r_s + \lambda(s,y))ds} \mid \mathcal{G}_t \right] du$.*

*Moreover, if the risk free rates $f(s,u)$ and the loss process $L_s$, for $t \le s \le u$, are $\mathcal{F}_t$-conditionally independent then $\gamma(t,y)$ in (20) can be replaced by*

$$\gamma(t,y) = \int_{T_0}^{T_n} f(t,u) p(t,u,y)\, du.$$

*Proof.* The value of the payment leg at time $t \le T_0$ is

$$\mathbb{E}\left[ \sum_{i=1}^n e^{-\int_t^{T_i} r_s ds} SH(L_{T_i}) \mid \mathcal{F}_t \right] = S \sum_{i=1}^n \int_{(x_1,x_2]} P(t,T_i,y) dy.$$

Next we use integration by parts to calculate

$$\begin{aligned} &\int_{T_0}^{T_n} e^{-\int_t^u r_s ds} dH(L_u) \\ &= e^{-\int_t^{T_n} r_s ds} H(L_{T_n}) - e^{-\int_t^{T_0} r_s ds} H(L_{T_0}) + \int_{T_0}^{T_n} r_u e^{-\int_t^u r_s ds} H(L_u) du. \end{aligned} \tag{21}$$

In view of (19), the (negative) value of the default leg at time $t \le T_0$ for the investor is thus

$$\int_{(x_1,x_2]} \Bigg( P(t,T_n,y) - P(t,T_0,y) + \int_{T_0}^{T_n} 1_{\{L_t \le y\}} \mathbb{E}\left[ r_u e^{-\int_t^u (r_s + \lambda(s,y))ds} \mid \mathcal{G}_t \right] du \Bigg) dy.$$

Summing up the two legs, we obtain (20).

The second part of the lemma follows from (21) since

$$\mathbb{E}\left[ r_u e^{-\int_t^u r_s\,ds} \mid \mathcal{F}_t \right] = f(t,u)P(t,u).$$

□

The *forward STCDO spread* $S_t^*$ prevailing at $t \le T_0$ is the spread which gives $\Gamma(t, S_t^*) = 0$. In view of (20) hence

$$S_t^* = \frac{\int_{(x_1,x_2]} 1_{\{L_t \le y\}} \left( p(t,T_0,y) - p(t,T_n,y) - \gamma(t,y) \right) dy}{\sum_{i=1}^n \int_{(x_1,x_2]} 1_{\{L_t \le y\}} p(t,T_i,y)\,dy}.$$

A STCDO *swaption* with strike spread $K$ gives the holder the right to enter the above STCDO with spread $K$ at swaption maturity $T_0$. Its value at $T_0$ is thus $\Gamma(T_0,K)^+$. Note that, since $\Gamma(T_0, S_{T_0}^*) = 0$, this swaption payoff can also be written as

$$\left( \sum_{i=1}^n \int_{(x_1,x_2]} 1_{\{L_t \le y\}} p(T_0,T_i,y)\,dy \right) \left( K - S_{T_0}^* \right)^+. \tag{22}$$

As it is the case for single name models, e.g. [8, 20, 4], there is no closed form solution for swaption prices available in general. See however Remark 5.3 below.

# 5 Affine Term Structure Models

In this section we consider an analytically tractable class of Markov factor models for the term structure movements (6)–(7). We assume that **(A5)** holds. Let $\mathcal{Z} \subset \mathbb{R}^d$ be some closed state space with non-empty interior and $Z_t$ some $\mathcal{Z}$-valued diffusion process satisfying

$$\begin{aligned} dZ_t &= \mu(Z_t)dt + \sigma(Z_t) \cdot dW_t, \\ Z_0 &= z \end{aligned} \tag{23}$$

where $\mu$ and $\sigma$ are continuous functions from $\mathbb{R}_+ \times \mathcal{Z}$ into $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively.

In what follows we consider *affine term structure* models of the form

$$\begin{aligned} f(t,T) &= A'(t,T) + B'(t,T)^\top \cdot Z_t \\ \phi(t,T,x) &= C'(t,T,x) + D'(t,T,x)^\top \cdot Z_t, \end{aligned}$$

that is, in terms of (6)–(7),

$$\begin{aligned} a(t,T) &= \partial_t A'(t,T) + \partial_t B'(t,T)^\top \cdot Z_t + B'(t,T)^\top \cdot \mu(Z_t) \\ b(t,T) &= B'(t,T)^\top \cdot \sigma(Z_t) \\ \alpha(t,T,x) &= \partial_t C'(t,T,x) + \partial_t D'(t,T,x)^\top \cdot Z_t + D'(t,T,x)^\top \cdot \mu(Z_t) \\ \beta(t,T,x) &= D'(t,T,x)^\top \cdot \sigma(Z_t) \end{aligned} \tag{24}$$

for some smooth functions $A'(t,T)$, $C'(t,T,x)$ and $B'(t,T)$, $D'(t,T,x)$ with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote

$$A(t,T) = \int_t^T A'(t,u)du,$$

and analogously $B(t,T)$, $C(t,T,x)$ and $D(t,T,x)$.

The following theorem gives a characterization of those affine term structure models which satisfy the no-arbitrage condition (8).

**Theorem 5.1.** *Assume that, for all $z \in \mathcal{Z}$, there exists a $\mathcal{Z}$-valued continuous solution $Z_t = Z_t^z$ of* (23) *such that the coefficients given in* (24) *satisfy* (9)–(10) *for all $t \le T$ and $x$ a.s. If the $d + \frac{d(d+1)}{2}$ functions in $(T,x)$,*

$$B_i(0,T)+D_i(0,T,x), \quad (B_k(0,T)+D_k(0,T,x))(B_l(0,T)+D_l(0,T,x)), \quad k \le l, \tag{25}$$

*are linearly independent, then $Z_t$ is necessarily* affine. *That is, drift and diffusion matrix are affine functions of $z = (z_1, \dots, z_d) \in \mathcal{Z}$:*

$$\mu(z) = \mu_0 + \sum_{i=1}^d z_i \mu_i, \quad \frac{1}{2}\sigma \cdot \sigma^\top(z) = \nu_0 + \sum_{i=1}^d z_i \nu_i \tag{26}$$

*for some vectors $\mu_i \in \mathbb{R}^d$ and matrices $\nu_i \in \mathbb{R}^{d\times d}$. Moreover, $A$, $B$, $C$ and $D$ solve the following system of* Riccati equations*, for $t \le T$,*

$$\begin{aligned} -\partial_t A(t,T) &= A'(t,t) + \mu_0^\top \cdot B(t,T) - B(t,T)^\top \cdot \nu_0 \cdot B(t,T) \\ -\partial_t B_i(t,T) &= B_i'(t,t) + \mu_i^\top \cdot B(t,T) - B(t,T)^\top \cdot \nu_i \cdot B(t,T) \\ -\partial_t C(t,T,x) &= C'(t,t,x) + \mu_0^\top \cdot D(t,T,x) \\ &\quad - (2B(t,T) + D(t,T,x))^\top \cdot \nu_0 \cdot D(t,T,x) \\ -\partial_t D_i(t,T,x) &= D_i'(t,t,x) + \mu_i^\top \cdot D(t,T,x) \\ &\quad - (2B(t,T) + D(t,T,x))^\top \cdot \nu_i \cdot D(t,T,x) \end{aligned} \tag{27}$$

*with terminal values*

$$A(T,T) = C(T,T,x) = 0, \quad B(T,T) = D(T,T,x) = 0 \tag{28}$$

*for all* $(T,x)$.

*Proof.* Note that

$$\int_t^T \partial_t A'(t,u)du = \partial_t A(t,T) + A'(t,t),$$

and analogously for $B'$, $C'$ and $D'$. Hence $\int_t^T$-integrating (9) and (10) yields

$$\begin{aligned}\partial_t A(t,T) + A'(t,t) + (\partial_t B(t,T) + B'(t,t))^\top \cdot Z_t + B(t,T)^\top \cdot \mu(Z_t)\\ = \frac{1}{2} B(t,T)^\top \cdot \sigma \cdot \sigma^\top(Z_t) \cdot B(t,T)\end{aligned} \tag{29}$$

and

$$\begin{aligned}\partial_t C(t,T,x) + C'(t,t,x) + (\partial_t D(t,T,x) + D'(t,t,x))^\top \cdot Z_t + D(t,T,x)^\top \cdot \mu(Z_t)\\ = \left(B(t,T) + \frac{1}{2} D(t,T,x)\right)^\top \cdot \sigma \cdot \sigma^\top(Z_t) \cdot D(t,T,x).\end{aligned} \tag{30}$$

Letting $t \downarrow 0$, by continuity, we obtain the respective equalities for $Z_t$ replaced by $z$, for all $T$, $x$ and $z$. Adding equations (29) and (30), we infer that

$$\begin{aligned}&(B(0,T) + D(0,T,x))^\top \cdot \mu(z)\\ &\quad + (B(0,T) + D(0,T,x))^\top \cdot \frac{\sigma \cdot \sigma^\top(z)}{2} \cdot (B(0,T) + D(0,T,x))\end{aligned}$$

is an affine function in $z$, for all $T$ and $x$. By assumption (25), we conclude that $\mu$ and $\sigma \cdot \sigma^\top/2$ must be affine functions of the form (26).

Plugging (26) back in (29)–(30) and separating first order terms in $z_i$, we obtain (27). □

The next theorem is the converse to Theorem 5.1 and gives sufficient conditions for the existence of an arbitrage-free affine term structure model.

**Theorem 5.2.** *Assume* $\mu$ *and* $\sigma\sigma^\top$ *are affine of the form* (26). *Let* $A'(t,t)$, $C'(t,t,x)$ *and* $B'(t,t)$, $D'(t,t,x)$ *be some given functions with values in* $\mathbb{R}^d$ *and* $\mathbb{R}^{d\times d}$, *respectively, jointly continuous in* $(t,x) \in \mathbb{R}_+ \times \mathcal{I}$, *and such that* $C'(t,t,x) + D'(t,t,x)^\top \cdot z$ *is decreasing in* $x$ *with* $C'(t,t,1) + D'(t,t,1)^\top \cdot z = 0$ *for all* $t$ *and* $z \in \mathcal{Z}$.

*Let* $A$, $B$, $C$, $D$ *be given as solutions of the Riccati equations* (27)–(28), *and let* $Z_t$ *be a continuous* $\mathcal{Z}$*-valued solution of* (23), *for some* $z \in \mathcal{Z}$. *Then the conclusions of Theorem 3.3 apply and*

$$P(t,T,x) = 1_{\{L_t \le x\}} e^{-A(t,T) - C(t,T,x) - (B(t,T) + D(t,T,x))^\top \cdot Z_t}$$

*defines an arbitrage-free* $(T,x)$*-bond market.*

*Proof.* It follows as in the proof of Theorem 5.1 that (9)–(10) are equivalent to (29)–(30), which again are implied by (27). Moreover, $\lambda(t,x) = C'(t,t,x) + D'(t,t,x)^\top \cdot Z_t$ satisfies the required properties in Theorem 3.3. Hence the conclusions of Theorem 3.3 apply. $\square$

**Remark 5.3.** *Using the affine toolbox, as developed in e.g. [8, 20, 4], and the fact that $r_t = A'(t,t) + B'(t,t)^\top \cdot Z_t$ and $\lambda(t,x) = C'(t,t,x) + D'(t,t,x)^\top \cdot Z_t$ are affine functions of the affine process $Z_t$, derivative prices such as in Lemma 4.1 and* (22) *can now efficiently be computed.*

## 5.1 Example

As simple example, we consider: $d = 1$, $\mathcal{Z} = \mathbb{R}_+$, $\mu_0 \geq 0$, $\mu_1 \in \mathbb{R}$, $\nu_1 = \sigma^2/2$, for some $\sigma > 0$. That is, $Z_t$ is a Feller square root process:

$$dZ_t = (\mu_0 + \mu_1 Z_t)dt + \sigma\sqrt{Z_t}dW_t, \quad Z_0 = z \in \mathbb{R}_+.$$

Moreover, we let $A'(t,t) \equiv r \geq 0$, $B'(t,t) \equiv 0$, $C'(t,t,x) = c(t,x)$, $D'(t,t,x) = d(x)$, for some $\mathbb{R}_+$-valued functions $c(t,x)$ and $d(x)$ which are decreasing in $x \in \mathcal{I}$ and vanishing at $x = 1$. That is, we have a constant risk free rate

$$r_t \equiv r, \quad \text{and} \quad \lambda(t,x) = c(t,x) + d(x)Z_t.$$

The Riccati equations (27)–(28) become

$$\begin{aligned}
A(t,T) &= (T-t)r \\
B(t,T) &= 0 \\
C(t,T,x) &= \int_t^T \left(c(s,x) + \mu_0 D(s,T,x)\right) ds \\
-\partial_t D(t,T,x) &= d(x) + \mu_1 D(t,T,x) - \frac{\sigma^2}{2} D(t,T,x)^2, \quad D(T,T,x) = 0.
\end{aligned}$$

The last equation for $D$ has the solution

$$D(t,T,x) \equiv D(T-t,x) = \frac{2d(x)\left(e^{\rho(x)(T-t)} - 1\right)}{\rho(x)\left(e^{\rho(x)(T-t)} + 1\right) - \mu_1\left(e^{\rho(x)(T-t)} - 1\right)}$$

where $\rho(x) = \sqrt{\mu_1^2 + 2\sigma^2 d(x)}$. Note that

$$\partial_T C(t,T,x) = c(T,x) + \mu_0 D(T-t,x).$$

Hence, we obtain

$$\begin{aligned}
f(t,T) &\equiv r \\
\phi(t,T,x) &= c(T,x) + \mu_0 D(T-t,x) + \partial_T D(T-t,x) Z_t.
\end{aligned}$$

Since the independence assumption in the second part of Lemma 4.1 is clearly met, we conclude that $\gamma(t, y)$ in (20) can be replaced by

$$\gamma(t, y) = r \int_{T_0}^{T_n} p(t, u, y) \, du.$$

Hence STCDO values, and thus spreads and swaptions, are efficiently computable via (20). We conclude with the remarkable fact that this simple model is capable of capturing any given initial spread curve $\phi(0, T, x)$ by an appropriate choice of the function $c(T, x)$.

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