Copulas and dependence measurement

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Abstract: Copulas are a general tool for assessing the dependence structure of random variables. Important properties as well as a number of examples are discussed, including Archimedean copulas and the Marshall-Olkin copula. As measures of the dependence we consider linear correlation, rank correlation, the coefficients of tail dependence and association.

Copulas are a tool for modeling and capturing the dependence of two or more random variables (rvs). In the work of [22] the term *copula* was used the first time; it is derived from the Latin word *copulare*, to connect or to join. Similar in spirit, Hoeffding studied distributions under "arbitrary changes of scale" already in the 1940s, see [7].

The main purpose of a copula is to disentangle the dependence structure of a random vector from its marginals. A d-dimensional copula is defined as a function $C : [0,1]^d \to [0,1]$ which is a cumulative distribution function (cdf) with uniform[1] marginals. On one side, this leads to the following properties:

1. $C(u_1, \ldots, u_d)$ is increasing in each component $u_i, i \in \{1, \ldots, d\}$;

2. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $1 \le i \le d$;

3. For $a_i \le b_i, 1 \le i \le d$, C satisfies the *rectangle inequality*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1,i_1}, \ldots, u_{d,i_d}) \ge 0,$$

where $u_{j,1} = a_j$ and $u_{j,2} = b_j$.

On the other side, every function satisfying i.–iii. is a copula. Furthermore, $C(1, u_1, \ldots, u_{d-1})$ is again a copula and so are all k-dimensional marginals with $2 \le k < d$.

The construction of *multivariate* copulas is difficult. There is a rich literature on uni- and bivariate distributions but many of this families do not have obvious multivariate generalizations[2]. Similar in spirit, it is not at all straightforward to generalize a two-dimensional copula to higher dimensions. For example, consider the construction of 3-dimensional copulas. A possible attempt is to try $C_1(C_2(u_1, u_2), u_3)$ where C_1, C_2 are bivariate copulas. However, already for $C_1 = C_2 = \max\{u_1 + u_2 - 1, 0\}$ (the countermonotonicity copula, introduced in Section 1.1) this procedure fails. See [18], Section 3.4 for further details. Also Chapter 4 in [11] gives an overview of construction multivariate copulas with different concepts. In particular it discusses the construction of a d-dimensional copulas given the set of $d(d-1)/2$ bivariate margins.

The class of Archimedean copulas (see also the following section on Archimedean copulas) is an important class for which the construction of multivariate copulas can be performed quite generally. A natural example of a 3-dimensional Archimedean copula is given by the following exchangeable Archimedean copula:

$$C(u_1, u_2, u_3) = \phi^{-1}(\phi(u_1) + \phi(u_2) + \phi(u_3)) \quad (1)$$

with appropriate generator ϕ. However, for appropriate ϕ_1, ϕ_2,

$$\phi_1^{-1}(\phi_2 \circ \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3)) \quad (2)$$

also gives a 3-dimensional copula (see [14], Section 5.4.3). It is of course possible that ϕ_1 and ϕ_2 are generators of different types of Archimedean copulas.

[1] Although standard, it is not necessary to consider uniform marginals (see **copulas**).

[2] One example is the exponential distributions whose multivariate extension leads to the Marshall-Olkin copula, introduced in a following paragraph.

The key to the separation of marginals and dependence structure is the *quantile transformation*. Let U be a standard uniform rv and $F^{-1}(y) := \inf\{x : F(x) \geq y\}$ be the generalized inverse of F. Then

$$P\big(F^{-1}(U) \leq x\big) = F(x). \tag{3}$$

This result is frequently used for simulation: the generation of uniform rvs is readily implemented in typical software packages and if we are able to compute F^{-1}, we can sample from F using (3).

On the contrary, the *probability transformation* is used to compute copulas implied from distributions, see the following section on "Copulas derived from distributions": consider X having a continuous distribution function F, then $F(X)$ is standard uniform. [3]

Sklar's theorem. It is not surprising that every distribution function inherently embodies a copula function. On the other side, any copula entangled with some marginal distributions in the right way, leads to a proper multivariate distribution function. This is the important contribution of Sklar's theorem [22]. $\operatorname{Ran} F$ denotes the range of F.

Theorem. *Consider a d-dimensional cdf F with marginals $F_1, \dots, F_d$. There exists a copula C, such that*

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big) \tag{4}$$

for all x_i in $[-\infty, \infty]$, $i = 1, \dots, d$. If F_i is continuous for all $i = 1, \dots, d$ then C is unique; otherwise C is uniquely determined only on $\operatorname{Ran} F_1 \times \dots \times \operatorname{Ran} F_d$*. On the other hand, consider a copula C and univariate cdfs $F_1, \dots, F_d$. Then F as defined in* (4) *is a multivariate cdf with marginals $F_1, \dots, F_d$.*

It is important to note that for discrete distributions copulas are not as natural as they are for continuous distributions, compare [8].

In the following we therefore concentrate on continuous F_i, $i = 1, \dots, d$. It is interesting to examine the consequences of representation (4) for the copula itself. Using that $F \circ F^{-1}(y) = y$ for any continuous cdf F, we obtain

$$C(\mathbf{u}) = F\big(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\big). \tag{5}$$

[3] See, for example [14], Proposition 5.2.

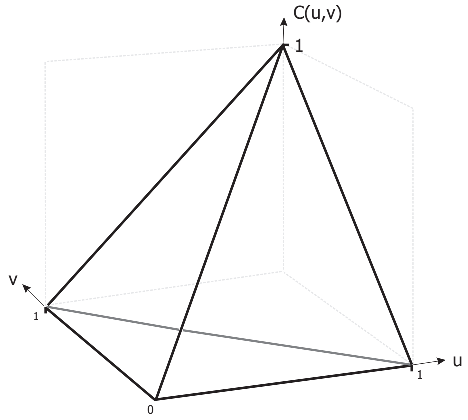

Figure 1: According to the Fréchet-Hoeffding bounds every copula has to lie inside of the pyramid shown in the graph. The surface given by the bottom and back side of the pyramid (the lower bound) is the countermonotonicity copula $C(u, v) = \max\{u + v - 1, 0\}$, while the front side (the upper bound) is the comonotonicity copula, $C(u, v) = \min(u, v)$.

While relation (4) usually is the starting point for simulations that are based on a given copula and given marginals, relation (5) rather proves as a theoretical tool to obtain the copula from any multivariate distribution function. This equation also allows to extract a copula directly from a multivariate distribution function.

Invariance under transformations. An important property of copulas is that it is invariant under strictly increasing tansformations: for strictly increasing functions $T_i : \mathrm{R} \to \mathrm{R}, i = 1, \dots, d$ the rvs $X_1, \dots, X_d$ and $T_1(X_1), \dots, T_d(X_d)$ have the same copula.

Bounds of copulas. Hoeffding and Fréchet independently derived that a copula always lies in between certain bounds, compare Figure 1. The reason for this is the existence of some extreme cases of dependency, co- and countermonotonicity. The so-called *Fréchet-Hoeffding bounds* are given by

$$\max\Big\{\sum_{i=1}^{d} u_i + 1 - d, 0\Big\} \leq C(\mathbf{u}) \leq \min\{u_1, \dots, u_d\}, \tag{6}$$

which holds for any copula C. However, whereas a comonotonic copula exists in any dimension d, there is no countermonotonicity copula in the case of dimensions greater than two[4].

1.1 Important copulas

First of all, the *independence copula* is given by

$$\prod_{i=1}^{d} u_i. \tag{7}$$

Random variables are independent if and only if their copula is the independence copula.

The *comonotononicity copula* or the *Fréchet-Hoeffding upper bound* is given by

$$\min\{u_1, \ldots, u_d\}. \tag{8}$$

Rvs $X_1, \ldots, X_d$ are called *comonotonic*, if their copula is as in (8). This is equivalent to $(X_1, \ldots, X_d)$ having the same distribution as $(T_1(Z), \ldots, T_d(Z))$ with some rv Z and strictly increasing functions $T_1, \ldots, T_d$. Hence, comonotonicity refers to perfect dependence in the sense where all rvs are, in an increasing and deterministic way, depending on Z.

The other case of perfect dependence is given by countermonotonicity. The *countermonotonicity copula* reads

$$\max\{u_1 + u_2 - 1, 0\}. \tag{9}$$

Two rvs with this copula are called *countermonotonic*. This is equivalent to (X_1, X_2) having the same distribution as $(T_1(Z), T_2(Z))$ for some rv Z and T_1 being increasing and T_2 being decreasing or vice versa. However, the *Fréchet-Hoeffding lower bound* as given in equation (6) is *not* a copula for $d > 2$, see [14], Example 5.21.

Copulas derived from distributions. The probability transformation[5] allows to obtain the copula inherent in multivariate distributions: for a multivariate cdf F with continuous marginals F_i the inherent copula is given by

$$C(\mathbf{u}) = F\big(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\big). \tag{10}$$

For example, for a multivariate normal distribution, the implied copula is called *Gaussian copula*. For a d-dimensional rv X the correlation matrix[6] $\mathbf{\Gamma}$ is obtained from the covariance matrix by scaling each component to variance 1. Hence $\mathbf{\Gamma}$ is given by the entries $\mathrm{Corr}(X_i, X_j)$, $1 \le i, j \le d$ (compare correlation and correlation risk). For such a *correlation matrix* $\mathbf{\Gamma}$ the Gaussian copula is given by

$$\Phi_{\mathbf{\Gamma}}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)). \tag{11}$$

In a similar fashion one obtains the *t-copula* or the *Student copula*

$$t_{\nu,\mathbf{\Gamma}}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)), \tag{12}$$

where $\mathbf{\Gamma}$ is a correlation matrix, t_ν is the cdf of the one dimensional t_ν distribution and $t_{\nu,\mathbf{\Gamma}}$ is the cdf of the multivariate $t_{\nu,\mathbf{\Gamma}}$ distribution. The mixing nature of the t-distribution leads to a dramatically different behavior in the tails, which is an important property in applications. See the following section on tail dependence.

Archimedean copulas. An important class of analytically tractable copulas are *Archimedean copulas*. For the bivariate case, consider a continuous and strictly decreasing function $\phi : [0, 1] :\to [0, \infty]$ with $\phi(1) = 0$. Then $C(u_1, u_2)$ given by

$$\begin{cases} \phi^{-1}(\phi(u_1) + \phi(u_2)) & \text{if } \phi(u_1) + \phi(u_2) \le \phi(0), \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

is a copula if and only if ϕ is convex; see [18], Theorem 4.1.4. If $\phi(0) = \infty$ the generator is said to be *strict* and $C(u_1, u_2) = \phi^{-1}(\phi(u_1) + \phi(u_2))$.

For the multivariate case there are different possibilities of generalization. Quite a special case is when the copula is of the form

$$\phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)). \tag{14}$$

These are so-called exchangeable Archimedean copulas and [15] give a complete characterization of such ϕ leading to a copula of the form (14). One may also consider asymmetric specifications of multivariate Archimedean copulas see [14] Section 5.4.2 and 5.4.3. We present some examples of

[4] See [14], Example 5.21, for a counter-example.

[5] For a rv X with continuous cdf F the rv $F(X)$ is standard uniform, see Section 1.

[6] See correlation and correlation risk.

Archimedean copulas in the bivariate case: From the generator $(-\ln u)^\theta$ one obtains the bivariate *Gumbel copula* or *Gumbel-Hougaard copula* :

$$\exp\Big(-\Big[(-\ln u_1)^\theta + (-\ln u_2)^\theta\Big]^{\frac{1}{\theta}}\Big), \tag{15}$$

where $\theta \in [1,\infty)$. For $\theta = 1$ it coincides with the independence copula, and for $\theta \to \infty$ it converges to the comonotonicity copula. The Gumbel copula has tail dependence in the upper right corner.

The *Clayton copula* is given by[7]

$$\big(\max\{u_1^{-\theta} + u_2^{-\theta} - 1, 0\}\big)^{-\frac{1}{\theta}}, \tag{16}$$

where $\theta \in [-1,\infty) \setminus \{0\}$. For $\theta \to 0$ it converges to the independence copula, and for $\theta \to \infty$ to the comonotonicity copula. For $\theta = -1$ we obtain the Fréchet-Hoeffding lower bound. The generator $\theta^{-1}(u^{-\theta} - 1)$ of the Clayton copula is strict only if $\theta > 0$. In this case

$$C_\theta^{Cl}(u_1, u_2) = \big(u_1^{-\theta} + u_2^{-\theta} - 1\big)^{-\frac{1}{\theta}} \tag{17}$$

The generator $\ln(e^{-\theta} - 1) - \ln(e^{-\theta u} - 1)$ leads to the *Frank copula* given by

$$-\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u_1} - 1)\cdot(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right), \tag{18}$$

for $\theta \in \mathrm{R} \setminus \{0\}$.

The *generalized Clayton copula* is obtained from the generator $\theta^{-\delta}(u^{-\theta} - 1)^\delta$:

$$\Big([(u_1^{-\theta} - 1)^\delta + (u_2^{-\theta} - 1)^\delta]^{\frac{1}{\delta}} + 1\Big)^{-\frac{1}{\theta}}, \tag{19}$$

with $\theta > 0$ and $\delta \geq 1$. Note that for $\delta = 1$ the standard Clayton copula is attained, compare [18] Example 4.19.

Further examples of Archimedean copulas may be found in [18], in particular one may consider Table 4.1 therein as well as Sections 4.5 and 4.6.

The Marshall-Olkin copula. The *Marshall-Olkin copula* is a copula with singular component. For intuition, consider two components which are subject to certain shocks which lead to failure of either one of them or both components. The shocks occur at times assumed to be independent and exponentially distributed. Denote the realized shock times by Z_1, Z_2 and Z_{12}. Then we obtain for the probability that the two components live longer than x_1 and x_2, respectively,

$$P(Z_1 > x_1)\, P(Z_2 > x_2)\, P(Z_{12} > \max\{x_1, x_2\}). \tag{20}$$

This extends to the multivariate case in a straightforward way, compare [4] and [18]. The related copula equals

$$\min\{u_2 \cdot u_1^{1-\alpha_1}, u_1 \cdot u_2^{1-\alpha_2}\}, \tag{21}$$

with $\alpha_i \in [0,1]$. A similar family is given by the *Cuadras-Augé* copulas

$$\min\{u_1, u_2\} \cdot \big(\max\{u_1, u_2\}\big)^\alpha, \tag{22}$$

$\alpha \in [0,1]$, see [3].

2 Measures of dependence

Measures of dependence summarize the dependence structures of rvs. There are three important concepts: linear correlation, rank correlation and tail dependence. A further concept of dependence is *association* see [6] or [17].

Linear correlation. Linear correlation is a well studied concept. It is a dependence measure which is useful only for *elliptical distributions* (see **multivariate distributions**). The reason for this is that elliptical distributions are fully described by mean vector, covariance matrix and a characteristic generator function. As mean and variances are determined by the marginal distributions, the copulas of elliptical distributions depend only on the covariance matrix and the generator function. Linear correlation therefore has a distinguished role in this class which it does not have in other multivariate models.

Rank correlation. Rank correlations describe the dependence structure of the ranks, i.e. the dependence structure of the considered rvs when

[7] For generating the Clayton copula it would be sufficient to use $(u^{-\theta} - 1)$ instead of $\frac{1}{\theta}(u^{-\theta} - 1)$ as generator. However, for $\theta < 0$ this function is increasing and the above result would not apply.

transformed to uniform marginals using the probability transformation. Most importantly, this implies a direct representation in terms of the underlying copula, compare (5). We consider *Kendall's tau* and *Spearman's rho*, which also play an important role in nonparametric statistics.

For rvs $\mathbf{X} = X_1, \dots, X_d$ with marginals $F_i,\ i = 1, \dots, d$, *Spearman's rho* is defined by

$$\rho_S(\mathbf{X}) := \mathrm{Corr}\,\big(F_1(X_1), \dots, F_d(X_d)\big); \tag{23}$$

Corr is the correlation matrix whose entries are given by $\mathrm{Corr}(F_i(X_i), F_j(X_j))$.

Consider an independent copy $\tilde{\mathbf{X}}$ of $\mathbf{X}$. Then *Kendall's tau* is defined by

$$\rho_\tau(\mathbf{X}) := \mathrm{Cov}\,\big[\,\mathrm{sign}\,\big(\mathbf{X} - \tilde{\mathbf{X}}\big)\big]. \tag{24}$$

For $d = 2$,

$$\begin{aligned}\rho_\tau(X_1, X_2) = P\big((X_1 - \tilde{X}_1)\cdot(X_2 - \tilde{X}_2) > 0\big)\\ - P\big((X_1 - \tilde{X}_1)\cdot(X_2 - \tilde{X}_2) < 0\big)\end{aligned} \tag{25}$$

which explains this measure of dependency.

Both measures have values in $[-1, 1]$; they are 0 for independent variables (while there might also be non-independent rvs with zero rank correlation) and they equal 1 (-1) for the comonotonic (countermonotonic) case. Moreover, they can directly be derived from the copula of $\mathbf{X}$, see [14], Proposition 5.29. For example,

$$\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 \big(C(u_1, u_2) - u_1 u_2\big)\, du_1\, du_2. \tag{26}$$

In the case of a bivariate Gaussian copula one obtains[8] $\rho_S(X_1, X_2) = \frac{6}{\pi} \arcsin \frac{\rho}{2}$ and a similar expression for ρ_τ.

For other examples and certain bounds which interrelate those two measures we refer to [18], Sections 5.1.1 – 5.1.3. For multivariate extensions see e.g. [20] and [23].

Tail dependence. One distinguishes between *upper* and *lower* tail dependence. Consider two rvs X_1 and X_2 with marginals F_1, F_2 and copula C. *Upper tail dependence* means intuitively, that with large values of X_1 also large values of X_2 are expected. More precisely, the *coefficient of upper tail dependence* is defined by

$$\lambda_u := \lim_{q \nearrow 1} P\big(X_2 > F_2^{-1}(q) \big| X_1 > F_1^{-1}(q)\big), \tag{27}$$

provided that the limit exists and $\lambda_u \in [0, 1]$. The *coefficient of lower tail dependence* is

$$\lambda_l := \lim_{q \searrow 0} P\big(X_2 \le F_2^{-1}(q) \big| X_1 \le F_1^{-1}(q)\big). \tag{28}$$

If $\lambda_u > 0$, X_1 and X_2 are called *upper tail dependend*, while for $\lambda_u = 0$ they are *asymptotically independent* in the upper tail; analogously for λ_l. For continuous cdfs Bayes' rule gives

$$\lambda_l = \lim_{q \searrow 0} \frac{C(q, q)}{q} \tag{29}$$

and

$$\lambda_u = 2 + \lim_{q \searrow 0} \frac{C(1-q, 1-q) - 1}{q}. \tag{30}$$

A Gaussian copula has no tail dependence if the correlation is not equal to 1 or -1. For the bivariate t-distribution

$$\lambda_l = \lambda_u = 2\, t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right) \tag{31}$$

provided that $\rho > -1$. Note that even for zero correlation this copula shows tail dependence.

Tail dependence is a key quantity for joint quantile exceedances, see Example 5.34 in [14]: a multivariate Gaussian distribution will give a much smaller probability to the event that all returns from a portfolio are below the 1% quantiles of their respective distributions than a multivariate t-distribution. The reason for this is the difference in the tail dependence.

Association. A somewhat stronger concept than correlation is the so-called association introduced in [6]. If $\mathrm{Cov}(X, Y) \ge 0$ then one would consider X and Y as somehow associated. If, moreover, $\mathrm{Cov}(f(X), g(Y)) \ge 0$ for all pairs of nondecreasing functions $f,\ g$, they would be considered more strongly associated. If $\mathrm{Cov}(f(X, Y), g(X, Y)) \ge 0$

[8] Compare [14], Theorem 5.36. ρ_S and ρ_τ for elliptic distributions are also covered.

for all pairs of functions f, g which are nondecreasing in each argument, an even stronger dependence holds. One therefore calls rvs $X_1, \dots, X_d =: \mathbf{X}$ *associated* if $\mathrm{Cov}(f(\mathbf{X}), g(\mathbf{X})) \geq 0$ for all f, g which are nondecreasing and the covariance exists. Examples of associated rvs include independent rvs, positively correlated normal variables and also the generalized exponential distribution turning up in the Marshall-Olkin copula.

3 Sampling from copulas

Consider given marginals $F_1, \dots, F_d$ and a given copula C. The first step is to simulate $(U_1, \dots, U_d)$ with uniform marginals and copula C. By (3), the vector $(F_1^{-1}(U_1), \dots, F_d^{-1}(U_d))$ has copula C and the desired marginals.

If the copula is inherited from a multivariate distribution, the task reduces to simulating this multivariate distribution, e.g. Gaussian or t-distribution.

If the copula is Archimedean, this task is more demanding and we refer to [14], Algorithm 5.48 for details.

4 Conclusion

On one side copulas are a very general tool to describe dependence structures and have been successfully applied in many cases. However, the immense generality is also a drawback in many applications and also the static character of this measure of dependence has been criticized; see the referenced literature. Needless to say, the application of copulas has been a great success to a number of fields, and especially in finance they are a frequently used concept. They serve as an excellent tool for calibrating dependence structures or stress testing portfolios or other products in finance and insurance as they allow to interpolate between extreme cases of dependence.

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5 Related articles

correlation and correlation risk; **default correlation and asset correlation**; **typology of risk exposures**; **operational risk**; **copulas (financial econometrics)**; **copulas and dependence concepts in insurance**

6 Literature

The literature on copulas is growing fast. The vital article on copulas **copulas** by P. Embrechts gives an excellent overview, guide to the literature and applications. An introduction to copulas which extends this note in many ways may be found in [21]. For a detailed exposition of copulas with different applications in view we refer to [14], [19] as well as to [4]. [2] gives additional examples and [13] analyzes extreme financial risks. Estimation of copulas is discussed in [14], Section 5.5, [1] and [10]. For an in-depth study of copulas consider [18] or [11]. Interesting remarks of the history and the development of copulas may be found in [7]. For more details on Marshall-Olkin copulas, in particular the multivariate ones, see [4] and [18]. In the modelling of Lévy processes (see **Lévy processes**) one considers dependency of jumps where the measure is no longer a probability measure. This leads to the development of so-called Lévy copulas, compare [12]. The mentioned pitfalls with linear correlation are discussed in detail in [5] or [14], Chapter 5.2.1. For a discussion on the general difficulties in the application of copulas we refer to [16] and [9].

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Name	Copula	Paramter range
Independence or product copula	$\Pi(\mathbf{u}) = \prod_{i=1}^{d} u_i$	
Comonotonicity copula or Fréchet-Hoeffding upper bound	$M(\mathbf{u}) = \min\{u_1, \ldots, u_d\}$	
Countermonotonicity copula or Fréchet-Hoeffding lower bound	$W(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$	
Gaussian copula*	$C^{Ga}_{\mathbf{\Gamma}}(\mathbf{u}) = \Phi_{\mathbf{\Gamma}}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))$	
t- or Student copula*	$C^{t}_{\nu,\mathbf{\Gamma}}(\mathbf{u}) = t_{\nu,\mathbf{\Gamma}}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d))$	
Gumbel copula or Gumbel-Hougaard copula	$C^{Gu}_{\theta}(u_1, u_2) = \exp\Big(- \Big[(-\ln u_1)^\theta + (-\ln u_2)^\theta\Big]^{\frac{1}{\theta}} \Big)$	$\theta \in [1, \infty)$
Clayton copula+	$C^{Cl}_{\theta}(u_1, u_2) = \Big(\max\{u_1^{-\theta} + u_2^{-\theta} - 1, 0\} \Big)^{-\frac{1}{\theta}}$	$\theta \in [-1, \infty)$
Generalized Clayton+ copula	$C^{Cl}_{\theta,\delta}(u_1, u_2) = \Big(\Big[(u_1^{-\theta} - 1)^\delta + (u_2^{-\theta} - 1)^\delta \Big]^{\frac{1}{\delta}} + 1 \Big)^{-\frac{1}{\theta}}$	$\theta \geq 0, \delta \geq 1$
Frank copula+	$C^{Fr}_{\theta}(u_1, u_2) = -\frac{1}{\theta} \ln \Big(1 + \frac{(e^{-\theta u_1} - 1) \cdot (e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \Big)$	$\theta \in \mathrm{R}$
Marshall-Olkin copula or generalized Cuadras-Augé copula	$C_{\alpha_1,\alpha_2}(u_1, u_2) = \min\{u_2 \cdot u_1^{1-\alpha_1}, u_1 \cdot u_2^{1-\alpha_2}\}$	$\alpha_1, \alpha_2 \in [0, 1]$

Table 1: List of some copulas. For the Gumbel, Clayton, Frank and the Marshall-Olkin copula only the bivariate versions are stated. References to the multivariate versions are given in the text. Further examples of copulas may be found in [18], Table 4.1 (page 94) and Sections 4.5 and 4.6.

*: where $\mathbf{\Gamma}$ is a correlation matrix, i.e. a covariance matrix where each variance is scaled to 1.
$^+$: for the (generalized) Clayton and the Frank copula the case $\theta = 0$ is given as the limit for $\theta \to 0$, which leads to the independence copula in both cases.