Shot-Noise Processes and the Minimal Martingale Measure

Thorsten Schmidt, Department of Mathematics, University of Leipzig, Germany[1] and
Winfried Stute, Department of Mathematics, University of Giessen, Germany[2]

April 18, 2007

Abstract

This article proposes a model for stock prices which incorporates shot-noise effects. This means, that sudden jumps in the stock price are allowed, but their effect may decline as time passes by. Our model is general enough to capture arbitrary effects of this type. Generalizing previous approaches to shot-noise we in particular allow the decay to be stochastic. This model describes an incomplete market, so that the martingale measure is not unique. We derive the minimal martingale measure via continuous time methods.

Keywords: Minimal martingale measure; shot-noise process; jump diffusion

1 Introduction

In financial markets information often comes as a surprise. This usually leads to a jump in stock prices, may it be upward or downward. Quite often this is an over-reaction and the effect partially vanishes as time passes by.

A by now classical approach to incorporate jumps in a stock price model is via so-called jump-diffusion, which originated from the work of Merton (1973). In a jump-diffusion model the stock prices may jump to a new level and then follow a geometric Brownian motion. This means that the jump effect persists in the model. In reality this need not be the case since, e.g., an upward jump may lead to profit taking. As a result, the jump effect may partially fade away.

In Arai (2004) a multidimensional jump-diffusion model has been considered and the minimal martingale measure was computed. This paper generalizes the one-dimensional case to shot-noise models in several aspects.

The setup of the paper is as follows. In section two we extend the model of Altmann, Schmidt and Stute (2006) by considering a general model for shot-noise processes. We analyze the model in continuous time and derive the minimal martingale measure. Martingale measures are important for pricing derivatives of financial assets. Actually, these prices may be obtained as expectations of the discounted payoffs at maturity w.r.t. a martingale measure. Finally we give conditions which ensure that the minimal martingale measure is indeed a probability measure. A small simulation study exhibits typical features of the model.

[1]Thorsten Schmidt, Department of Mathematics, University of Leipzig, D-04081 Leipzig. Email: Thorsten.Schmidt@math.uni-leipzig.de

[2]Winfried Stute, Department of Mathematics, University of Giessen, Arndtstr. 2, D-35392 Giessen, Germany. Email: Winfried.Stute@math.uni-giessen.de

2 Setup

Consider a sequence of iid random variables (rv) $\eta^i, i = 1, 2, \ldots$ with values in $\mathbb{R}^d$ and a sequence of independent Brownian motions $B^i, i = 1, 2, \ldots$. Let $(\lambda_t)_{t\geq 0}$ be a positive hazard process, such that $\mathbb{P}(\int_0^T \lambda_t dt < \infty) = 1$ for a fixed horizon T. Finally, let $\tilde{N}$ be a standard Poisson process independent of all η^i, B^i and λ. Define $\Lambda_t := \int_0^t \lambda_s ds$ and set $N_t := \tilde{N}_{\Lambda_t}$. Then N is a Cox process with intensity λ. Denote its jump times by τ_i.

Let $a : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$, $b : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^+$ be smooth functions, and let the processes J^i, $i = 1, 2, \ldots$ be given by the unique strong solutions of the stochastic differential equations (SDE)

$$J^i(t) = J^i(0) + \int_0^t a(s, J^i(s), \eta^i)\, ds + \int_0^t b(s, J^i(s), \eta^i)\, dB^i(s), \tag{1}$$

where $J^i(0)$ are iid real valued rv. Then we call

$$Y_t := \sum_{i=1}^{N_t} J^i(t - \tau_i) \tag{2}$$

a general shot-noise process. To shorten notation, we write $a^i_t := a(t-\tau_i, J^i(t-\tau_i), \eta^i)$ and $b^i_t := b(t-\tau_i, J^i(t-\tau_i), \eta^i)$. Note that we do not require that $J^i(0)$ and η^i are independent. It will turn out to be useful at some time to have η equal to $J^i(0)$.

We first mention some important special cases of (1):

1. The processes J^i are of the form $U_i \cdot h(t)$, with iid U_i and a deterministic differentiable function h. We assume $h(0) = 1$. Then

$$J^i(t) = U_i + \int_0^t U_i h'(s)\, ds.$$

   Indeed, this is a special case of (1) with $b \equiv 0$, $\eta^i = U_i = J^i(0)$ and $a(s, J, \eta^i) = \eta^i h'(s)$. An important example for h is $h(t) = e^{-at}$, in which case the process is Markovian.

2. In the previous example the decay speed was fixed. As an extension, we may want the decay speed to be random. For this, consider a differentiable function $h : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$, with $h(0) = 1$ and $J^i(t) = U_i \cdot h(t, \tilde{\eta}^i)$, with iid $\tilde{\eta}^i$. Assume J^i equals

$$J^i(t) = U_i + \int_0^t U_i h'(s, \tilde{\eta}^i)\, ds.$$

   This time we take $\eta^i = (U_i, \tilde{\eta}^i)$ and the choice of a, b is obvious. We would typically assume that the investor observes η^i at the ith jump τ_i of N. The decline is random, but if the jump occurs, the future decay is known. Some kind of regime-switching could be incorporated by letting $h(t, x) = \mathbb{1}_{\{t\in[0,x]\}}$. In this case the above setting would imply that at the upward-jump it is also immediately known, when the process jumps down again.

3. Up to now, b was always 0. In the next example we consider a stochastic noise effect. For example, if some unexpected event occurs investors may overreact due to various reasons. As time passes by, further information my arise so that the market comes back to more reasonable levels. To incorporate this effect we assume a mean reversion of the noise effect. Let J^i be equal to

$$J^i(t) = J^i(0) + \int_0^t \kappa\big(\tilde{\eta}^i - J^i(s)\big)ds + \int_0^t \sigma dB^i(s).$$

Intuitively, this means that the process Y jumps up by $J^i(0)$ at τ_i and then diffuses to the level $Y(\tau_i-) + \tilde{\eta}^i$. Also the volatility after the jump will decline exponentially such that the effect of coming back is captured. Other examples are immediately found: if J^i is required to be nonnegative, one may set $b(t, J, \eta) = \sigma\sqrt{J}$ so that

$$dJ^i(t) = \kappa(\theta - J^i(t))dt + \sigma\sqrt{J^i(t)}dB^i(t). \tag{3}$$

The process specified by (3) is a so-called Cox, Ingersoll and Ross (1985) process. For more details we refer to Björk (2003).

The main tool for obtaining the minimal martingale measure is the semimartingale representation of Y. The increment of Y equals

$$Y_{t+\Delta} - Y_t = \sum_{i=1}^{N_t} \big(J^i(t+\Delta-\tau_i) - J^i(t-\tau_i)\big) + \sum_{i=N_t+1}^{N_{t+\Delta}} \big(J^i(t+\Delta-\tau_i) - J^i(t-\tau_i)\big).$$

Letting Δ go to zero one obtains that

$$dY_t = \sum_{i=1}^{N_{t-}} dJ^i(t-\tau_i) + d\Big(\sum_{i=1}^{N_t} J^i(0)\Big).$$

Denoting $m_1 := \mathbb{E}(J^1(0))$ we obtain the semimartingale decomposition:

$$\begin{aligned} dY_t &= \Big(\sum_{i=1}^{N_{t-}} a_t^i\Big)dt + \sum_{i=1}^{N_{t-}} b_t^i dB_t^i + d\Big(\sum_{i=1}^{N_t} J^i(0)\Big) \qquad (4)\\ &= \Big(\sum_{i=1}^{N_{t-}} a_t^i + \lambda_t m_1\Big)dt + \sum_{i=1}^{N_{t-}} b_t^i dB_t^i + d\Big(\sum_{i=1}^{N_t} J^i(0) - \int_0^t \lambda_s m_1\, ds\Big), \end{aligned}$$

where the last two terms are local martingales.

3 The Model

In this section we consider a model for stock prices which utilizes the previously introduced general shot-noise process. First we show how to add shot-noise effects to a standard model for stock prices. Assume that $\tilde{S}$ is a stochastic process for stock prices. For example, $\tilde{S}$ could be a geometric Brownian motion. Set

$$S_t = \tilde{S}_t \cdot \exp\big(Y_t\big). \tag{5}$$

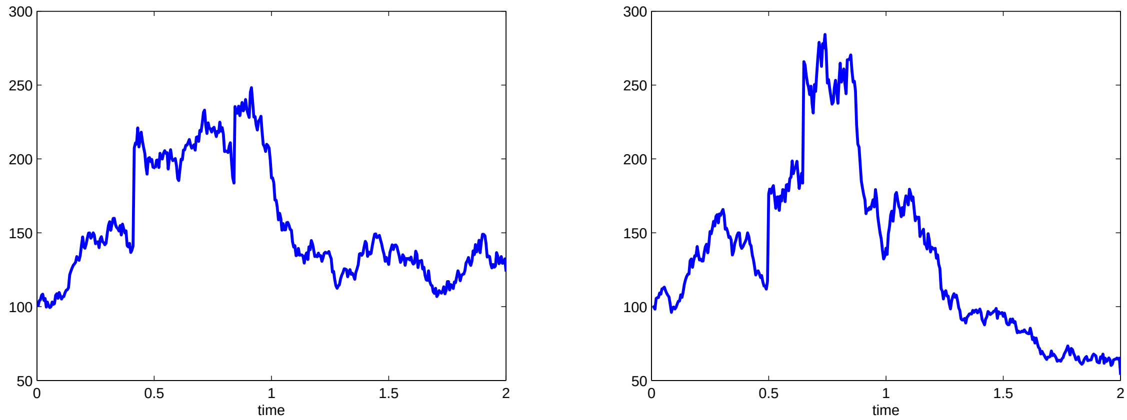

Figure 1: Two simulations of shot-noise processes according to (5). $\tilde{S}$ is a geometric Brownian motion starting in 100 with $\sigma = 0.4$. Left: shot-noise model with $J^i(t) = U^i \exp(-2t)$ with jumps at 0.4 and 0.9; right: shot-noise model with $J^i(t)$ according to (3).

We give some examples in Figure 1. Both simulations start from the model given by (5). The left graph uses a shot-noise component of the form $J^i(t) = U^i \exp(-at)$. After the jumps (at $t = 0.5$ and $t = 0.9$) the model tends towards the pre-jump level. The right plot incorporates a shot-noise specification with a stochastic mean-reversion effect according to (3), where $\kappa = 2$, $\theta = 0.1$ and $\sigma = 0.4$. After the jumps the process shows an increasing volatility which decreases after some time. This is due to the non-zero volatility of J^i.

Remark 3.1. The model presented in Altmann, Schmidt and Stute (2006) assumes

$$\begin{aligned} S_t &= S_0 \exp\Big[(\mu - \frac{\sigma^2}{2})t - \sigma B_t\Big] \prod_{i=1}^{N_t}(1 + U_i h(t-\tau_i)) \\ &= S_0 \exp\Big[(\mu - \frac{\sigma^2}{2})t - \sigma B_t\Big] \exp\Big(\sum_{i=1}^{N_t} \ln(1 + U_i h(t-\tau_i))\Big). \end{aligned}$$

This is a special case of (5), with J^i chosen as

$$J^i(t) = \ln(1+U_i) + \int_0^t \frac{U_i h'(s)}{1 + U_i h(s)}\, ds$$

and $\tilde{a}_t \equiv \mu + \sigma^2/2, \tilde{b}_t \equiv \sigma$.

A special case: exponential decay. The shot-noise process considered so far has a quite general behaviour which typically leads to non-Markovian processes. Considering $J^i(t) = U_i h(t)$ there are important special cases where the process is Markovian. For example, the classical jump-diffusion is obtained setting $h \equiv 1$. More interestingly, the process is also Markovian if $h(s+t) = h(s)h(t)$, i.e., h has the form e^{-at}. The main reason for this is that the shot-noise process then takes on the following form

$$S_t = \sum_{i=1}^{N_t} U_i h(t - \tau_i) = h(t) \sum_{i=1}^{N_t} U_i h(-\tau_i).$$

This process behaves deterministically between the jump times, with velocity h' and jumps at τ_i. Moreover, as has been shown in Gaspar and Schmidt (2005), the only h for which Markovianity holds, is $h(t) = e^{-at}$.

4 The minimal martingale measure

We assume for simplicity that the market interest rate is zero. The *minimal martingale measure* $\hat{Q}$ as proposed in Föllmer and Schweizer (1990) can be described by its density L_T, where $d\hat{Q} := L_T dP$ and P is the objective measure. The density is determined as follows. Assume that S has the semimartingale representation $S_t = A_t + M_t$ with a local martingale M and an increasing process A of bounded variation. Furthermore, assume there exists a process $\hat{\lambda}$ which satisfies

$$A_t = \int_0^t \hat{\lambda}_{s-} \, d\langle M \rangle_s. \tag{6}$$

Then the minimal martingale density is given by

$$L_T = \mathcal{E}\Big(- \int_0^{\cdot} \hat{\lambda}_{s-} \, dM_s \Big)_T, \tag{7}$$

where $\mathcal{E}(\cdot)_t$ denotes the Doléans-Dade exponential (see, for example, Protter (2004)).

Recall that $S_t = \tilde{S}_t \exp(Y_t)$. Consider a Brownian motion B which is independent of Y and two positive processes $(\tilde{a}_t)_{t\geq 0}$ and $(\tilde{b}_t)_{t\geq 0}$ which are adapted to the natural filtration of B. We assume that $\tilde{S}_t$ is the strong solution of

$$d\tilde{S}_t = \tilde{S}_t(\tilde{a}_t dt + \tilde{b}_t dB_t).$$

Proposition 4.1. *Set* $m_e := \mathbb{E}[\exp(J^1(0)) - 1]$ *and* $m_2 := \mathbb{E}[(\exp(J^i(0)) - 1)^2]$ *and assume both are finite. The minimal martingale measure is given by the density* L_T *in* (7)*, where*

$$\hat{\lambda}_t = \frac{1}{S_{t-}} \cdot \frac{\tilde{a}_t + \sum_{i=1}^{N_{t-}} a_t^i + \lambda_t m_e + \frac{1}{2}\sum_{i=1}^{N_{t-}} (b_t^i)^2}{(\tilde{b}_t)^2 + \sum_{i=1}^{N_{t-}} (b_t^i)^2 + \lambda_t m_2}.$$

Proof. First, we compute the semimartingale representation of S using Itô's formula. See Protter (2004) for a suitable version. From

$$d(\exp(Y_t)) = \exp(Y_{t-})\Big\{ dY_t^c + \frac{1}{2} d\langle Y^c \rangle_t + d\Big[\sum_{i=1}^{N_t} \Big(\exp(J^i(0)) - 1 \Big) \Big] \Big\}$$

we obtain by (4) and the independence of $\tilde{S}$ and Y:

$$\begin{aligned}
dS_t &= \exp(Y_{t-})\tilde{S}_t \Big(\tilde{a}_t dt + \tilde{b}_t dB_t \Big) \\
&\quad + \tilde{S}_t \exp(Y_{t-}) \Big[\sum_{i=1}^{N_{t-}} a_t^i dt + \sum_{i=1}^{N_{t-}} b_t^i dB_t^i + \frac{1}{2} \sum_{i=1}^{N_{t-}} (b_t^i)^2 dt + d\Big[\sum_{i=1}^{N_t} \Big(\exp(J^i(0)) - 1 \Big) \Big] \Big] \\
&= S_{t-} \Big[\Big(\tilde{a}_t + \sum_{i=1}^{N_{t-}} a_t^i + \lambda_t m_e + \frac{1}{2} \sum_{i=1}^{N_{t-}} (b_t^i)^2 \Big) dt \qquad (8) \\
&\quad + \tilde{b}_t dB_t + \sum_{i=1}^{N_{t-}} b_t^i dB_t^i + d\Big[\sum_{i=1}^{N_t} \Big(\exp(J^i(0)) - 1 \Big) - \int_0^t \lambda_s m_e \, ds \Big] \Big].
\end{aligned}$$

With the semimartingale representation $S_t = A_t + M_t$ at hand we now compute $\langle M \rangle$. Using the independence of B and B^i we obtain that

$$d\langle M \rangle_t = S_{t-}^2 \Big[(\tilde{b}_t)^2 + \sum_{i=1}^{N_{t-}} (b_t^i)^2 + \lambda_t m_2 \Big] dt. \tag{9}$$

Using (8) and (9) in (6) we get the stated representation of $\hat{\lambda}$. ■

Once $\hat{\lambda}$ is obtained the density follows immediately:

$$\begin{aligned} L_T &= \prod_{i=1}^{N_T} \Big(1 - \hat{\lambda}_{\tau_i} \Delta M_{\tau_i}\Big) E_T = \prod_{i=1}^{N_T} \Big(1 - \lambda_{\tau_i} S_{\tau_i -} \Big(e^{J^i(0)} - 1\Big)\Big) E_T \\ &= \prod_{i=1}^{N_T} \Big(1 - l_{\tau_i} \Big(e^{J^i(0)} - 1\Big)\Big) E_T, \end{aligned} \tag{10}$$

where we set

$$l_t := \frac{\tilde{a}_t + \sum_{i=1}^{N_{t-}} \big(a_t^i + \frac{1}{2}(b_t^i)^2\big) + \lambda_t m_e}{(\tilde{b}_t)^2 + \sum_{i=1}^{N_{t-}} (b_t^i)^2 + \lambda_t m_2}, \tag{11}$$

$$E_T := \exp\Big(- \int_0^T \hat{\lambda}_s S_{s-} dM_s^c - \frac{1}{2} \int_0^T \hat{\lambda}_s S_{s-} d\langle M^c \rangle_s \Big), \tag{12}$$

and M^c is the continuous part of M.

Conditions for $\hat{Q}$ to be a probability measure. It is well known, that the minimal martingale measure need not be a probability measure. It will turn out that this is often the case for the considered model. On the other side, there are special cases which guarantee that the minimal martingale measure is a probability measure. Note that the denominator in (11) is always positive. For example, consider $J_t^i = U_i$, i.e., the case of a jump diffusion. In this case the numerator equals $\tilde{a}_t + \lambda_t m_e$ and so $\hat{Q}$ is a probability measure if

$$\frac{\tilde{a}_t + \lambda_t m_e}{(\tilde{b}_t)^2 + \lambda_t m_2} \Big(e^{J^i(0)} - 1\Big) < 1. \tag{13}$$

This condition corresponds to Assumption 3.1 from Arai (2004). One possibility to guarantee positivity of L in a jump-diffusion model is therefore to assume that $J^i(0) < \bar{J}$ and choose the parameters such that the above inequality (13) is satisfied.

In the general case we have the following result.

Proposition 4.2. *Assume that $J^i(0) \leq \bar{J}$ with $\bar{J} > 0$ and set $K := 1/(\exp(\bar{J}) - 1)$. Then $L_T > 0$ a.s., if the following inequalities are satisfied for all $i \geq 1$ and $t \in [0, T]$:*

$$a_t^i + \frac{(b_t^i)^2}{2} \geq 0, \quad \tilde{a}_t + \lambda_t m_e \geq 0 \tag{14}$$

$$a_t^i + (b_t^i)^2 \left(\frac{1}{2} - K\right) \leq 0 \tag{15}$$

$$\tilde{a}_t + \lambda_t m_e - K\Big(\tilde{b}_t + \lambda_t m_2\Big) < 0. \tag{16}$$

Proof. First, $K > 0$. It follows from (15) that

$$\sum_{i=1}^{N_{t-}} \left(a_t^i + (b_t^i)^2 \left(\frac{1}{2} - K \right) \right) < 0.$$

From (16) we obtain that

$$K\big(\tilde{b}_t + \lambda_t m_2\big) - \tilde{a}_t - \lambda_t m_e > 0 \geq \sum_{i=1}^{N_{t-}} \left(a_t^i + (b_t^i)^2 \left(\frac{1}{2} - K \right) \right)$$

and therefore

$$\tilde{a}_t + \lambda_t m_e + \sum_{i=1}^{N_{t-}} \left(a_t^i + \frac{(b_t^i)^2}{2} \right) < K\Big(\tilde{b}_t + \lambda_t m_2 + \sum_{i=1}^{N_{t-}} (b_t^i)^2 \Big).$$

Comparing the above inequality with (11), we obtain $l_t < K$. Hence, if $J^i(0) \geq 0$ we have that

$$l_{\tau_i} \big(\exp(J^i(0)) - 1 \big) < 1. \tag{17}$$

On the other side, consider the case $J^i(0) < 0$. First, (14) ensures that $l_t \geq 0$. Together with $\exp(J^i(0)) - 1 < 0$ we have that $l_{\tau_i} \big(\exp(J^i(0)) - 1 \big) \leq 0 < 1$, i.e. (17) also holds in this case.

Inserting (17) in (10) guarantees that $L_T > 0$, and the proof is completed. ■

Stochastic noise. This paragraph describes an example with stochastic noise effect, i.e., b^i is not zero. Furthermore the model is constructed such that the minimal martingale measure is an equivalent probability measure. Assume that $J_t^i = \eta^i - I_t^i$, where each I^i follows a Cox-Ingersoll-Ross process with mean reversion level zero:

$$dI^i(t) = -\kappa I^i(t) dt + \sigma \sqrt{I^i(t)} dB_t^i \tag{18}$$

where $\kappa, \sigma > 0$ and random positive $I^i(0)$. This means that the process J jumps up at τ_i by $\eta^i + I^i(0)$ and then diffuses to the level η^i.

The economic intuition behind the above model is as follows: due to a surprising event, i.e. new information, catastrophes, etc., the market changes abruptly. This goes hand in hand with an overreaction which is modeled by I^i. After a certain time the overreaction vanishes totally as I^i diffuses to 0.

Recalling $K = 1/(\exp(\bar{J}) - 1)$, we have the following result.

Proposition 4.3. *Assume that* (16) *as well as* $\tilde{a}_t + \lambda_t m_e \geq 0$ *holds and, furthermore,*

$$\kappa + \sigma^2 \left(\frac{1}{2} - K \right) < 0. \tag{19}$$

Then $L_T > 0$ *with probability 1.*

Proof. First, it is well-known that zero is an absorbing boundary for I^i. Second, observe that $a_{\tau_i+t}^i = \kappa(\eta^i - J_t^i) \geq 0$ and $(b_{\tau_i+t}^i)^2 = \sigma^2(\eta^i - J_t^i)$. Hence (14) holds. From (19) it follows that (15) holds. Applying Proposition 4.2 shows the assertion. ■

Example 4.4. Consider a Black-Scholes model for $\tilde{S}_t$, i.e., $\tilde{a}_t = \mu \geq 0$, and assume that $\lambda_t \equiv \lambda > 0$. Then $K(\sigma + \lambda m_2) > \mu + \lambda m_e$ implies (16). If additionally $\mu + \lambda m_e \geq 0$ and (19) holds, then $\hat{Q}$ is a probability measure.

Example 4.5. For $\sigma = 0$ and $\eta^i \equiv 0$ we obtain the classical shot-noise process with negative jumps as a special case, in which $J_t^i = \eta^i - I_0^i \exp(-\kappa t) = -I_0^i \exp(-\kappa t)$.

References

Altmann, T., Schmidt, T. and Stute, W. (2006), A shot noise model for financial assets, *submitted*.

Arai, T. (2004), Minimal martingale measures for jump diffusion processes, *J. Appl. Prob.* **41**, 263–270.

Björk, T. (2003), *Arbitrage Theory in Continuous Time*. (Oxford University Press, 2nd ed.).

Cox, J. C., Ingersoll, J. W. and Ross, S. A. (1985), A theory of the term structure of interest rates, *Econometrica* **54**, 385–407.

Föllmer, H. and Schweizer, M. (1990), Hedging of contingent claims under incomplete information. In M. H. A. Davis and R. J. Elliott (Eds.), *Applied Stochastic Analysis*, Volume 5, pp. 389–414. Gordon and Breach, London/New York.

Gaspar, R. M. and Schmidt, T. (2005), Quadratic portfolio credit risk models with shot-noise effects, *Stockholm School of Economics Working Paper Series - N0. 616*.

Merton, R. (1973), Theory of rational option pricing, *Bell J. of Econ. and Management Science* **4**, 141–183.

Protter, P. (2004), *Stochastic Integration and Differential Equations*. (Springer Verlag, Berlin Heidelberg New York, 2nd ed.).