An Infinite Factor Model for Credit Risk

THORSTEN SCHMIDT[1]

Department of Mathematics, University of Leipzig,
Augustusplatz 10/11, D-04109 Leipzig, Germany.

email: thorsten.schmidt@math.uni-leipzig.de
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Abstract

The defaultable term structure is modeled using stochastic differential equations in Hilbert spaces. This leads to an infinite dimensional model, which is free of arbitrage under a certain drift condition. Furthermore, the model is extended to incorporate ratings based on a Markov chain.

1 Introduction

The demand for risky investments in areas different from the stock markets has increased enormously due to their recent struggles. On possibility to achieve this, is to take credit risk in exchange for an attractive yield and as a result methodologies for pricing and hedging credit derivatives as well as for risk management of credit risky assets became very important. The efforts of the Basel Committee is just one of many examples to substantiate this. For an introduction into this area consider the surveys by Giesecke (2004) and Schmidt and Stute (2004) or one of the excellent textbooks Schönbucher (2003), Duffie and Singleton (2003), Lando (2004) or McNeil, Frey, and Embrechts (2005).

There are basically two classes of models for credit risk, the structural and the intensity based ones. The latter gives a direct connection to interest rate models and therefore allows to use the powerful machinery developed for interest rate models. For this reason, in this article we focus on intensity based models.

A recent branch in interest rate theory deals with models allowing for infinitely many factors and provides a lot of new insights. For example, Pang (1998) shows that the calibration is advantageous over Heath, Jarrow, and Morton (1992) (henceforth HJM) models. Hedges more akin to practice have been derived by Carmona and Tehranchi (2004) and De Donno and Pratelli (2004).

In comparison to the interest rate market, the credit market is much more volatile. Quite obvious, this is a property of investments at higher risk. Traditionally term structure

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models are defined via diffusions driven by a low number of factors, usually three. This choice enables analytical tractability and is usually justified with a view towards the empirical fact that the first three principal components describe 95% of the observed variance. However, Rebonato (2002, Section 13.2.5) observed, if including the period of the Russian crisis in the analysis at least 10 factors will be needed to explain only 90% of the variability. The additional factors are mainly due to the credit riskiness of the investments in this markets. Viewed in this light it seems delicate not to consider a sufficient number of factors. Furthermore, as pointed out in Cont (2005), dealing with derivatives typically involves expectations of non-linear functions of the forward rate curve. Therefore, a model which might explain the variance of the forward rate quite well may still lack principal components which have a significant effect on the price of such derivatives. So it is natural to analyse these markets within an infinite dimensional framework.

Modeling credit risk in an intensity based model may extend HJM to credit risk. There are several ways to do this, and in the next two sections we present an approach in the framework due to Duffie and Singleton (1999). We start by formulating the extension of HJM using stochastic differential equations on Hilbert spaces. Our presentation uses the so-called Musiela-parametrization which was originated in Musiela (1993). See Bagchi and Kumar (2001), Filipović (2001) or Björk (2003) for related results in the interest rate case.

In Section 4 we present an approach based on credit ratings. We use a Markov model in combination with two different recovery structures. For a ratings-based recovery of market value approach with a finite number of factors, see Acharya, Das, and Sundaram (2002), and for a ratings-based recovery of treasury value approach, see Bielecki and Rutkowski (2000). We extend both models using SDEs on Hilbert spaces.

The arbitrage-free conditions are presented in a fashion which clarifies the connection between the defaultable spot rate, default intensity and the recovery structure. It seems remarkable, that in a setting with credit risk other than the recovery of market value the drift condition involves the whole term structure, compare Theorems 3.3, 4.3 and 4.5. However, this only makes sense if one considers a functional approach using SDEs on Hilbert spaces. Therefore, it is necessary in an arbitrage-free model with credit risk to use the general framework presented here.

At this point the question naturally arises, where are the advantages of infinite dimensional models as the term structure in typical credit risky markets seems to be quite simple – usually there are only a low number of bonds issued per company. First, if these bonds are pooled in a portfolio and analysed, for example, in a factor model, the number of maturities to be considered will be quite high. A good model for the term structure of credit spreads is of need. Second, more maturities come into play if one considers government bonds or credit default swaps. The latter are the most liquid underlying in credit markets.

Another argument towards infinite dimensional models is discussed in Schmidt (2004), namely that a calibration based on such a model may show better numerical results and may help to avoid frequent re-calibrations, while analytical tractability is maintained.

2 An Infinite Factor HJM Extension

To develop our model with credit risk in infinite dimensions, we first discuss the methodology in the case without credit risk. Kennedy (1994) gives an interest rate formulation with Gaussian random fields. This approach was extended to more general models using SDEs on Hilbert spaces by Goldstein (2000), Santa-Clara and Sornette (1997) and Bagchi and Kumar (2001). The framework we are presenting includes these models.

The idea of the HJM approach is to model the dynamics of the forward rates itself rather than to model the dynamics of the instantaneous interest rate and then derive the dynamics of the forward rates. The forward rates have a one-to-one correspondence to bond prices, which in the continuous-time case amounts to

$$B(t,T) \quad = \quad \exp\Big(- \int_t^T f(t,u)\,du \Big).$$

Usually, the forward rate is modeled via an n-dimensional Brownian motion W as

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\cdot dW_t, \qquad 0 \le t \le T, \tag{1}$$

where $\alpha(t,T) \in \mathbb{R}$ and $\sigma(t,T) \in \mathbb{R}^n$ form predictable processes. By $\cdot$ we denote the scalar product in $\mathbb{R}^n$.

Noticing that the forward-rate curve at time t, denoted by $f(t,\cdot) : [t, t+T^{**}] \mapsto \mathbb{R}$, is a function (of T), one could model a stochastic process $f(t)$ which itself takes values in a functional space. So the question arises which functional space, say H, should be choosen.

Fix a finite time horizon T^*. There are forward rates up to a maximum time-to-maturity in the market, say T^{**}, which is typically 30 years. Consequently, one can express the term structure of forward rates by the stochastic process $f(t,t+x) : [0,T^*] \times [t, t+T^{**}] \mapsto \mathbb{R}$. This leads to the so-called Musiela parametrization, namely by considering

$$r_t(x) := f(t, t+x).$$

The stochastic process $(r_t)_{t\in[0,T^*]}$ takes values in a functional space $\mathbb{R}^{[0,T^{**}]}$. Sometimes we use r_t also to denote the spot rate, which is more precisely $r_t(0)$.

Therefore, we choose for H a separable Hilbert space, which consists of real-valued functions on the interval $[0, T^{**}]$. Using a Hilbert space allows to apply the well developed methodology for stochastic differential equations on Hilbert spaces. We first present some basic facts about Wiener processes on Hilbert spaces and the Itô formula in this framework. For a concrete choice of H examine Filipović (2001).

2.1 Wiener Processes on Hilbert Spaces

We state some basic facts about Wiener processes in Hilbert spaces and the Itô formula in this framework. For more details, see Da Prato and Zabczyk (1992).

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a D-Wiener process $(X_s)_{s\geq 0}$ with values in H. X is a process with stationary and independent increments, such that for $s < t$ the increment $X_t - X_s$ is normally distributed with zero mean and covariance operator $(t-s)D$. We always denote by $<,>$ the inner product of H and by $\{e_k : k \in \mathbb{N}\}$ and $\{\lambda_k : k \in \mathbb{N}\}$ the system of eigenvectors, and eigenvalues of D, respectively. The following decomposition plays an important role.

Proposition 2.1. *Consider a D-Wiener process $(X_s)_{s\geq 0}$ and define*

$$\beta_k(s) :=< X_s, e_k > .$$

Then, for any $k \in \mathbb{N}$ such that $\lambda_k > 0$, $\frac{1}{\sqrt{\lambda_k}}\beta_k(s)$ are mutually independent Brownian motions. Moreover, we have the decomposition

$$X_s = \sum_{k=1}^{\infty} \beta_k(s) e_k, \tag{2}$$

and the series in (2) *converges in $L^2(\Omega, \mathcal{A}, \mathbb{P})$.*

Denote the norm on H by $||\cdot||$. For a covariance operator D the space $H_0 := D^{\frac{1}{2}}(H)$ is again a Hilbert space. By $L_2(H_0, H)$ we denote the space of all Hilbert-Schmidt operators from H_0 into H, that is, continuous, linear operators T with $\sum_k < Te_k^0, Te_k^0 >^2$ being finite, where $\{e_k^0 : k \in \mathbb{N}\}$ is an orthonormal basis of H_0.

For a predictable process $(\Phi(s))_{s\in[0,T^*]}$ with values in $L_2(H_0, H)$ the stochastic integral $\int_0^t \Phi(s) \cdot dX_s$ is a square-integrable martingale if

$$|||\Phi|||_{T^*} := \Big[\mathbb{E}\Big(||\int_0^{T^*} \Phi(s) \cdot dX_s||^2\Big)\Big]^{\frac{1}{2}} < \infty,$$

where we denote $\Phi(s) \cdot h$ for $\Phi(s)(h)$ whenever $h \in H_0$.

The Itô-formula for H-valued semimartingales is a consequence of Taylor's formula[2].

Theorem 2.2. *For an open subset A of the Hilbert space H, let $f : A \mapsto H$ be a function, whose first and second derivative is uniformly continuous on bounded subsets of A. Assume that the process $(Y(t))_{t\in[0,T^*]}$ admits the following representation, with $\int_0^{T^*} ||\alpha_u||\, du < \infty$ $\mathbb{P}$-a.s. and $|||\Phi|||_{T^*} < \infty$,*

$$Y(t) = Y(0) + \int_0^t \alpha_u \, du + \int_0^t \Phi(u) \cdot dX_u.$$

Then, for $t \in [0, T^*]$, we have $\mathbb{P}$-a.s.

$$\begin{aligned} f\big(Y(t)\big) &= f\big(Y(0)\big) + \int_0^t \mathcal{D}f\big(Y(u)\big) \cdot dY(u) \\ &+ \frac{1}{2}\int_0^t \sum_{k=1}^{\infty} \lambda_k \mathcal{D}^2 f\big(Y(u)\big)\big(\Phi(u)\cdot e_k, \Phi(u)\cdot e_k\big)\, du. \end{aligned} \tag{3}$$

[2]This version of the Itô formula is derived, for example, in Filipović (2001) or Schmidt (2003). Note, that the second derivative $D^2 f$ is a continuous bilinear mapping of $H \times H$ into H. In this case we use the notation $D^2 f(f, g)$.

Our presentation will frequently use processes which are *mild solutions* of stochastic partial differential equations, see Da Prato and Zabczyk (1992) for a thorough discussion. Denote by A the generator of the shift-semigroup and consider

$$dY_t = (AY_t + F(t, Y_t))\,dt + B(t, Y_t)\,dX_t. \tag{4}$$

Let $\{S(t)|t \in \mathbb{R}_+\}$ denote the semigroup of right-shifts, defined by $S(t)g(x) = g(x+t)$, for any function $g : \mathbb{R}_+ \mapsto \mathbb{R}$. Then a predictable, H-valued process $(Y_t)_{t\in[0,T]}$ is the mild solution of (4), if

$$Y_t = S(t)Y_0 + \int_0^t S(t-u)F(u, Y_u)\,du + \int_0^t S(t-u)B(u, Y_u)\,dX_u.$$

Sufficient conditions for the existence of such a mild solution are given in Da Prato and Zabczyk (1992, Theorem 7.4).

2.2 The Model

First, we restate the dynamics of the forward rates in terms of $r_t(x)$. To make a notational difference, we write $\alpha_t(x)$ for $\alpha(t, t+x)$ and $\sigma_t(x)$ for $\sigma(t, t+x)$. Therefore, we obtain from (1), setting $x := T - t$

$$r_t(x) \;=\; r_0(t+x) + \int_0^t \alpha_u(t-u+x)\,du + \int_0^t \sigma_u(t-u+x)\cdot dW_u.$$

Recall that S was the right-shift operator. This enables us to obtain a consistent formulation within a functional setting by

$$\begin{aligned} r_t(x) &= S(t)r_0(x) + \int_0^t S(t-u)\alpha_u(x)\,du + \int_0^t S(t-u)\sigma_u(x)\cdot dW_u \\ \Leftrightarrow \qquad r_t &= S(t)r_0 + \int_0^t S(t-u)\alpha_u\,du + \int_0^t S(t-u)\sigma_u\cdot dW_u, \end{aligned}$$

where r_0, α_u are themselves elements of $\mathbb{R}^n$ while σ_u is a $n\times n$-matrix. In this formulation the shift operator arises naturally, as forward rates with fixed maturity correspond to forward rates with decreasing time-to-maturity.

In the following we will generalize the dW-integral to Wiener processes on the Hilbert space H. Consider stochastic processes $\alpha : [0, T^*]\times\Omega \mapsto H$ and $\sigma : [0, T^*]\times\Omega \mapsto L_2(H_0; H)$, both predictable w.r.t. $(\mathcal{F}_t)_{t\geq 0}$, satisfying $\mathbb{P}(\int_0^{T^*}\alpha(s)\,ds < \infty) = 1$ and $|||\sigma|||_{T^*} < \infty$. Further on, assume that $(X(t))_{t\geq0}$ is a D-Wiener process. Assume the forward rate dynamics to follow

$$r_t = S(t)r_0 + \int_0^t S(t-u)\alpha_u\,du + \int_0^t S(t-u)\sigma_u\cdot dX_u. \tag{5}$$

Note that r_t takes values in H, so it represents the whole forward-rate curve, otherwise denoted by $f(t, t+x)$. For α we could explicitly write $\alpha_t(x)$ while this is not possible for σ. Still, even if the index x does not appear directly, it is not obsolete. As the last integral is an element of H for all t we can write it either as

$$\int_0^t S(t-u)\sigma_u \cdot dX_u =: I(t) \in H \tag{6}$$

or directly as $I(t, x)$.

We will make use of the eigenvalue expansion of X, see Equation (2), and therefore denote

$$\sigma_k(u, v) := \big(\sigma(u) \cdot e_k\big)(v),$$

where $\{e_k : k \in \mathbb{N}\}$ are the orthonormalized eigenvectors of D .

We derive the analogue of the drift condition of Heath, Jarrow, and Morton (1992) in an infinite dimensional setting. Starting with a model under some measure Q, which is assumed to be equivalent to the objective measure P, we derive a condition under which Q is a martingale measure. Then Q is called an equivalent martingale measure (EMM) and, as shown by Björk, di Masi, Kabanov, and Runggaldier (1997), the market is free of arbitrage. In the above notation, we get the following

Theorem 2.3. *Set $\sigma_k^*(t, x) := \int_0^x \sigma_k(u, v)\, dv$. Then all discounted bond prices are martingales iff*

$$\alpha_t(x) = \sum_{k=1}^{\infty} \lambda_k\, \sigma_k^*(t, x)\, \sigma_k(t, x) \qquad \forall t \in [0, T^*], x \in [0, T^{**}]. \tag{7}$$

Equation (7) is often referred to as the *drift condition*. Note that the drift condition derived by Heath, Jarrow, and Morton (1992) is the special case corresponding to $\lambda_k = 0$ after some finite number n. Filipović (2001, Lemma 4.3.3) obtains a similar result, using a different method which mainly relies on the eigenvalue expansion (2) to obtain the proof.

Intuitively, the drift condition means that, once the volatility (and dependence) structure is specified, the dynamics under the arbitrage-free measure are fixed. As a change of measure does not change the volatility structure, this could be estimated using historical data, cf. Shreve (2004, Sec. 10.3.6) or Schmidt (2004).

Forward rates observed in the market have a time-to-maturity of up to 20 years or more, while the time horizon for credit derivatives is relatively small. This implies for our model that $T^{**} > T^*$, which plays a role, for example, in the drift condition.

If the drift-condition is satisfied, then the market is free of arbitrage. Completeness follows if the equivalent martingale measure is unique. To the best of our knowledge conditions under which this holds true in the above setting are not yet available.

Proof of Theorem 2.3. In the Musiela parametrization, the bond price equals

$$B(t, T) = \exp\Big(-\int_0^{T-t} r_t(v)\, dv\Big).$$

As we want to consider $\int_0^{T-t} r_t(v)\,dv$, we introduce

$$y(t,x) = F(r_t,x) := \int\limits_0^x r_t(v)\,dv,$$

where $F : H \times \mathbb{R}^+ \mapsto H$. Consider x as fixed. Then, as F is linear, we immediately obtain that the Frèchet-derivative of F is F itself. Of course, the second derivative is zero. Thus, applying the Itô -formula (3) yields

$$F(r_t,x) = F(r_0,x) + \int\limits_0^t \mathcal{D}F \cdot \frac{\partial}{\partial x} r_u\,du + \int\limits_0^t \mathcal{D}F \cdot \alpha_u\,du + \int\limits_0^t \mathcal{D}F \cdot\, \sigma_u \cdot dX_u.$$

We suppress the dependence of the derivative on x. For example, $\mathcal{D}F \cdot \alpha_u = \int_0^x \alpha_u(v)\,dv$. Defining $\Phi : [0,T^*]^2 \times \Omega \mapsto L(H;H)$ by

$$\Phi(u,x) \cdot f := \int\limits_0^x [\sigma_u \cdot f](v)\,dv,$$

and $\alpha^*(t,x) := F(\alpha_t,x) = \int_0^x \alpha_t(v)\,dv$, we obtain the dynamics of $F(r_t,x)$ as

$$dF(r_t,x) = \Big[r_t(x) - r_t(0) + \alpha^*(t,x)\Big]\,dt + \Phi(t,x)\cdot dX_t.$$

The second step is to derive the dynamics of the bond price $B(t,T) = \exp(-y(t,T))$. To apply Itô's formula, we define

$$\tilde{F} : A \mapsto H, \qquad g(\cdot) \to \exp(g(\cdot)).$$

Here A is chosen in such a way, that $\exp(g(\cdot))$ is again an element of H. Then we have $B(t,\cdot) = [\tilde{F}(-y(t))](\cdot)$ or $B(t) = \tilde{F}(-y(t))$, respectively.

We calculate the first and second derivative of $\tilde{F}$. First, define for $f,g \in H$ the product of f and g by $(f \times g)(\cdot) := f(\cdot)g(\cdot)$. Then $\tilde{F}(g(\cdot)) = \sum_{k=1}^{\infty} \frac{g(\cdot)^k}{k!}$. The derivative of g^2 is

$$\mathcal{D}(g^2)(x) = 2g(x) \times \mathrm{id},$$

where id is the identity on H. The derivative of g^n is easily obtained by induction and we conclude

$$\mathcal{D}\tilde{F}(g) = \tilde{F}(g) \times \mathrm{id} \quad \text{and} \quad \mathcal{D}^2\tilde{F}(g) = \tilde{F}(g) \times \mathrm{id} \times \mathrm{id}\,.$$

It is very important to distinguish between the bond dynamics in Musiela parametrization and the canonical dynamics. To clarify this, we write just for the following formula $B(t,t+x)$ and $B_0(t,T)$, respectively. Then, the connection of both formulations is

$$dB_0(t,T) = dB(t,x)\Big|_{x=T-t} - \frac{\partial}{\partial x}B(t,x)\Big|_{x=T-t}\,dt.$$

Therefore, applying Itô's formula yields for the canonical dynamics

$$
\begin{aligned}
dB(t,T) &= \mathcal{D}\tilde{F}\big(B(u)\big)\cdot\Big[-\big(r_t(T-t)-r_t(0)+\alpha^*(t,T-t)\big)\,dt-\Phi(t,T-t)\cdot dX_t\Big]\\
&\quad+\frac{1}{2}\Big[\sum_{k=1}^{\infty}\lambda_k\mathcal{D}^2\tilde{F}\big(B(t)\big)\,\big(\Phi(t,T-t)\cdot e_k,\Phi(t,T-t)\cdot e_k\big)+r_t(T-t)B(t,T-t)\Big]\,dt\\
&= B(t,T)\Big[\big(r_t(0)-\alpha^*(t,T-t)\big)\,dt-\Phi(t,T-t)\cdot dX_t\\
&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(t,T-t)]^2\,dt\Big].
\end{aligned}
$$

Define the discounting process $D_t := \exp\big(-\int_0^t r_u\,du\big)$. Note that D_t is of finite variation. Applying the common Itô-formula therefore yields

$$
\begin{aligned}
d[D_tB(t,T)] \quad = \quad & D_tB(t,T)\Big\{\Big[r_t(0)-r_t(0)-\alpha^*(t,T-t)+\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(t,T-t)]^2\Big]\,dt\\
&-\Phi(t,T-t)\,dX_t\Big\}.
\end{aligned}
\tag{8}
$$

Note that we stress the dependence on $(r_t(0)-r_t)$, which is in this case equal to 0. In the case with credit risk we consider $\bar{r}_t(0)$ instead of the $r_t(0)$ and this term will not vanish. Consequently the discounted bond price is a martingale under $|||\sigma|||_{T^*}<\infty$, iff

$$
\alpha^*(t,T-t)=\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k[\sigma_k^*(t,T-t)]^2\,dt, \qquad \forall T\in[t,t+T^{**}],
$$

which is equivalent to

$$
\int_0^x \alpha_t(v)\,dv=\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k\Big[\int_0^x\sigma_k(t,v)\,du\Big]^2\,dv\,dt. \tag{9}
$$

Taking the partial derivative w.r.t. x, we arrive at (7). Conversely, (9) immediately follows from (7) and hence the conclusion. ■

We are not able to conclude that the market is complete, because there is, to our best knowledge, yet no result on uniqueness available.

2.3 Change of Measure

Up to now we considered the model under a measure Q and obtained conditions, under which Q is a martingale measure. In fact, the observed dynamics take place under the objective measure P, and we have to perform a change of measure to obtain the risk-neutral dynamics, which are necessary for pricing and hedging.

The main tool for doing this is Girsanov's Theorem, which might be found in Da Prato and Zabczyk (1992, Section 10.2.1) in a formulation suitable to our framework. Once the drift condition is obtained, the procedure for obtaining Q is similar throughout all models. Thus, if $(X_s)_{s\in[0,T^*]}$ is a D-Wiener process under P,

$$\tilde{X}_s := X_s + \int_0^s \mu_u \, du$$

is a D-Wiener process under Q, where $dQ := \mathcal{E}(\mu)dP$, $(\mu_s)_{s\in[0,T^*]}$ is a predictable process with values in $H_0 = D^{\frac{1}{2}}(H)$. Denoting $\Phi(s)(\cdot) := <\mu_s, \cdot>_0$, the density has the form

$$\mathcal{E}(\mu) := \exp\Big(- \int_0^{T^*} \Phi(s) \cdot dX_s - \frac{1}{2} \int_0^{T^*} |\mu_s|_0^2 \, ds \Big).$$

We obtain the following

Proposition 2.4. *If a predictable process $(\mu_s)_{s\in[0,T^*]}$ exists, which satisfies $\mathbb{E}(\mathcal{E}(\mu)) = 1$ and*

$$\big[\sigma_t \cdot \mu_t\big](x) = \alpha_t(x) - \sum_{k=1}^{\infty} \lambda_k \, \sigma_k^*(t,x) \, \sigma_k(t,x),$$

for all $t \in [0,T^*]$ and $x \in [0,T^{**}]$, then the measure Q as defined above is an equivalent martingale measure.

Proof. The process $(\tilde{X}_s)_{s\geq 0}$ is a D-Wiener process under Q and the dynamics of the forward rates equal

$$\begin{aligned} r_t &= S(t)r_0 + \int_0^t S(t-u)\alpha_u \, du + \int_0^t S(t-u)\, \sigma_u \cdot d\big(\tilde{X}_u - \int_0^u \mu(v)\, dv\big) \\ &= S(t)r_0 + \int_0^t S(t-u)\big[\alpha_u - \sigma_u \cdot \mu(u)\big] \, du + \int_0^t S(t-u)\, \sigma_u \cdot dX_u. \end{aligned}$$

Therefore, Q is an equivalent martingale measure, if the drift condition is satisfied for the drift $(\alpha - \sigma \cdot \mu)$, that is

$$\alpha_t(x) - \big[\sigma(t) \cdot \mu(t)\big](x) = \sum_{k=1}^{\infty} \lambda_k \, \sigma_k^*(t,x) \, \sigma_k(t,x).$$

■

If credit risk is incorporated into this setting, the change of measure furthermore results in a change of the intensity. This is also true for the ratings model of Section 4, cf. Bielecki and Rutkowski (2002, Sections 4.4 and 7.2).

3 Models with Credit Risk

At this point we incorporate default risk into our model. In the finite-dimensional HJM framework this was first considered by Duffie and Singleton (1999). In the following we extend their results to infinite dimensions.

The following is a basic assumption for the next two sections and summarizes the infinite dimensional setting for the defaultable forward rates. As before, we consider a separable Hilbert space H, whose elements are assumed to be functions $f : [0, T^{**}] \mapsto \mathbb{R}$.

Assumption (A1): Let $\bar{\alpha} : [0, T^*] \times \Omega \mapsto H$ and $\bar{\sigma} : [0, T^*] \times \Omega \mapsto L(H; H)$ be stochastic processes, which are predictable w.r.t. $(\mathcal{F}_t)_{t \geq 0}$, with finite $|||\bar{\sigma}|||_{T^*}$ and $\mathbb{P}(\int_0^{T^*} \bar{\alpha}(s)\, ds < \infty) = 1$. Furthermore, the defaultable forward rate follows

$$\bar{r}_t = S(t)\bar{r}_0 + \int_0^t S(t-u)\, \bar{\alpha}_u\, du + \int_0^t S(t-u)\, \bar{\sigma}_u \cdot d\bar{X}_u,$$

where $(\bar{X}_s)_{s \in [0,T^*]}$ is a $\bar{D}$-Wiener process.

Consider a hazard-rate model, that is, for a given filtration $(\mathcal{G}_t)_{t \geq 0}$ of general market information, the default time τ has an intensity $(\lambda_t)_{t \geq 0}$ which is adapted to $(\mathcal{G}_t)_{t \geq 0}$. Here $\mathcal{G}_t := \sigma(\bar{X}_s, X_s : s \leq t)$ with the usual augmentation and $\mathcal{F}_t := \mathcal{G}_t \vee \sigma(1_{\{\tau \leq s\}} : s \leq t)$. For details see, for example, Lando (1998) or Schmidt (2003).

3.1 Recovery of Market Value

For methods using SDEs the *recovery of market value* model is particularly well suited. In this model the dynamics before a default occurs are modeled analogously to the risk-free case. If a default occurs, say at τ, the bond loses a random fraction q_τ of its pre-default value, where $(q_s)_{s \in [0,T^*]}$ is a predictable process with values in $[0, 1]$. The remaining value is immediately paid to the bond holder, and therefore no longer at risk to the default.

The pre-default dynamics of the bond $\bar{B}$ are modeled by specifying the dynamics of the forward rates, denoted by $\bar{r}_t(x)$. Hence,

$$1_{\{\tau > t\}} \bar{B}(t,T) = 1_{\{\tau > t\}} \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du \Big).$$

If the bond defaults within its lifetime its value at default is assumed to become

$$1_{\{\tau \leq T\}} \bar{B}(\tau, T) = 1_{\{\tau \leq T\}} (1 - q_\tau) \bar{B}(\tau-, T).$$

In contrast to other recovery models the value of the bond immediately before default has some influence on the repayment, which seems reasonable.

The value of $(1 - q_\tau)\bar{B}(\tau-, T)$ is immediately available to the bond owner at default and no more at risk. Therefore, the value of the defaultable bond can be represented by

$$\bar{B}(t,T) = 1_{\{\tau > t\}} \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du \Big) + 1_{\{\tau \leq t\}} \exp\Big(\int_\tau^t r_u\, du \Big)(1 - q_\tau)\bar{B}(\tau-, T).$$

Denote by $\{\bar{e}_k : k \in \mathbb{N}\}$ and $\{\bar{\lambda}_k : k \in \mathbb{N}\}$ the eigenvectors and eigenvalues of $\bar{D}$. Set $\bar{\sigma}_k(u, v) := (\bar{\sigma}_u \cdot \bar{e}_k)(v)$ and $\bar{\sigma}_k^*(u, T) := \int_0^{T-u} \bar{\sigma}_k(u, v)\, dv$. Then we can state the following

Theorem 3.1. *Assume that $\bar{\alpha}_s(x)$ is continuous in s for any $x \in [0, T^{**}]$ and assumption* (A1) *holds. Under the recovery of market value model, discounted bond prices are martingales, iff the following two conditions hold on $\{\tau > t\}$:*

(i) For any $t \in [0, T^*]$, $x \in [0, T^{**}]$

$$\bar{\alpha}_t(x) = \sum_{k=1}^{\infty} \bar{\lambda}_k \, \bar{\sigma}_k^*(t, x) \, \bar{\sigma}_k(t, x). \tag{10}$$

(ii) For any $t \in [0, T^*]$

$$\bar{r}_t(0) = r_t(0) + q_t \lambda_t. \tag{11}$$

Proof. If we denote the discounting factor by $D_t = \exp\big(-\int_0^t r_u \, du\big)$, the discounted gains process $G(t, T) := D_t \bar{B}(t, T)$ equals

$$\begin{aligned} G(t,T) &= 1_{\{\tau>t\}} D_t \bar{B}(t,T) + 1_{\{\tau \le t\}} \exp\Big(-\int_0^\tau r_u \, du\Big)(1 - q_\tau)\bar{B}(\tau-, T) \\ &= 1_{\{\tau>t\}} D_t \bar{B}(t,T) + 1_{\{\tau \le t\}} D_\tau (1 - q_\tau)\bar{B}(\tau-, T) \\ &= 1_{\{\tau>t\}} D_t \bar{B}(t,T) + \int_0^t D_s (1 - q_s) \bar{B}(s-, T) \, d\Lambda_s. \end{aligned}$$

For the last representation we set $\Lambda_s := 1_{\{\tau \le s\}}$. The t-dynamics of $G(t, T)$ becomes

$$dG(t,T) = d\big((1 - \Lambda_t) D_t \bar{B}(t,T)\big) + (1 - q_t) D_t \bar{B}(t-, T) \, d\Lambda_t =: (1) + (2).$$

Taking into account that Λ_t is of finite variation the first summand equals

$$(1) = -d\Lambda_t \, D_t \bar{B}(t,T) + (1 - \Lambda_t) \, d\big(D_t \bar{B}(t,T)\big).$$

The calculation of the discounted bond's dynamics is analogous to the risk-free case. Using expression (8) with $\bar{\lambda}_k, \bar{\beta}_k$, respectively and setting $\tilde{\alpha}(u, T) := \int_0^{T-u} \bar{\alpha}(u, v) \, dv$ as well as $\tilde{\sigma}_k(u, T) := \int_0^{T-u} \bar{\sigma}_k(u, v) \, dv$, we obtain

$$\begin{aligned} d[D_t \bar{B}(t,T)] &= D_t \bar{B}(t,T) \Big\{ \Big[\bar{r}_t(0) - r_t - \tilde{\alpha}(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \bar{\lambda}_k [\tilde{\sigma}_k(t,T)]^2 \Big] \, dt \\ &\qquad - \sum_{k=1}^{\infty} \tilde{\sigma}_k(t,T) \, d\bar{\beta}_k(t) \Big\}. \end{aligned} \tag{12}$$

$\bar{r}_s(\cdot)$ is continuous in s, because $\bar{\alpha}(s, \cdot)$ is continuous by assumption and $\bar{X}_s$ by definition. Therefore, on $\{\tau > t\}$, we have $\bar{B}(t-, T) = \bar{B}(t, T)$.

By definition of $(\lambda_s)_{s \ge 0}$, we have that $\Lambda_s - \int_0^{s \wedge \tau} \lambda_s \, ds$ is a $\mathcal{F}$-martingale, which implies that

$$d\tilde{M}_t := d\Lambda_t - 1_{\{t \le \tau\}} \lambda_t \, dt = d\Lambda_t - (1 - \Lambda_t) \lambda_t \, dt$$

is the differential of an $\mathcal{F}$-martingale, see Bielecki and Rutkowski (2002, Lemma 4.2.1). This leads to

$$\begin{aligned}
dG(t,T) &= \Big(- D_t\bar{B}(t,T) + (1-q_t)D_t\bar{B}(t,T)\Big)\, d\Lambda_t \\
&+ (1-\Lambda_t)\, D_t\bar{B}(t,T)\bigg\{ \Big[\bar{r}_t(0) - r_t - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty} \bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Big]\, dt \\
&\qquad - \sum_{k=1}^{\infty} \tilde{\sigma}_k(t,T)\, d\bar{\beta}_k(t)\bigg\} \\
&= D_t\bar{B}(t,T)\bigg\{ - q_t d\tilde{M}_t - \sum_{k=1}^{\infty} \tilde{\sigma}_k(t,T)\, d\bar{\beta}_k(t) \\
&+ (1-\Lambda_t)\Big[- q_t\lambda_t + \bar{r}_t(0) - r_t - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty} \bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Big]\, dt\bigg\}.
\end{aligned}$$

Hence the dt-term represents the drift. As $(G(t,T))_{t\geq 0}$ is a martingale, iff the drift is zero, it is a martingale, iff for all $t \leq T$

$$1_{\{\tau>t\}}\Big[- q_t\lambda_t + \bar{r}_t(0) - r_t - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty} \bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Big] = 0. \tag{13}$$

Note that this is needed only for $t < \tau$. This is due to the assumption that the recovery value is immediately paid to the bond holder and therefore there is no risky dynamics after default. Consequently, equation (13) is true under (10) and (11).

For the converse, since this equation must hold for any $t \leq \tau \wedge T$ and the $\tilde{\alpha}$ and $\tilde{\sigma}$ -terms equal zero if $T = t$ we obtain (10) and then (11). ■

Remark 3.2. If one prefers a drift condition which does not depend on a particular realization of τ, one can assume that conditions (10) and (11) hold true for any $t \in [0,T^*]$ and $x \in [0,T^{**}]$. Then Q is an EMM as the discounted bond prices are martingales. Of course, Q being an EMM still yields (10) and (11) on $\{\tau > t\}$ only.

3.2 Recovery of Treasury

There are different models of recovery, as discussed, for example, in Schmidt (2003). An alternative to the recovery of market value is the *recovery of treasury* formulation. In this model, the default entails a reduction of the face value by a pre-specified constant. The reduced face value, denoted by δ, is assumed to be no longer at the default risk and is paid to the bond holder at maturity T. This is certainly equivalent to paying $\delta B(\tau,T)$ immediately at default.

Therefore the value of the defaultable bond in this model is

$$\bar{B}(t,T) = 1_{\{\tau>t\}} \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du\Big) + 1_{\{\tau\leq t\}}\delta B(t,T), \qquad 0 \leq t \leq T.$$

Theorem 3.3. *Assume a recovery of treasury model and the riskless bond market to be arbitrage-free. Under assumption* (A1)*, discounted defaultable bond prices are martingales, iff on* $\{\tau > t\}$ *for any* $t \in [0, T^*], T \in [t, t + T^{**}]$

$$\bar{r}_t(0) = r_t + \lambda_t\Big(1 - \delta\Big) \tag{14}$$

and the following drift condition holds

$$\alpha(t, x) = \sum_{k=1}^{\infty} \bar{\lambda}_k \sigma_k^*(t, x) \bar{\sigma}_k(t, x) + \delta \lambda_t \, F(r_t, \bar{r}_t)(x). \tag{15}$$

Here F *is a mapping taking values in* H *defined for suitable* $g, h \in H$ *by*

$$F(g, h)(x) := \frac{\partial}{\partial x} \exp\Big(\int_0^x \big(h(u) - g(u)\big)\, du \Big). \tag{16}$$

In the HJM setup the SDEs on $f(t, T)$ are treated separately for each T. Note that the above drift condition relates the forward rates for different T to each other (via the term with F), so a joint formulation is necessary. This term with F has a simple interpretation, as

$$F(r_t, \bar{r}_t)(x) = \frac{\partial}{\partial x} \frac{B(t, t+x)}{\bar{B}(t, t+x)} = \big(\bar{r}_t(x) - r_t(x)\big) \frac{B(t, t+x)}{\bar{B}(t, t+x)}.$$

Of course, the situation is different, if one uses the decomposition of $\bar{B}$ in $(1 - \delta)$ zero-recovery and δ risk-free bonds. Then, just a condition on the zero-recovery bonds is needed and this is given by Theorem 3.1.

Proof. With the notation of the previous proof, the discounted gains process in this model becomes

$$G(t, T) = D_t \bar{B}(t, T) = (1 - \Lambda_t) D_t \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du \Big) + \Lambda_t \delta D_t B(t, T)$$

with dynamics

$$\begin{aligned} dG(t, T) = {} & (1 - \Lambda_t)\, d\Big[D_t \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du \Big) \Big] \\ & - D_t \exp\Big(- \int_0^{T-t} \bar{r}_t(u)\, du \Big)\, d\Lambda_t + \Lambda_t \delta d\big(D_t B(t, T)\big) + \delta D_t B(t, T) d\Lambda_t. \end{aligned}$$

Taking into account that on $\{\tau > t\}$, $\exp\big(- \int_0^{T-t} \bar{r}_t(u)\, du \big) = \bar{B}(t, T)$, the value of

$d\big(D_t\bar{B}(t,T)\big)$ is given in equation (12). This yields

$$dG(t,T) = (1-\Lambda_t)D_t\bar{B}(t,T)\Bigg\{\Big[\bar{r}_t(0) - r_t(0) - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Big]\,dt$$

$$- \sum_{k=1}^{\infty}\tilde{\sigma}_k(t,T)\,d\bar{\beta}_k(t)\Bigg\} + \Lambda_t\delta d\big(D_tB(t,T)\big)$$

$$+ \Big[-D_t\exp\big(-\int_0^{T-t}\bar{r}_t(u)\,du\big) + \delta D_tB(t,T)\Big]\Big[d\tilde{M}_t + (1-\Lambda_t)\lambda_t\,dt\Big]$$

$$= (1-\Lambda_t)D_t\bar{B}(t,T)\Bigg\{\bar{r}_t(0) - r_t(0) - \lambda_t - \tilde{\alpha}(t,T)$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty}\bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Bigg\}\,dt + \delta D_tB(t,T)\,(1-\Lambda_t)\lambda_t\,dt + d\bar{M}_t,$$

where we denote the sum of all martingale terms by $\bar{M}_t$. Note that $D_tB(t,T)$ is a martingale, as we assumed the riskless bond market to be free of arbitrage. Consequently the drift of $(G(t,T))_{t\geq 0}$ is zero, iff on $\{\tau > t\}$ for all $0 \leq t \leq T$

$$0 = \bar{B}(t,T)\Bigg\{\bar{r}_t(0) - r_t(0) - \lambda_t - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2\Bigg\} + \delta B(t,T)\lambda_t$$

$$\Leftrightarrow\ 0 = \bar{r}_t(0) - r_t - \lambda_t + \delta\lambda_t\frac{B(t,T)}{\bar{B}(t,T)} - \tilde{\alpha}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\bar{\lambda}_k[\tilde{\sigma}_k(t,T)]^2.$$

Setting $t = T$ yields (14). Taking partial derivatives in the above equation we arrive at

$$\alpha(t,x) = \sum_{k=1}^{\infty}\bar{\lambda}_k\sigma_k^*(t,x)\bar{\sigma}_k(t,x) + \delta\lambda_t\frac{B(t,t+x)}{\bar{B}(t,t+x)}\Big(\bar{r}_t(x) - r_t(x)\Big).$$

Note that the last term involves the whole term structure at time t. We write it more concisely as $F(r_t,\bar{r}_t)(x)$ with

$$F(g,h)(x) := \frac{\partial}{\partial x}\exp\Big(\int_0^x h(u) - g(u)\,du\Big) = \Big(h(x) - g(x)\Big)\exp\Big(\int_0^x h(u) - g(u)\,du\Big).$$

Similar arguments as for Theorem 3.1 yield the desired result. ■

3.3 Models with Infinite Factors

In this section we present several applications of infinite factor models, two of them in the interest rate context.

First, assuming that $\bar{\sigma} : [0,T^*] \mapsto L(H;H)$ is deterministic immediately results in a Gaussian model. In analogy to Vargiolu (1998) a historical estimation of the covariance structure using the Karhunen-Loève decomposition is possible. The procedure requires two steps. Firstly, the covariance operator is estimated using historical data. In the second step the first eigenvectors /values are obtained, say up to a number $\bar{N}$. This results in a $\bar{N}$-factor HJM model which is used as an approximation of the infinite factor model.

Let us consider the procedure in further detail. The covariance operator of $\bar{r}(t)$ equals[3]

$$\mathrm{Var}\left(\bar{r}(t)\right) = \int_0^t \left(\bar{\sigma}(s)D^{\frac{1}{2}}\right)\left(\bar{\sigma}(s)D^{\frac{1}{2}}\right)^* ds.$$

Assuming we consider a time interval which is small enough so that variations of $\bar{\sigma}(s)$ do not play a significant role, one could use[4]

$$\frac{D_n(t_n)}{t_n - t_1} := \frac{1}{n}\sum_{i=1}^{n} \bar{r}(t_i) \otimes \bar{r}(t_i)$$

as an estimator of $(\bar{\sigma}(t)D^{\frac{1}{2}})\,(\bar{\sigma}(t)D^{\frac{1}{2}})^*$, where $t = t_n$ (or t_1, respectively $t_{n/2}$).

Focusing on the error of a finite dimensional approximation rather than pre-specifying the dimension naturally involves the Karhunen-Loève decomposition in the following way. The first $\bar{N}$ eigenvalues and eigenvectors of $D_n(t_n)$ can be obtained as follows. Fix $k_0 \in H$ and define

$$k^{n+1} := D_n(t_n) \cdot k^n.$$

Then k^{n+1} itself is an element of H. Vargiolu (1998) shows that

$$k^n \to e_1 \qquad \text{and} \qquad \frac{\| k^{n+1} \|}{\| k^n \|} \to \lambda_1, \qquad \text{as } n \to \infty.$$

Using $D_1 := D_n(t_n) - \lambda_1 e_1 \otimes e_1$, and applying the procedure to D_1 yields e_2 and λ_2 and so on. This is the classical Mises-Geiringer procedure.

The number of eigenvectors, $\bar{N}$, will be chosen such that the desired precision is obtained. Finally, we approximate

$$\left(\bar{\sigma}(t)D^{\frac{1}{2}}\right) \simeq \sum_{k=1}^{\bar{N}} \lambda_k^{\frac{1}{2}} e_k,$$

and this represents the approximating $\bar{n}$-factor classical HJM model.

Second, in the work of Gisdakis (2004) the decomposition in Theorem 2.1 is used for a parsimonious model of so-called shape factors. This model is a special case of ours and seems to be a promising approach for applications in credit risk pricing.

Third, Cont (2005) proposed a model using stochastic processes in Hilbert spaces and showed that certain statistical features of the term structure of interest rates, which were observed in empirical studies, can be reproduced. In particular, the model captures the

[3]With $\left(\bar{\sigma}(s)D^{\frac{1}{2}}\right)^*$ we denote the adjoint operator of $\bar{\sigma}(s)D^{\frac{1}{2}}$.

[4]Here $\otimes$ denotes the tensor product of elements of H. The decomposition of a linear operator D into its eigenvectors e_k and eigenvalues λ_k then can be written in the form $D = \sum_{k=1}^{\infty} \lambda_k\, e_k \otimes e_k$.

imperfect correlation between maturities, mean reversion and the structure of principal components of term structure deformations.

Fourth, Collins-Dufresne and Goldstein (2003) generalized the affine framework to random field models. Their focus is on models which posses quasi-analytic solutions for the characteristic functions. Therefore, closed-form solutions for many types of fixed income derivatives can be derived.

4 Models Using Ratings

As ratings are readily available and a widely used tool in markets subject to credit risk, a model should be capable of using this information. In this section we lay out the framework for a model in infinite dimensions that incorporates different rating classes. We present two alternative recovery structures with recovery levels dependent on the pre-default rating.

The basic assumption of the next two sections describes the behavior of the defaultable forward rates with respect to the current rating.

Assumption (A2). Assume that there are $K-1$ ratings, where 1 denotes the highest rating and $K-1$ the lowest, while K is associated with default. Denoting by $\mathcal{K} = \{1, \ldots, K-1\}$ the set of possible ratings and putting $\bar{\mathcal{K}} = \mathcal{K} \cup \{K\}$, we assume that the rating i forward rate satisfies for $t \in [0, T^*]$

$$r_t^i = S(t) r_0^i + \int_0^t S(t-u) \alpha^i(u)\, du + \int_0^t S(t-u) \sigma^i(u) \cdot dX^i(u),$$

where $(X^i(t))_{t \in [0,T^*]}$ is a D^i-Wiener process. Furthermore, $\alpha^i : [0, T^*] \times \Omega \mapsto H$ and $\sigma^i : [0, T^*] \times \Omega \mapsto L(H; H)$ are stochastic processes, which are predictable and satisfy $\int_0^{T^*} \bar{\alpha}^i(s)\, ds < \infty$ a.s. and $|||\bar{\sigma}^i|||_{T^*} < \infty$, for all $i \in \mathcal{K}$.

To exclude arbitrage we furthermore assume that

$$r_t^{K-1}(x) > \cdots > r_t^1(x) > r_t(x) \qquad \forall x \in [0, T^{**}].$$

This corresponds to the fact that higher rated bonds are more expensive than lower rated ones. If this were not the case the rating of the bond would seem to be wrong. This could happen because of speculative behavior or if the rating is delayed by some other effects and is not modeled here.

The above relation could equivalently be stated by the condition that the inter-rating spreads must be positive, see Acharya, Das, and Sundaram (2002).

The process which describes the current rating of the bond, $(C^1(t))_{t \geq 0}$, takes values in $\bar{\mathcal{K}}$ and is assumed to be a Markov process at this state. Intuitively this means that for this particular bond the "history of ratings" does not influence the price nor default risk of the bond, only the current rating does. We denote by $C^2(t)$ the previous rating before $C^1(t)$.

If there were no changes in the rating up to time t we set $C^2(t) = C^1(t)$. The default τ occurs at the first time, when the state K is reached, $\tau := \inf\{t \geq 0 : C^1(t) = K\}$.

Denote the conditional infinitesimal generator of C^1 given $\mathcal{G}_t$ under the measure Q by

$$\begin{pmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \lambda_{13}(t) & \cdots & \lambda_{1K}(t) \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Each $(\lambda_{ij}(t))_{t\geq 0}$ is a $(\mathcal{G}_t)_{t\geq 0}$-adapted process satisfying the condition

$$\lambda_{ii}(t) = -\sum_{i,j\in\bar{\mathcal{K}},j\neq i} \lambda_{ij}(t), \qquad \text{for all } t \geq 0. \tag{17}$$

We state the following proposition which is proven, for example, in Bielecki and Rutkowski (2002, Prop. 11.3.1).

Proposition 4.1. *For any function $f : \bar{\mathcal{K}} \mapsto \mathbb{R}$ the following process is a martingale:*

$$\tilde{M}(t) \;=\; f(C^1(t)) - \int_0^t \sum_{j=1}^K \lambda_{C^1(u),j} f(j)\, du. \tag{18}$$

For the rating transition to the default state, using equation (11.51) of Bielecki and Rutkowski (2002), we immediately conclude

Proposition 4.2. *The process $(M^i(t))_{t\geq 0}$ is a martingale for any $i \in \mathcal{K}$:*

$$M^i(t) \;=\; 1_{\{C^2(t)=i,C^1(t)=K\}} - \int_0^t \lambda_{iK}(u) 1_{\{C^1(u)=i\}}\, du. \tag{19}$$

4.1 Rating Based Recovery of Market Value

Assume that for $i \in \mathcal{K}$, the rating i recovery rate $(q^i(t))_{t\geq 0}$ to be a nonnegative stochastic process which is predictable for all $i \in \mathcal{K}$. In extension to Section 3.1 we model the defaultable bond with rating transitions for all $t \in [0, T^*]$ and $T \in [t, t + T^{**}]$ by

$$\begin{aligned} \bar{B}(t,T) = 1_{\{C_1(t)\neq K\}} \exp\Big(-\int_0^{T-t} r_t^{C^1(t)}(u)\, du\Big) \\ + 1_{\{C^1(t)=K\}} q_\tau^{C^2(t)} \bar{B}(\tau-, T) \exp\Big(\int_\tau^t r_u\, du\Big). \end{aligned} \tag{20}$$

We call this recovery modeling *rating based recovery of market value.* This may be compared to the case without ratings in Section 3.1. The advantages of the recovery of market value model carry through to this model.

At this point we can compute the defaultable forward rate, the forward rate offered by the bond $\bar{B}(t,T) = \bar{B}(t,t+x)$

$$
\begin{aligned}
\bar{r}_t(x) &= -\frac{\partial}{\partial x}\ln \bar{B}(t,t+x) \\
&= \frac{-1}{\bar{B}(t,t+x)}\frac{\partial}{\partial x}\Big\{ 1_{\{C^1(t)\neq K\}} \exp\Big(-\int_0^x r_t^{C^1(t)}(u)\,du\Big) \\
&+ 1_{\{C^1(t)=K\}} q_\tau^{C^2(t)} \bar{B}(\tau-, t+x) \exp\Big(\int_\tau^t r_u\,du\Big)\Big\}.
\end{aligned}
$$

Computing the derivative yields

$$
\bar{r}_t(x) = 1_{\{C^1(t)\neq K\}} r_t^{C^1(t)}(x) + 1_{\{C^1(t)=K\}} r_\tau^{C^2(t)}(x+t-\tau).
$$

Interestingly, this expression does not depend on the different recovery rates, which is due to the fact that the forward rates describe the behavior of *relative* price changes. So the defaultable forward rate equals the forward rate with respect to the bond's rating. If the bond defaulted, the forward rate curve remains static, as the post-default movement is parallel to the risk-free interest.

For any $i \in \mathcal{K}$ we denote by $\{e_k^i : k \in \mathbb{N}\}$ and $\{\lambda_k^i : k \in \mathbb{N}\}$ the eigenvectors and eigenvalues of D^i and set $\sigma_k^i(u,v) := (\sigma^i(u)\cdot \bar{e}_k^i)(v)$ and $\sigma_k^{i*}(u,T) := \int_0^{T-u} \sigma_k^i(u,v)\,dv$. Set

$$
B^i(t,T) := \exp\Big(-\int_0^{T-t} r_t^i(u)\,du\Big)
$$

and recall the mapping F from (16). Then we can state the following

Theorem 4.3. *Assume that* (A2) *and* (20) *hold under the measure Q. Then discounted defaultable bond prices are martingales under Q iff the following two conditions are satisfied on $\{\tau > t\}$:*

(i) For any $t \in [0,T^*]$, $T \in [t, t+T^{**}]$,

$$
r_t^{C^1(t)}(0) = r_t + \big(1 - q_t^{C^1(t)}\big)\lambda_{C^1(t),K}(t). \tag{21}
$$

(ii) For any $t \in [0,T^*]$, $x \in [0,T^{**}]$,

$$
\begin{aligned}
\alpha^{C^1(t)}(t,x) &= \sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \sigma_k^{C^1(t)*}(t,x)\sigma_k^{C^1(t)}(t,x) \\
&+ \sum_{j=1}^{K-1} \lambda_{C^1(t),j}(t)\, F(r^j, r^{C^1(t)})(x).
\end{aligned} \tag{22}
$$

Under the conditions of the above theorem and, if Q is equivalent to the objective measure P, Q is an equivalent martingale measure and so the market is free of arbitrage.

Proof. Recall that $G(t,T) = D_t \bar{B}(t,T)$. Using equation (20), we determine the discounted gains process

$$\begin{aligned} G(t,T) &= D_t \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} B^i(t,T) \\ &+ D_\tau \, 1_{\{C^1(t)=K\}} \sum_{i=1}^{K-1} 1_{\{C^2(t)=i\}} q^i_\tau \bar{B}(\tau-,T). \end{aligned}$$

Note that the indicators have finite variation, just like $(D_t)_{t\geq 0}$, and therefore Itô's formula yields the dynamics

$$\begin{aligned} dG(t,T) &= \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} d\big(D_t B^i(t,T)\big) + \sum_{i=1}^{K-1} D_t B^i(t,T) d1_{\{C^1(t)=i\}} \\ &+ d\Big(\sum_{i=1}^{K-1} 1_{\{C^1(t)=K, C^2(t)=i\}}\Big) q^i_\tau \bar{B}(\tau-,T) D_\tau. \end{aligned}$$

For the last term,

$$q^i_\tau \bar{B}(\tau-,T) D_\tau d1_{\{C^1(t)=K, C^2(t)=i\}} = q^i_t \, B^i(t,T) D_t d1_{\{C^1(t)=K, C^2(t)=i\}},$$

as the indicator changes only at $t=\tau$. Furthermore, due to the continuity of the forward rates, $\bar{B}(\tau-,T) = B^{C^2(\tau)}(\tau,T)$. Using (18) with $f^i(x) = 1_{\{x=i\}}$ for $i \in \mathcal{K}$, we have

$$\begin{aligned} d1_{\{C^1(t)=i\}} &= d\Big(\tilde{M}^i(t) + \int_0^t \sum_{j=1}^{\mathcal{K}} \lambda_{C^1(u),j} f^i(j)\, du\Big) \\ &= d\Big(\tilde{M}^i(t) + \int_0^t \lambda_{C^1(u),i}\, du\Big) = d\tilde{M}^i(t) + \lambda_{C^1(t),i}\, dt. \end{aligned} \tag{23}$$

Analogously to the default-free case (see 12) the dynamics of each i-rated bond for $t \in [0,T^*]$ and $T \in [t, t+T^{**}]$ can be expressed as

$$\begin{aligned} d\big(D_t B^i(t,T)\big) = D_t B^i(t,T) \Big\{ \Big(r^i_t(0) - r_t(0) - \tilde{\alpha}^i(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \lambda^i_k [\tilde{\sigma}^i_k(t,T)]^2 \Big)\, dt \\ - \sum_{k=1}^{\infty} \tilde{\sigma}^i_k(t,T)\, d\beta^i_k(t) \Big\}, \end{aligned} \tag{24}$$

with $\tilde{\alpha}^i(t,T) = \int_0^{T-t} \alpha^i(t,u)\, du$ and $\tilde{\sigma}^i_k(t,T) := \int_0^{T-t} \sigma^i_k(t,v)\, dv$.

Use (19) to obtain

$$\begin{aligned} dG(t,T) &= \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} D_t B^i(t,T) \Big\{ \Big[r_t^i(0) - r_t(0) - \tilde{\alpha}^i(t,T) \\ &\qquad + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^i [\tilde{\sigma}_k^i(t,T)]^2 \Big] dt - \sum_{k=1}^{\infty} \tilde{\sigma}_k^i(t,T)\, d\beta_k^i(t) \Big\} \\ &+ \sum_{i=1}^{K-1} D_t B^i(t,T) \Big(d\tilde{M}^i(t) + \lambda_{C^1(t),i}(t)\, dt \Big) \\ &+ \sum_{i=1}^{K-1} q_t^i\, B^i(t,T) D_t \Big(dM^i(t) + \lambda_{i,K}(t) 1_{\{C^1(t)=i\}}\, dt \Big) \\ &= \sum_{i=1}^{K-1} D_t B^i(t,T) \Big\{ 1_{\{C^1(t)=i\}} \Big[r_t^i(0) - r_t(0) + q_t^i\, \lambda_{i,K}(t) \\ &\qquad - \tilde{\alpha}^i(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^i [\tilde{\sigma}_k^i(t,T)]^2 \Big] + \lambda_{C^1(t),i}(t) \Big\}\, dt + d\bar{M}_t, \end{aligned}$$

where we denoted the sum of the martingale parts by $\bar{M}_t$. The dt-term yields the drift, and $G(t,T)$ is a martingale, iff the drift is zero. Again, we separate the terms depending on t only from the terms depending also on T. Note that the drift-term is equal to

$$\begin{aligned} 1_{\{C^1(t)\neq K\}} \Big\{ & B^{C^1(t)}(t,T) \Big[r_t^{C^1(t)}(0) - r_t(0) + q_t^{C^1(t)} \lambda_{C^1(t),K}(t) - \tilde{\alpha}^{C^1(t)}(t,T) \\ & + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^{C^1(t)} [\tilde{\sigma}_k^{C^1(t)}(t,T)]^2 \Big] + \sum_{j=1}^{K-1} B^j(t,T) \lambda_{C^1(t),j}(t) \Big\}. \end{aligned}$$

The drift has to be zero. Setting $T = t$ and using (17) we arrive at (21). For the remaining part of the drift we obtain on $\{C^1(t) = c\}$ with $c \neq K$

$$\tilde{\alpha}^c(t,T) = \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^c [\tilde{\sigma}_k^c(t,T)]^2 + \sum_{j=1}^{K-1} \Big[\frac{B^j(t,T)}{B^c(t,T)} - 1 \Big] \lambda_{c,j}(t).$$

We take the partial derivative w.r.t. T and get

$$\alpha^c(t,x) = \sum_{k=1}^{\infty} \lambda_k^c \sigma_k^{c^*}(t,x) \sigma_k^c(t,x) + \sum_{j=1}^{K-1} \lambda_{c,j}(t)\, F(r^j, r^c)(x),$$

such that (26) follows. The converse is easily seen. ■

Remark 4.4. Again, if one prefers a drift condition not depending on a particular realization of $(C^1(t))_{t\geq 0}$, equivalence in Theorem 4.3 cannot be maintained, see Remark 3.2. In this case we require the above equations to be satisfied for any $i \in \mathcal{K}$, which leads to the following conditions for $t \in [0, T^*]$, $T \in [t, t + T^{**}]$

(i) $$r_t^i(0) = r_t(0) + (1 - q_t^i) \lambda_{i,K}(t).$$

(ii) $$\alpha_t^i(x) = \sum_{k=1}^{\infty} \lambda_k^i\, \sigma_k^{i*}(t,x)\, \sigma_k^i(t,x) + \sum_{j=1}^{K-1} \lambda_{i,j} F(r^j, r^i)(x).$$

4.2 Ratings-Based Recovery of Treasury

Another recovery model is the recovery of treasury model proposed by Bielecki and Rutkowski (2000). With the notations of the previous section, the defaultable bond with is modeled for $t \in [0, T^*]$ and $T \in [t, t + T^{**}]$ by

$$\begin{aligned} \bar{B}(t,T) &= 1_{\{C^1(t) \neq K\}} \exp\Big(- \int_0^{T-t} r_t^{C^1(t)}(u)\, du \Big) \\ &+ 1_{\{C^1(t) = K\}} \delta_{C^2(t)} B(t,T). \end{aligned} \tag{25}$$

The rating i-recovery rate δ_i is assumed to be constant. This recovery modeling is referred to as *rating based recovery of treasury.*

Calculating the defaultable forward rate in this model yields

$$\bar{r}_t(x) = 1_{\{C^1(t) \neq K\}} r_t^{C^1(t)}(x) + 1_{\{C^1(t) = K\}} r_t(x).$$

This is similar to the ratings-based recovery of market value setting, and, of course, differences appear just for the behavior after default. In particular, the post-default forward rate equals the default-free rate, but part of the invested money is lost.

Theorem 4.5. *Assume that* (A2) *and* (25) *hold under the measure Q. Recall the mapping F from* (16) *and define the following $\mathbb{R}^K$ valued processes:*

$$\begin{aligned} \boldsymbol{\lambda}_t &:= \big(\lambda_{C^1(t),1}\,,\ \dots\ ,\ \lambda_{C^1(t),K}\big)^\top, \\ \mathbf{F}_t(x) &:= \big(F(r^1, r^{C^1(t)})(x)\,,\ \dots\ ,\ F(r^{K-1}, r^{C^1(t)})(x)\,,\ \delta_{C^1(t)} F(r, r^{C^1(t)})(x)\big)^\top. \end{aligned}$$

Then discounted defaultable bond prices are martingales under Q, iff for any $t \in [0, T^*]$, $T \in [t, t + T^{**}]$ on $\{\tau > t\}$

$$r_t^{C^1(t)}(0) = r_t + \lambda_{C^1(t),K}(t)\big(1 - \delta_{C^1(t)}\big) \tag{26}$$

and the following drift condition holds:

$$\alpha^{C^1(t)}(t,x) = \sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \sigma_k^{C^1(t)*}(t,x)\, \sigma_k^{C^1(t)}(t,x) + \boldsymbol{\lambda}_t \cdot \mathbf{F}_t(x). \tag{27}$$

where the mapping F is defined in (16).

Proof. Using the notation of Theorem 4.3, the dynamics of $\bar{B}(t,T)$ become

$$\begin{aligned} d\bar{B}(t,T) = \sum_{i=1}^{K-1} &\Big[1_{\{C^1(t)=i\}} dB^i(t,T) + B^i(t,T)\, d1_{\{C^1(t)=i\}} \Big] \\ + \sum_{i=1}^{K-1} &\Big[1_{\{C^1(t)=K, C^2(t)=i\}} \delta_i\, dB(t,T) + B(t,T) \delta_i\, d1_{\{C^1(t)=K, C^2(t)=i\}} \Big]. \end{aligned}$$

Note that the differentials of the indicators are -1 or 1 when a jump occurs and zero otherwise. Using (23), we have

$$\sum_{i=1}^{K-1} B^i(t,T) d1_{\{C^1(t)=i\}} = \sum_{i=1}^{K-1} B^i(t,T)(d\tilde{M}^i(t) + \lambda_{C^1(t),i}\, dt).$$

Furthermore, use (19) and (24) to obtain

$$\begin{aligned} d\bar{B}(t,T) = \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} B^i(t,T) \Bigg\{ \Bigg[r_t^i(0) - \tilde{\alpha}^i(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^i [\tilde{\sigma}_k^i(t,T)]^2 \Bigg] dt \\ - \sum_{k=1}^{\infty} \tilde{\sigma}_k^i(t,T)\, d\beta_k^i(t) \Bigg\} \\ + \sum_{i=1}^{K-1} \delta_i B(t,T) \left[\lambda_{i,K}(t) 1_{\{C^1(t)=i\}}\, dt + dM_t^i \right] \\ + \sum_{i=1}^{K-1} 1_{\{C^1(t)=K, C^2(t)=i\}} \delta_i\, dB(t,T) \\ + \sum_{i=1}^{K-1} B^i(t,T) \Big[\lambda_{C^1(t),i}(t) dt + d\tilde{M}_t^i \Big]. \end{aligned}$$

Separating the drift and martingale parts, this leads to

$$\begin{aligned} d\bar{B}(t,T) &= \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} B^i(t,T) \Big[r_t^i(0) - \tilde{\alpha}^i(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^i [\tilde{\sigma}_k^i(t,T)]^2 \Big] dt \\ &+ \sum_{i=1}^{K-1} B^i(t,T) \lambda_{C^1(t),i}(t) dt + 1_{\{C^1(t)=K, C^2(t)=i\}} \delta_i\, dB(t,T) \\ &+ \sum_{i=1}^{K-1} \delta_i B(t,T) \lambda_{i,K}(t) 1_{\{C^1(t)=i\}}\, dt \\ &- \sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}} B^i(t,T) \sum_{k=1}^{\infty} \tilde{\sigma}_k^i(t,T)\, d\beta_k^i(t) \\ &+ \sum_{i=1}^{K-1} B^i(t,T) d\tilde{M}^i(t) + \delta_i B(t,T)\, dM^i(t). \end{aligned}$$

If we denote the discounting factor by D_t the discounted bond price equals

$$
\begin{aligned}
d(D_t\bar{B}(t,T)) &= (-r_t)D_t\bar{B}(t,T)\,dt + D_t d\bar{B}(t,T)\\
&= -r_t D_t\Big[\sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}}B^i(t,T) + \sum_{i=1}^{K-1} 1_{\{C^1(t)=K,C^2(t)=i\}}\delta_i B(t,T)\Big]\,dt\\
&+ D_t\Bigg\{\sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}}B^i(t,T)\Big[r_t^i(0) - \tilde{\alpha}^i(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^i[\tilde{\sigma}_k^i(t,T)]^2\Big]\,dt\\
&+ \sum_{i=1}^{K-1} B^i(t,T)\lambda_{C^1(t),i}(t)dt + 1_{\{C^1(t)=K,C^2(t)=i\}}\delta_i\,dB(t,T)\\
&+ \sum_{i=1}^{K-1}\delta_i B(t,T)\lambda_{i,K}(t)1_{\{C^1(t)=i\}}\,dt\Bigg\} + d\tilde{\tilde{M}}_t,
\end{aligned}
$$

where we added the martingale parts up to $d\tilde{\tilde{M}}_t$. As the discounted risk-free bond is a martingale by assumption, we conclude that the $1_{\{C^1(t)=K,C^2(t)=i\}}$-terms sum up to a martingale. We have

$$
\begin{aligned}
d(D_t\bar{B}(t,T)) = D_t\Bigg\{&\sum_{i=1}^{K-1} 1_{\{C^1(t)=i\}}B^i(t,T)\Big[-r_t(0) + r_t^i(0) - \tilde{\alpha}^i(t,T)\\
&+ \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^i[\tilde{\sigma}_k^i(t,T)]^2\Big]\,dt + \sum_{i=1}^{K-1} B^i(t,T)\lambda_{C^1(t),i}(t)dt\\
&+ \sum_{i=1}^{K-1}\delta_i B(t,T)\lambda_{i,K}(t)1_{\{C^1(t)=i\}}\,dt\Bigg\} + d\bar{M}_t,
\end{aligned}
$$

denoting the martingale part by $\bar{M}_t$.

Therefore, $D_t\bar{B}(t,T)$ is a martingale, iff on $\{C^1(t) \neq K\}$

$$
\begin{aligned}
0 = {}& B^{C^1(t)}(t,T)\Big[r_t^{C^1(t)}(0) - r_t(0) - \tilde{\alpha}^{C^1(t)}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{C^1(t)}[\tilde{\sigma}_k^{C^1(t)}(t,T)]^2\Big]\\
&+ \delta_{C^1(t)}B(t,T)\lambda_{C^1(t),K} + \sum_{j=1}^{K-1} B^j(t,T)\lambda_{C^1(t),j}(t).
\end{aligned} \tag{28}
$$

Again, we separate the above equation by setting $T = t$. This leads to condition (26). The remaining part yields

$$
\begin{aligned}
\tilde{\alpha}^{C^1(t)}(t,T) = {}& \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{C^1(t)}[\tilde{\sigma}_k^{C^1(t)}(t,T)]^2 + \delta_{C^1(t)}\Big[\frac{B(t,T)}{B^{C^1(t)}(t,T)} - 1\Big]\lambda_{C^1(t),K}\\
&+ \sum_{j=1}^{K-1}\Big[\frac{B^j(t,T)}{B^{C^1(t)}(t,T)} - 1\Big]\lambda_{C^1(t),j}(t).
\end{aligned}
$$

Taking the partial derivative w.r.t. T we finally arrive at

$$\begin{aligned}\alpha^{C^1(t)}(t,x) &= \sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \sigma_k^{C^1(t)*}(t,x)\, \sigma_k^{C^1(t)}(t,x) + \delta_{C^1(t)} \lambda_{C^1(t),K} F(r_t, r^{C^1(t)})(x) \\ &+ \sum_{j=1}^{K-1} \lambda_{C^1(t),j}(t) F(r^j, r^{C^1(t)})(x).\end{aligned}$$

Using the short notation with $\boldsymbol{\lambda}$ and $\mathbf{F}$ we arrive at (27). As in the previous proofs the converse follows easily. ■

Remark 4.6. Similar to Remark 3.2, we obtain the following condition, which does not depend on C^1 but implies an arbitrage-free market together with $r_t^i(x) = r_t + \lambda_{i,K}(t)(1-\delta_i)$:

$$\alpha^i(t,x) = \sum_{k=1}^{\infty} \lambda_k^i \sigma_k^{i*}(t,x)\, \sigma_k^i(t,x) + \boldsymbol{\lambda}_t^i \cdot \mathbf{F}_t^i(x),$$

where $\boldsymbol{\lambda}^i$ and $\mathbf{F}^i$ are obtained from $\boldsymbol{\lambda}$ and $\mathbf{F}$ by simply replacing $C^1(t)$ with i.

It seems natural that condition (11) extends to the rating model. Equation (26) represents the no-arbitrage relationship between the interest offered by a bond rated i, the likelihood of rating changes and recovery while the inter-relationship of risky and default-free term structures enters into (27).

An equivalent but more concise version of (28) is obtained on $\{C^1(t) = i\}$ setting

$$a^i(t,T) := -r_t(0) + r_t^i(0) - \tilde{\alpha}^i(t,T) + \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^i [\tilde{\sigma}_k^i(t,T)]^2.$$

Recall that $i \in \mathcal{K}$. Substituting $\lambda_{ii}(t) = -\sum_{j=1, j\neq i}^{K} \lambda_{ij}(t)$, we obtain

$$\begin{aligned} 0 &= B^i(t,T) a^i(t,T) + \delta_i B(t,T) \lambda_{i,K}(t) \\ &+ \sum_{j=1, j\neq i}^{K-1} B^j(t,T) \lambda_{i,j}(t) - B^i(t,T) \sum_{j=1, j\neq i}^{K} \lambda_{i,j}(t) \end{aligned}$$

and hence have the equivalent representation of (28),

$$\begin{aligned} 0 &= B^i(t,T) a^i(t,T) + \sum_{j=1, j\neq i}^{K-1} \big(B^j(t,T) - B^i(t,T)\big) \lambda_{i,j}(t) \\ &+ \big(\delta_i B(t,T) - B^i(t,T)\big) \lambda_{i,K}(t). \end{aligned}$$

The first part of this expression relates to the drift of the bond itself, while the other parts refer to the possible changes into a different rating class. A change of the rating immediately entails a change of the bond's price. These are multiplied with the rate, that such a change may happen, see also Prop. 4.1. Note, however, that this is not a drift condition. It is rather a general condition which identifies an arbitrage-free model.

4.3 Pricing

Pricing in credit risky models is usually done via calculation of the expectation of the discounted contingent claim, see for example Lando (1998), Duffie and Singleton (1999) or Bielecki and Rutkowski (2002). A series of examples where we were able to obtain closed form solutions in a Gaussian random field setup may be found in Schmidt (2004).

5 Conclusion

The paper is a starting point for modeling infinite factor models for credit risk with stochastic differential equations on Hilbert spaces. Several models were proposed. A first class of models reflects different recovery scenarios, while a second class of models also incorporates rating migration based on a Markov chain. These models are shown to be free of arbitrage opportunities under certain drift conditions.

The main goal is to provide a substantially new framework for credit risk models which allow for an infinite dimensional factor structure. These models have been successfully applied to interest rate models and seem promising for the credit risk framework.

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