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# Credit risk with infinite dimensional Lévy processes

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**Summary:** The forward rate curve is assumed to follow a stochastic differential equation w.r.t. a Lévy process with infinite dimensions. Conditions under which the market is free of arbitrage are provided for both the interest rate case and for the case of credit risk with ratings. A simulation shows that typical movements of the yield curve are well captured by the model.

## 1 Introduction

A zero-coupon bond is an asset which pays one unit of money at a prespecified maturity date $T$. If the pay-off takes place with probability one, we say that the bond is default-free. In contrast, we call this asset a defaultable bond, if there is some positive probability of getting less than one unit of money. Zero-coupon bonds issued by corporates are the most typical examples for defaultable bonds. The probability of default depends on economic and firm-specific variables. This probability is usually reflected by a rating class which is assigned by commercial rating agencies.

Various approaches for modeling defaultable bonds have been studied in literature and they can roughly be divided into two main groups: the firm-value models and the intensity-based models. Recent literature stresses the advantages of intensity-based models which build on interest rate models, see e.g. the survey [23] or one of the books [17], [1].

Up to now, mainly models based on real-valued Brownian motions were considered. This type of models does not reflect real-world data sufficiently, since the implied normal distribution has exponentially decaying tails in contrast to the semi-heavy tails of real-world data. Several studies on this topic can be found—we refer to [6] as one example. A number of credit risk models which provide more flexibility have been developed, see e.g. [5], [24, Section 2.7], or [10]. Whereas in [7] a model based on the very flexible class of Lévy processes is introduced.

Newer interest rate models are based on Hilbert-space valued processes in order to model forward rate curves directly. One way of applying Hilbert-space valued Brownian

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motions to credit risk can be found in [22]. Although the credit derivatives data is still scarce, the market is believed to grow rapidly, see, for example, [11]. So calibration procedures will be of high importance. For the special case of Gaussian random fields explicit pricing formulas for various credit derivatives were derived in [21]. Also two different calibration methodologies were proposed, each one serving different needs. The first one is believed to avoid frequent re-calibration while the second one adjusts for the scarcity of data by incorporating historical information.

This paper provides an approach which combines the advantages of the approaches in [7] and [22] by presenting a model that is based on Lévy processes in Hilbert spaces. There are several reasons for using a infinite dimensional model. While typically it is argued that the characteristic movements of yield curves are covered by three factors, for pricing of derivatives and hence calibration it is very important to cover as many factors as possible. This is because derivatives are non-linear objects and a factor with a small explanatory power for the variance might be highly significant in pricing, as argued in [4]. If times of a crisis are included in an historical analysis of interest rates, it has also been observed that ten factors were needed to explain only 90 % of the variability, compare [20, Section 13.2.5]. As credit risky markets are much more volatile than interest rate markets, it seems to be a good motivation to study high-dimensional models.

If a model with ratings is considered it will turn out that the drift condition interrelates the forward rates in quite a complicated way. More precisely, the risk-neutral drift depends on the whole forward rate curve. Therefore SDEs for the forward rates can not be viewed as a set of SDEs indexed by $T$ as pioneered in [9]. Rather a functional formulation is necessary to give the risk-neutral dynamics the appropriate meaning, compare Corollary 4.6.

The organisation of the paper is as follows. In the next section we revisit some basic facts from Hilbert-space valued Lévy processes. In Section 3 we introduce the underlying default-free interest rate models based on Hilbert-space valued Lévy processes, and Section 4 describes the model for the defaultable zero-coupon bond. Finally, we give some simulations and the conclusion.

# 2 Preliminaries

Consider a separable Hilbert space $H$ with inner product $\langle\cdot,\cdot\rangle$ and denote the norm on $H$ by $\|\cdot\|$. $H$ is a subspace of $L^2(\mathbb{R}^+)$. Elements of $H$ will represent the whole yield curve at a certain time. There are several ways to choose $H$ depending on the required smoothness, see [19] or [8].

Denote by $L(H)$ the Banach space of linear operators on $H$. If $h \in H$ and $\Phi \in L(H)$ we write $\Phi \cdot h$ for $\Phi(h)$.

Furthermore, we consider a finite time horizon $T^*$ and a Lévy process $(L_s)_{s\in[0,T^*]}$ taking values in $H$. The value of the Lévy process at a fixed time $s$ is an element of $H$ and thus may be stated either as $L(s)$ or $L(s,t) := L(s)(t)$. This is why we also speak of a Lévy random field.

The Lévy random field admits the characteristic function

$$\phi_{L_1}(z) = \exp\Big(\mathrm{i}\langle b_L, z\rangle - \frac{1}{2}\langle D_L \cdot z, z\rangle + \int_H \Big[\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle 1_B(x)\Big]\, F(\mathrm{d}x)\Big),$$

where $b_L \in H$, $D_L$ is an element of $L(H)$, the covariance operator of the continuous part of $(L_t)$, and $B := \{h \in H : \|h\| < 1\}$. The Lévy measure $F$ is a measure on $H$ with $\int_H (\|x\|^2 \wedge 1)\, F(\mathrm{d}x) < \infty$ and $F(\{0\}) = 0$.

The following decomposition of a Lévy process $(L(t))_{t\geq 0}$ with values in $H$ will be useful for interchanging integration w.r.t. forward rates.

**Proposition 2.1** *Suppose $(L_t)_{t\in[0,T^*]}$ is a Lévy process with values in $H$ and $\mathbb{E}\|L_t\|^2 < \infty$ for all $t \in [0, T^*]$. If $\{e_k : k \in \mathbb{N}\}$ is an arbitrary orthonormal basis of $H$, we have the following decomposition*

$$L_t(u) = \sum_{k=1}^{\infty} \langle L(t), e_k\rangle e_k(u), \tag{2.1}$$

*where the series converges in $L^2$. Furthermore, for any $k \in \mathbb{N}$ the process $(l_k(t))_{t\in[0,T^*]}$ defined by $l_k(t) := \langle L_t, e_k\rangle$ is a real-valued Lévy process.*

**Proof:** First, consider fixed $t \in [0, T^*]$. Then for any $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=1}^{m} e_k\langle L_t, e_k\rangle\right\|^2 &= \sum_{k,j=1}^{m} \mathbb{E}\Big(\langle L_t, e_k\rangle\langle L_t, e_j\rangle\Big)\langle e_k, e_j\rangle \\ &= \sum_{k=1}^{m} \mathbb{E}\Big(\langle L_t, e_k\rangle^2\Big) \\ &\leq \mathbb{E}\,\|L_t\|^2.\end{aligned}$$

The last inequality follows from the Bessel inequality, see e.g. [27], and so the series converges in $L^2$.

Second, as $l_k(t_2) - l_k(t_1) = \langle L(t_2) - L(t_1), e_k\rangle$ the processes $l_k$ have stationary and independent increments, because $L$ already has stationary and independent increments. □

In the following, we therefore always assume that the considered Lévy process $(L_t)_{t\in[0,T^*]}$ has second moments, in the sense, that $\mathbb{E}\|L_t\|^2 < \infty$ for all $t \in [0, T^*]$.

We refer to [18] for more details on stochastic integration w.r.t. Hilbert-valued Lévy processes. However, a somewhat different approach using a series representation like (2.1) is taken in [26]. First, we introduce a suitable class of integrands.

**Definition 2.2** *Consider a Lévy process $(L_t)_{t\in[0,T^*]}$ with values in $H$. With the decomposition from Proposition 2.1 we call $\mathcal{L}_{T^*}(H)$ the space of all predictable processes $(\sigma_t)_{t\in[0,T^*]}$ taking values in $L(H)$ such that the process $(\sigma_t)_{t\in[0,T^*]}$ is locally bounded.*[1]

Then, for $\sigma \in \mathcal{L}_T$ the process $(\int_0^t \sigma(u) \cdot \mathrm{d}L_u)_{t\in[0,T^*]}$ is an $H$-valued semi-martingale and it is a local martingale, if $(L_t)$ is.

**Lemma 2.3** *Consider a $H$-valued Lévy -process with second moments. Then, for $\sigma \in \mathcal{L}_T$ its stochastic integral w.r.t. $(L_t)_{t\geq 0}$ is a local martingale.*

**Proof:** As $(\sigma)$ is locally bounded, there exists a sequence of stopping times $(T_n)_{n\geq 1}$, converging to infinity, such that $(\sigma_{t\wedge T_n})$ is bounded. Considering

$$I_t^n = \int_0^{t\wedge T_n} \sigma(u) \cdot \mathrm{d}L_u$$

this is a sequence of martingales, and the claim follows. □

From [15] we obtain the following Itô-formula for Lévy random fields.

**Theorem 2.4** *Let $(L_t)_{t\in[0,T^*]}$ be a Lévy process with values in the Hilbert space $H$ and $(\sigma(t))_{t\in[0,T^*]} \in \mathcal{L}_{T^*}(H)$. Set $X_t := \int_0^t \sigma(u) \cdot \mathrm{d}L_u$ for all $t \in [0, T^*]$ and denote by $\lambda_k$ and $e_k$, $k \in \mathbb{N}$ the eigenvalues and eigenvectors, respectively, of $D_L$. For an open subset $A \subset H$ and a twice differentiable function $F : A \to H$ with uniformly continuous second derivative on bounded subsets of $H$ it holds, that*

$$\begin{aligned} F(X_t) = F(X_0) &+ \int_0^t DF(X_{u-}) \cdot \mathrm{d}X_u \\ &+ \frac{1}{2}\int_0^t \sum_{k=1}^{\infty} \lambda_k D^2 F(X_{u-}) \cdot (\sigma(u) \cdot e_k, \sigma(u) \cdot e_k)\, \mathrm{d}u \\ &+ \sum_{s\leq t} \big[\Delta F(X_s) - \big(DF(X_{s-}) \cdot \Delta X_s\big)\big]. \end{aligned} \tag{2.2}$$

Note that the second derivative, $D^2F(\cdot)$ is a bilinear mapping, and we therefore write $D^2F(\cdot) \cdot (a, b)$ for this derivative evaluated at $a$ and $b$.

# 3 The default-free case

We consider a filtered probability space $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{0\leq t\leq T^*})$, and derive conditions under which the measure $Q$ is a martingale measure. If $Q$ is furthermore equivalent to the objective measure $P$, it is an equivalent martingale measure and the market is free of arbitrage, see [2].

[1] Here we consider the space $L(H)$ endowed with the operator topology, which makes it a Banach space.

The parameterization of [3] turns out to be useful in the considered context. Denote by

$$r_t(x) := f(t, t + x)$$

the instantaneous forward rate at time $t$ with time-to-maturity $x$. The stochastic process $(r_t)_{t\geq 0}$ is assumed to take values in the Hilbert space $H$ which is a subspace of $L^2(\mathbb{R}^+)$. The bond prices depend on the forward rates in the following way:

$$B(t, T) = \exp\left(-\int_0^{T-t} r_t(v)\,\mathrm{d}v\right).$$

It is important to state the dependence on $T$ instead of $x$ for the bond prices. If we consider for a moment a bond with time-to-maturity $x$ denoted by $B_t(x)$ then, for fixed $x$, $(B_t(x))_{t\geq 0}$ refers to a price process which does not exist in this form in the market. This is because $B_{t_1}(x)$ and $B_{t_2}(x)$ are different bonds (for $t_1 \neq t_2$).

Let $\{S_t : t \in \mathbb{R}_+\}$ denote the semigroup of right shifts, defined by $S_t g(\cdot) = g(\cdot + t)$, for any function $g : \mathbb{R}_+ \mapsto \mathbb{R}$. Assume that $(A(t))_{t\in[0,T^*]}$ is a predictable stochastic process with values in the Hilbert space $H$ and $(\sigma(t))_{t\in[0,T^*]}$ is an element of $\mathcal{L}_{T^*}(H)$, see Definition 2.2. The shift operator acts as follows: by $S_{t-u}A(u)$ we mean the element in $H$ obtained by shifting $A(u)$ for fixed $u$, i.e. $v \mapsto A(u)(v + t - u)$.

The above introduced parametrization suggests the following dynamics, compare [3] or [8],

$$r_t = S_t r_0 + \int_0^t S_{t-u}A(u)\,\mathrm{d}u + \int_0^t S_{t-u}\sigma(u)\cdot \mathrm{d}L_u, \tag{3.1}$$

where furthermore, $(L_t)_{t\in[0,T^*]}$ is a Lévy process with values in $H$. We assume that $(L_t)$ is a martingale, more precisely that $(L_t)$ has no drift and the jump part is already compensated. If $\mathbb{E} \parallel L_1 \parallel < \infty$, then $(L_t)$ can be stated in the following form:

$$L_t = W_t + \int_0^t \int_H x \left(\mu^L - \nu^L\right)(\mathrm{d}s, \mathrm{d}x),$$

where $W$ is a $D^L$-Wiener process on $H$ and $\mu^L$ is the random measure of jumps with $Q$-compensator $\nu^L(\mathrm{d}s, \mathrm{d}x) = \mathrm{d}s\, F(\mathrm{d}x)$. That is, for any Borel set $\mathcal{T}$ of $\mathbb{R}^+$ and any Borel set of $H$, $\mu^L$ denotes the number of jumps in the time interval $\mathcal{T}$ which have sizes in $\Lambda$,

$$\mu^L(\mathcal{T}, \Lambda) = \sum_{s\in\mathcal{T}} 1_\Lambda(\Delta L_s).$$

We will need exponential moments for $(L)$, which are guaranteed by the following condition.

$$\int \mathbb{1}_{\{\|x\|>1\}} \mathrm{e}^{\langle c,x\rangle}\nu(\mathrm{d}x) < \infty, \quad \forall c \in H. \tag{3.2}$$

It has been shown in [13] that under some mild conditions the HJM-drift condition already ensures the existence of exponential moments.

To shorten the notation, we define $A^*(u, T) := \int_0^{T-u} A(u, v)\,\mathrm{d}v$ and also $\sigma_k^*(u, T) := \int_0^{T-u} [\sigma(u) \cdot e_k](v)\,\mathrm{d}v$.

Furthermore, we assume in the following that $\int_0^{T^*} A(u)\,\mathrm{d}u \in H$ almost surely and $\sigma \in \mathcal{L}_{T^*}(H)$. The processes $(\sigma_k^*(t, T))_{t\in[0,T^*]}$ will appear in the dynamics of $B$, such they need to be locally bounded for each $T \in [0, T^*]$. Note that this follows from $\sigma \in \mathcal{L}_{T^*}(H)$.

Define the discounting process $\beta_t := \exp(-\int_0^t r_u(0)\,\mathrm{d}u)$, such that in the considered market the numeraire is given by $\beta^{-1}(t)$. By the fundamental theorem of asset pricing, the market is free of arbitrage if there exists an equivalent measure under which discounted assets are local martingales.

In this paper we use the so-called martingale approach, i.e. we directly consider a measure $Q$ which is equivalent to the objective probability measure and state conditions under which $Q$ is also a martingale measure. So the following theorem allows to classify the martingale measures.

**Theorem 3.1** *All discounted bond prices are local martingales, iff for all $(t, T)$ with $0 \le t \le T \le T^*$ the following condition holds $Q$-a.s.:*

$$\begin{aligned}0 = &-A^*(t, T) + \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \left[\sigma_k^*(t, T)\right]^2 \\ &+ \int_H \left[\exp\left(\int_0^{T-t} [\sigma(t)\cdot x](v)\,\mathrm{d}v\right) - 1 - \int_0^{T-t} [\sigma(t)\cdot x](v)\,\mathrm{d}v\right] F(\mathrm{d}x). \qquad (3.3)\end{aligned}$$

**Proof:** Denote

$$y(t, T) := -\int_0^{T-t} r_t(v)\,\mathrm{d}v.$$

First, we want to derive the dynamics of the process $(y(t))_{t\ge 0}$. Using the dynamics of $(r_t)$, (3.1), we obtain

$$\begin{aligned}y(t, T) = -\int_0^{T-t} \Bigg[ &r_0(v + t) + \int_0^t A(u, v + t - u)\,\mathrm{d}u \\ &+ \left[\int_0^t S_{t-u}\sigma(u)\cdot \mathrm{d}L_u\right](v)\Bigg]\mathrm{d}v\,. \qquad (3.4)\end{aligned}$$

Note that

$$\begin{aligned}-\int_0^{T-t} r_0(v + t)\,\mathrm{d}v &= y(0, T) + \int_0^T r_0(v)\,\mathrm{d}v - \int_0^{T-t} r_0(v + t)\,\mathrm{d}v \\ &= y(0, T) + \int_0^t r_0(v)\,\mathrm{d}v. \qquad (3.5)\end{aligned}$$

As we need to show that discounted bond prices are martingales under $Q$, it is convenient when the short rate explicitly appears in the dynamics of $y$. From (3.1) we deduce

$$\int_0^t r_v(0)\,\mathrm{d}v = \int_0^t r_0(v)\,\mathrm{d}v + \int_0^t \int_0^v A(u, v-u)\,\mathrm{d}u\,\mathrm{d}v + \int_0^t \left[\int_0^v S_{v-u}\sigma(u)\cdot \mathrm{d}L_u\right](0)\,\mathrm{d}v. \tag{3.6}$$

Inserting (3.5) and (3.6) into (3.4) yields

$$\begin{aligned} y(t,T) = {} & y(0,T) + \int_0^t r_v(0)\,\mathrm{d}v \\ & - \int_0^t \int_0^v A(u, v-u)\,\mathrm{d}u\,\mathrm{d}v - \int_t^T \int_0^t A(u, v-u)\,\mathrm{d}u\,\mathrm{d}v \\ & - \int_0^{T-t} \left[\int_0^t S_{t-u}\sigma(u)\cdot \mathrm{d}L_u\right](v)\,\mathrm{d}v - \int_0^t \left[\int_0^v S_{v-u}\sigma(u)\cdot \mathrm{d}L_u\right](0)\,\mathrm{d}v. \end{aligned}$$

Using Fubini's theorem we can interchange the order of the $A$-integrals, which leads to the following expression for the sum of the $A$-integrals

$$-\int_0^t \int_u^T A(u, v-u)\,\mathrm{d}v\,\mathrm{d}u. \tag{3.7}$$

Interchanging the order in the last two terms requires using the eigenvalue expansion of $L$, which yields

$$\begin{aligned} \left[\int_0^t S_{t-u}\sigma(u)\cdot \mathrm{d}L_u\right](v) &= \sum_{k=1}^{\infty} \int_0^t [\sigma(u)\cdot e_k](v+t-u)\,\mathrm{d}l_k(u) \\ &= \sum_{k=1}^{\infty} \int_0^t \sigma_k(u, v+t-u)\,\mathrm{d}l_k(u), \end{aligned}$$

where we set $\sigma_k(u,v) := [\sigma(u)\cdot e_k](v)$. This holds, because $\sigma \in \mathcal{L}_{T^*}(H)$.

Furthermore, this condition allows to apply the stochastic Fubini theorem, and we get the following

$$\begin{aligned} & \int_0^{T-t} \left[\int_0^t S_{t-u}\sigma(u)\cdot \mathrm{d}L_u\right](v)\,\mathrm{d}v \\ &= \int_0^{T-t} \sum_{k=1}^{\infty} \int_0^t \sigma_k(u, v+t-u)\,\mathrm{d}l_k(u)\,\mathrm{d}v \\ &= \sum_{k=1}^{\infty} \int_0^t \int_t^T \sigma_k(u, v-u)\,\mathrm{d}v\,\mathrm{d}l_k(u) \end{aligned}$$

as well as

$$\int_0^t \left[\int_0^v S_{v-u}\sigma(u)\cdot \mathrm{d}L_u\right](0)\,\mathrm{d}v = \sum_{k=1}^{\infty} \int_0^t \int_u^t \sigma_k(u, v-u)\,\mathrm{d}v\,\mathrm{d}l_k(u).$$

These expressions can be summed up and we obtain a considerably easier formulation of $y$,

$$y(t,T) = y(0,T) + \int_0^t r_v(0)\,\mathrm{d}v - \int_0^t \int_u^T A(u, v-u)\,\mathrm{d}v\,\mathrm{d}u \\ - \sum_{k=1}^{\infty} \int_0^t \int_u^T \sigma_k(u, v-u)\,\mathrm{d}v\,\mathrm{d}l_k(u).$$

We use the abbreviations $A^*(u,T)$ and $\sigma_k^*(u,T)$ defined just before the theorem and obtain the dynamics of $y$,

$$y(t,T) = y(0,T) + \int_0^t r_v(0)\,\mathrm{d}v - \int_0^t A^*(u,T)\,\mathrm{d}u \\ - \sum_{k=1}^{\infty} \int_0^t \sigma_k^*(u,T)\,\mathrm{d}l_k(u). \tag{3.8}$$

Note that the existence of these integrals is ensured by the assumption that $\sigma_k^*$ are locally bounded. To apply the Itô-formula (2.2) we need a more functional analytic representation of the above equation. Therefore, define $\Phi : [0, T^*] \times \Omega \to L(H, H)$ by

$$[\Phi(u) \cdot f](\cdot) := \int_u^{\cdot} [\sigma(u) \cdot f](v-u)\,\mathrm{d}v.$$

Then

$$\begin{aligned} \int_0^t [\Phi(u) \cdot \mathrm{d}X(u)](T) &= \sum_k \int_0^t [\Phi(u) \cdot e_k](T)\,\mathrm{d}l_k(u) \\ &= \sum_k \int_0^t \int_u^T [\sigma(u) \cdot e_k](v-u)\,\mathrm{d}v\,\mathrm{d}l_k(u) \\ &= \sum_k \int_0^t \sigma_k^*(u,T)\,\mathrm{d}l_k(u). \end{aligned}$$

Setting $m(u,\cdot) := r_u(0) - A^*(u,\cdot)$ we obtain

$$y(t) = y(0) + \int_0^t m(u)\,\mathrm{d}u - \int_0^t \Phi(u) \cdot \mathrm{d}L_u. \tag{3.9}$$

We define

$$F : H \to H, \quad g(\cdot) \mapsto \exp(g(\cdot)),$$

where $\exp(g(\cdot))$ is the function $h$, s.t. $h(x) = \exp(g(x)), \forall x \in \mathbb{R}^+$. Then we have $B(t,\cdot) = F(y(t,\cdot))$. For two real-valued functions $g, h$ we denote $(g \times h)(\cdot) := g(\cdot)h(\cdot)$. Then it is easily seen, that $DF(x) = F(x) \times \mathrm{id}$ and $D^2F(x) = F(x) \times \mathrm{id} \times \mathrm{id}$. Thus, applying Itô's formula yields

$$
\begin{aligned}
B(t) \;=\; & B(0) + \int_0^T DF(y(u-)) \cdot [m(u)\,\mathrm{d}u - \Phi(u) \cdot \mathrm{d}L(u)] \\
& + \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} \lambda_k \, D^2 F(y(u-)) \cdot (\Phi(u) \cdot e_k, \Phi(u) \cdot e_k)\,\mathrm{d}u \\
& + \sum_{s \le t} \big[ F(y_s) - F(y_{s-}) - DF(y(s-)) \cdot \Phi(s) \cdot \Delta L_s \big],
\end{aligned}
$$

and we obtain inserting the derivatives of $F$

$$
\begin{aligned}
B(t) \;=\; & B(0) + \int_0^t B(u-) \times m(u)\,\mathrm{d}u - \int_0^t B(u-) \times \Phi(u) \cdot \mathrm{d}L_u \\
& + \frac{1}{2} \int_0^t \sum_k \lambda_k B(u-) \times (\Phi(u) \cdot e_k) \times (\Phi(u) \cdot e_k)\,\mathrm{d}u \\
& + \sum_{s \le t} \big[ \Delta B(s) - B(s-) \times [\Phi(s) \cdot \Delta L_s] \big].
\end{aligned}
$$

Evaluating $B(t, \cdot)$ at maturity $T$ reveals

$$
\begin{aligned}
B(t, T) \;=\; & B(0, T) + \int_0^t B(u-, T)[r_u(0) - A^*(u, T)]\,\mathrm{d}u \\
& - \sum_k \int_0^t B(u-, T) \sigma_k^*(u, T)\,\mathrm{d}l_k(u) \\
& + \frac{1}{2} \int_0^t B(u-, T) \sum_k \lambda_k [\sigma_k^*(u, T)]^2\,\mathrm{d}u \\
& + \sum_{s \le t} \big[ B(s, T) - B(s-, T) - B(s-, T)\,[\Phi(s) \cdot \Delta L(s)](T) \big].
\end{aligned}
$$

Now we have to find the martingale parts in this expression. Of course, if we consider the discounted bond price the term with $r_u(0)$ vanishes in the above dynamics.

Because of $B(s) = F(y(s))$ we obtain $B(s)/B(s-) = \exp(\Phi(s)\Delta L_s)$ and thus

$$
\begin{aligned}
\Delta B(s, T) \;=\; & B(s-, T) \left[ \frac{B(s, T)}{B(s-, T)} - 1 \right] \\
=\; & B(s-, T) \Big( \exp \Big( [\Phi(s) \cdot \Delta L_s](T) \Big) - 1 \Big).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
& \sum_{s \le t} \Delta B(s, T) - B(s-, T)[\Phi(s) \cdot \Delta L(s)](T) \\
& \quad = \sum_{s \le t} B(s-, T) \Big( \exp \Big( [\Phi(s) \cdot \Delta L_s](T) \Big) - 1 - [\Phi(s) \cdot \Delta L_s](T) \Big).
\end{aligned}
$$

This term can also be expressed as

$$\int_0^t \int_H B(s-,T)\Big(\exp\Big([\Phi(s)\cdot x](T)\Big) - 1 - [\Phi(s)\cdot x](T)\Big)\,\mu^L(\mathrm{d}s,\mathrm{d}x).$$

Note that $(\beta_t)_{t\geq 0}$ is real valued and of finite variation. Then, by Itô's formula,

$$\mathrm{d}[\beta_t B(t,T)] = (-r_{t-}(0))\beta_{t-}B(t-,T)\,\mathrm{d}t + \beta_{t-}\,\mathrm{d}B(t,T)$$

and therefore

$$\begin{aligned}\beta(t)B(t,T) = \beta(0)B(0,T) &- \int_0^t \beta(u-)B(u-,T)A^*(u,T)\,\mathrm{d}u \\ &- \sum_k \int_0^t \beta(u-)B(u-,T)\sigma_k^*(u,T)\,\mathrm{d}l_k(u) \\ &+ \frac{1}{2}\int_0^t \beta(u-)B(u-,T)\sum_k \lambda_k\left[\sigma_k^*(u,T)\right]^2\,\mathrm{d}u \\ &+ \int_0^t \int_H \beta(u-)B(u-,T)\Big[\exp([\Phi(u)\cdot x](T)) - 1 \\ &\qquad - [\Phi(u)\cdot x](T)\Big]\Big(\mu^L(\mathrm{d}u,\mathrm{d}x) - \nu^L(\mathrm{d}u,\mathrm{d}x)\Big) \\ &+ \int_0^t \int_H \beta(u-)B(u-,T)\Big[\exp([\Phi(u)\cdot x](T)) - 1 \\ &\qquad - [\Phi(u)\cdot x](T)\Big]F(\mathrm{d}x)\,\mathrm{d}u.\end{aligned} \tag{3.10}$$

The Lévy processes $l_k$ are local martingales by assumption. The integral w.r.t. $\mu^L - \nu^L$ is a real-valued square integrable martingale according to [14, Theorem 3.4.5] if the integrand is bounded. A localizing argument reveals that this is indeed only a local martingale under the weaker assumption on $\sigma$. Noticing, that $[\Phi(u)\cdot x](T) = \int_t^T [\sigma(u)\cdot x](v)\,\mathrm{d}v$, we conclude. □

It is a suitable property to the forward rates that they are positive. If this is the case, the bond prices will also be true martingales as they are bounded.

# 4 An infinite factor Lévy model of credit risk

In this section we want to incorporate default risk in the previous model. We directly present a model of default ratings, as the model without ratings is a special case.

**Assumption 4.1** *Assume that there are $K-1$ ratings, where $1$ denotes the highest rating and $K-1$ the lowest, while $K$ is associated with default. Denoting by $\mathcal{K} = \{1,\ldots,K-1\}$ the set of possible ratings and putting $\bar{\mathcal{K}} = \mathcal{K}\cup\{K\}$, we assume that the rating $i$ forward rate satisfies*

$$r_t^i = S_t r_0^i + \int_0^t S_{t-u}A^i(u)\,\mathrm{d}u + \int_0^t S_{t-u}\,\sigma^i(u)\cdot\mathrm{d}L^i(u), \quad i\in\mathcal{K},$$

*where $(L^i(t))_{t\in[0,T^*]}$ are $H$-valued Lévy processes. The covariance operator of the continuous part of $L^i$ is denoted by $D^i$ and the jump measure by $\nu^i$, respectively. Furthermore, $A^i : [0,T^*]\times\Omega \mapsto H$ and $\sigma^i : [0,T^*]\times\Omega \mapsto L(H)$ are stochastic processes, which are predictable w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ and satisfy $\mathbb{P}(\int_0^{T^*} A^i(s)\,\mathrm{d}s \in H) = 1$ and $\sigma^i \in \mathcal{L}_{T^*}(H)$, for all $i \in \mathcal{K}$.*

As previously, we introduce the terms $A^{i*}(u,T) := \int_0^{T-u} A^i(u,v)\,\mathrm{d}v$ and $\sigma_k^{i*}(u,T) := \int_0^{T-u}[\sigma^i(u)\cdot e_k](v)\,\mathrm{d}v$. To ensure existence of the appearing integrals, we assume that the processes $(\sigma^i(t))_{t\in[0,T^*]} \in \mathcal{L}_{T^*}$ for each $i \in \mathcal{K}$.

The process describing the current rating of the bond, $(C^1(t))_{t\in[0,T^*]}$, takes values in $\bar{\mathcal{K}}$ and is assumed to be a Markov process. We denote by $C^2(t)$ the previous rating before $C^1(t)$. If there were no changes in rating up to time $t$ we set $C^2(t) = C^1(t)$. The default $\tau$ occurs at the first time, when the state $K$ is reached, $\tau = \inf\{t \in [0,T^*] : C^1(t) = K\}$. We set $\tau = T^* + 1$ if the inf is empty.

We follow [7] to enlarge the probability space $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{0\leq t\leq T^*})$ to $(\tilde{\Omega}, \mathcal{G}, \tilde{Q}, (\mathcal{G}_t)_{0\leq t\leq T^*})$. The filtration $(\mathcal{G}_t)_{0\leq t\leq T^*}$ is obtained by adding the information on $(C_t^1)$ to $(\mathcal{F}_t)$.

Denote the conditional infinitesimal generator of $C^1$ given $\mathcal{G}_t$ under the measure $\tilde{Q}$ by

$$\Lambda_t = \begin{pmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \lambda_{13}(t) & \cdots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \lambda_{23}(t) & \cdots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

for $t \in [0,T^*]$. Each $(\lambda_{ij}(t))$ is a $(\mathcal{G}_t)$-adapted process, satisfying

$$\lambda_{ii}(t) = -\sum_{j\neq i}\lambda_{ij}(t), \qquad \text{for all } t \in [0,T^*]. \tag{4.1}$$

Assume the rating $i$ recovery rate $(q^i(t))_{t\in[0,T^*]}$ to be a nonnegative stochastic process which is predictable w.r.t. $\mathcal{F}$ for all $i \in \mathcal{K}$. In extension to the previous section, we model the defaultable bond with rating transitions by

$$\begin{aligned} \bar{B}(t,T) = \mathbb{1}_{\{C^1(t)\neq K\}} \exp\left(-\int_0^{T-t} r_t^{C^1(t)}(u)\,\mathrm{d}u\right) \\ + \mathbb{1}_{\{C^1(t)=K\}} q_\tau^{C^2(t)} \bar{B}(\tau-,T)\exp\left(\int_\tau^t r_u\,\mathrm{d}u\right). \end{aligned} \tag{4.2}$$

We call this recovery modeling *rating based recovery of market value*. See [22] for a *rating based recovery of treasury*. The proposed methods also apply to this case.

Denote $B^i(t,T) := \exp\big[-\int_0^{T-t} r_t^i(u)\,\mathrm{d}u\big]$.

**Theorem 4.2** *Assume a rating based recovery of market value model and that Assumption 4.1 holds under the measure $\tilde{Q}$. Then all discounted bond prices are local martingales under $\tilde{Q}$, iff the following conditions are satisfied $\tilde{Q}$-a.s. on $\{\tau > t\}$.*

*1. For any* $(t, T)$ *such that* $t \in [0, T^*]$ *and* $T \geq t$*,*

$$r^{C^1(t)}(0) \;=\; r_t(0) + \left(1 - q_t^{C^1(t)}\right) \lambda_{C^1(t),K}(t). \tag{4.3}$$

*2. For any* $(t, T)$ *such that* $t \in [0, T^*]$ *and* $T \geq t$*,*

$$\begin{aligned} 0 \;=\; & -A^{C^1(t)*}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \left[\sigma_k^{C^1(t)*}(t,T)\right]^2 \\ & + \sum_{j=1,\, j\neq C^1(t)}^{K-1} \left[1 - \frac{B^j(t,T)}{B^{C^1(t)}(t,T)}\right] \lambda_{C^1(t),j}(t). \\ & + \int_H \Bigg[ \exp\left(\int_0^{T-t} \left[\sigma^{C^1(t)}(t)\cdot x\right](v)\,\mathrm{d}v\right) - 1 \\ & \qquad - \int_0^{T-t} \left[\sigma^{C^1(t)}(t)\cdot x\right](v)\,\mathrm{d}v \Bigg] F^{C^1(t)}(\mathrm{d}x). \end{aligned} \tag{4.4}$$

**Proof:** The rating based recovery of market value leads to the discounted gains process

$$\begin{aligned} G(t,T) \;=\; & \beta(t) \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=i\}} B^i(t,T) \\ & + \beta(\tau)\, \mathbb{1}_{\{C^1(t)=K\}} \sum_{i=1}^{K-1} \mathbb{1}_{\{C^2(t)=i\}} q_\tau^i \bar{B}(\tau-, T). \end{aligned}$$

Note that the indicators have finite variation, just like $(\beta_t)$, and therefore Itô's formula yields the dynamics

$$\begin{aligned} dG(t,T) \;=\; & \sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=i\}} d(\beta(t) B^i(t,T)) \\ & + \sum_{i=1}^{K-1} \beta(t-) B^i(t-, T) \mathrm{d}\mathbb{1}_{\{C^1(t)=i\}} \\ & + \mathrm{d}\left(\sum_{i=1}^{K-1} \mathbb{1}_{\{C^1(t)=K, C^2(t)=i\}}\right) q_\tau^i\, \beta(\tau-) \bar{B}(\tau-, T). \end{aligned}$$

For the last term,

$$\begin{aligned} & q_\tau^i\, \beta(\tau-) \bar{B}(\tau-, T) \mathrm{d}\mathbb{1}_{\{C^1(t)=K, C^2(t)=i\}} \\ = & \, q_t^i\, \beta(t-) B^i(t-, T) \mathrm{d}\mathbb{1}_{\{C^1(t)=K, C^2(t)=i\}}, \end{aligned}$$

as the indicator changes only at $t = \tau$. Furthermore with probability one we have $\bar{B}(\tau-, T) = B^{C^2(\tau)}(\tau-, T)$.

Proposition 11.3.1 in [1] yields that the following processes,

$$M^i(t) := \mathbb{1}_{\{C^1(t)=i\}} - \int_0^t \lambda_{C^1(u),i}\,\mathrm{d}u \tag{4.5}$$

and

$$\tilde{M}^i(t) := \mathbb{1}_{\{C^2(t)=i, C^1(t)=K\}} - \int_0^t \lambda_{iK}\mathbb{1}_{\{C^1(u)=i\}}\,\mathrm{d}u, \tag{4.6}$$

are martingales.

$$\begin{aligned}
\beta(t)B^i(t,T) \;=\; & \beta(0)B^i(0,T) \\
& \cdot \int_0^t \beta(u-)B^i(u-,T)\left(r_u^i(0) - r_u(0) - A^{i*}(u,T)\right)\,\mathrm{d}u \\
& - \sum_{k=1}^{\infty}\int_0^t \beta(u-)B^i(u-,T)\sigma_k^{i*}(u,T)\,\mathrm{d}l_k^i(u) \\
& + \frac{1}{2}\int_0^t \beta(u-)B^i(u-,T)\sum_{k=1}^{\infty}\lambda_k^i\left[\sigma_k^{i*}(u,T)\right]^2\,\mathrm{d}u \\
& + \int_0^t\int_H \beta(u-)B^i(u-,T)\big[\exp([\Phi^i(u)\cdot x](T)) - 1 \\
& - [\Phi^i(u)\cdot x](T)\big]\mu^i(\mathrm{d}u,\mathrm{d}x).
\end{aligned}$$

Combining this with the equations (4.5) and (4.6) it leads to

$$\begin{aligned}
\mathrm{d}G(t,T) = & \sum_{i=1}^{K-1}\mathbb{1}_{\{C^1(t)=i\}}\beta(t-)B^i(t-,T)\Bigg\{\left(r_t^i(0) - r_t(0)\right)\,\mathrm{d}t \\
& - A^{i*}(t,T)\,\mathrm{d}t + \frac{1}{2}\lambda_k^i\left[\sigma_k^{i*}(t,T)\right]^2\,\mathrm{d}t + q_t^i\lambda_{i,K}(t)\,\mathrm{d}t \\
& + \int_H\left[\exp\left(\left[\Phi^i(t)\cdot x\right](T)\right) - 1 - \left[\Phi^i(t)\cdot x\right](T)\right]F^i(\mathrm{d}x)\,\mathrm{d}t\Bigg\} \\
& + \sum_{i=1}^{K-1}\beta(t-)B^i(t-,T)\lambda_{C^1(t),i}\,\mathrm{d}t + \mathrm{d}M_t,
\end{aligned}$$

where we added the martingale terms up to $M_t$. Under the integrability conditions, $(G(t,T))_{t\geq 0}$ is a martingale, iff the drift term equals zero. As $\beta(t)B^i(t,T) > 0$ this is equivalent to

$$
\begin{aligned}
0 = {}& B^{C^1(t)}(t,T)\Big\{ r_t^{C^1(t)}(0) - r_t(0) - A^{C^1(t)*}(t,T) + q_t^{C^1(t)} \lambda_{C^1(t),K}(t) \\
& + \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \Big[\sigma_k^{C^1(t)*}(t,T)\Big]^2 \\
& + \int_H \Big[\exp\big(\big[\Phi^{C^1(t)}(t)\cdot x\big](T)\big) - 1 - \big[\Phi^{C^1(t)}(t)\cdot x\big](T)\Big] F^{C^1(t)}(\mathrm{d}x)\Big\} \\
& + \sum_{i=1}^{K-1} B^i(t,T)\lambda_{C^1(t),i}.
\end{aligned} \tag{4.7}
$$

At this point it is important to observe that (4.7) has to hold for all $(t, T)$ with $0 \leq t \leq T^*$ and $t \leq T$. Note that all terms depending on $T$ equal zero, if we set $t = T$. This allows us to split the above equation into two parts. But first we have to note that by equation (4.1) we obtain

$$
\begin{aligned}
& \sum_{i=1}^{K-1} \frac{B^i(t,T)}{B^{C^1(t)}(t,T)} \lambda_{C^1(t),i}(t) \\
&\qquad = \sum_{i=1, i\neq C^1(t)}^{K-1} \frac{B^i(t,T)}{B^{C^1(t)}(t,T)} \lambda_{C^1(t),i}(t) + \lambda_{C^1(t),C^1(t)}(t) \\
&\qquad = \sum_{i=1, i\neq C^1(t)}^{K-1} \left[\frac{B^i(t,T)}{B^{C^1(t)}(t,T)} - 1\right] \lambda_{C^1(t),i}(t) - \lambda_{C^1(t),K}(t).
\end{aligned}
$$

The first part is

$$
D_1(t) := B^{C^1(t)}(t,T)\Big[r_t^{C^1(t)}(0) - r_t(0) + q_t^{C^1(t)}\lambda_{C^1(t),K}(t)\Big]
$$

while we the second part contains the terms also depending on $T$

$$
\begin{aligned}
D_2(t,T) := {}& A^{C^1(t)*}(t,T) + \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k^{C^1(t)} \Big[\sigma_k^{C^1(t)*}(t,T)\Big]^2 \\
& + \sum_{i=1, i\neq C^1(t)}^{K-1} \left[\frac{B^i(t,T)}{B^{C^1(t)}(t,T)} - 1\right] \lambda_{C^1(t),i}(t) \\
& + \int_H \Big[\exp\big(\big[\Phi^{C^1(t)}(t)\cdot x\big](T)\big) - 1 - \big[\Phi^{C^1(t)}(t)\cdot x\big](T)\Big] F^{C^1(t)}(\mathrm{d}x)
\end{aligned}
$$

and we have $D_2(t,t) = 0$. Furthermore, if condition (4.4) holds, then $D_2$ is zero and of course $D_1(t) = 0$ is equivalent to (4.3). We conclude that if both $D_1$ and $D_2$ equal zero, which holds under the presented conditions, discounted bond prices are local martingales.

If, for the converse, $\tilde{Q}$ is a martingale measure, (4.7) holds. As $D_1$ does not depend on $T$, we have that, setting $T = t$, $D_1$ equals zero for all $t \leq T^*$ and hence must $D_2(t, T) = 0$ for all considered $(t, T)$ and we conclude. $\square$

It is possible to relax the needed assumptions, if the model is considered directly in the HJM-parametrization as done in [12].

**Remark 4.3** The credit risky model without ratings is a special case of the presented framework. This may be seen by setting $K := 2$ and

$$\Lambda_t := \begin{pmatrix} -\lambda(t) & \lambda(t) \\ 0 & 0 \end{pmatrix}.$$

Thus, $(\lambda_t)_{t\in[0,T^*]}$ is the default intensity as introduced, for example, in [16].

From Theorem 4.2 we immediately obtain

**Corollary 4.4** *If $\tilde{Q}$ is an arbitrage-free measure, the defaultable bond price satisfies the risk-neutral valuation formula*

$$\begin{aligned}\bar{B}(t,T) = \mathbb{E}_t^{\tilde{Q}}\Big[&\exp\left(-\int_t^T r_u \,\mathrm{d}u\right)\mathbb{1}_{\{C^1(T)\neq K\}} \\ &+ \exp\left(-\int_t^\tau r_u\,\mathrm{d}u\right)\mathbb{1}_{\{C_1(t)=K\}} q_\tau^{C^2(\tau)}\bar{B}(\tau-,T)\Big].\end{aligned}$$

**Remark 4.5** If a drift condition, not depending on a particular realization of $(C^1(t))$, is preferred we require the above equations to be satisfied for any rating $C^1(t)$, which leads to the following conditions:

1. For $t \in [0, T^*]$ and $T \geq t$,

$$r^i(0) = r_t(0) + (1 - q_t^i)\lambda_{i,K}(t).$$

2. For $t \in [0, T^*]$ and $T \geq t$,

$$\begin{aligned} 0 = &-A^{i*}(t,T) + \frac{1}{2}\sum_k \lambda_k^i \left[\sigma_k^{i*}(t,T)\right]^2 \\ &+ \sum_{j=1,j\neq i}^{K-1}\left[1 - \frac{B^j(t,T)}{B^i(t,T)}\right]\lambda_{i,j}(t) \\ &+ \int_H \left[\exp\left(\int_0^{T-t}\left[\sigma^i(t)\cdot x\right](v)\,\mathrm{d}v\right) - 1\right. \\ &\left. - \int_0^{T-t}\left[\sigma^i(t)\cdot x\right](v)\,\mathrm{d}v\right] F^i(\mathrm{d}x). \end{aligned} \tag{4.8}$$

Although conditions similar to (4.4) have been stated in literature, it was not yet pointed out that this condition relates the drift of the forward rate to the whole yield curve. This is not the case for the defaultable model without ratings or the risk-free case. Therefore, a functional setting for $r_t$ is needed to make sense of the model under which the drift condition holds.

Simply deriving (4.8) leads to the following corollary. Of course, a formulation which relates to the drift condition (4.4) is easily obtained proceeding similarly.

**Corollary 4.6** *The drift condition* (4.8) *is equivalent to the following condition on* $A$

$$
\begin{aligned}
A^i(t, T-t) = &\sum_k \lambda_k^i \sigma_k^i(t, T-t)\, \sigma_k^{i*}(t, T) \\
&+ \int_H \left[\sigma^i(t)\cdot x\right](T-1)\left(\mathrm{e}^{\int_0^{T-t}[\sigma^i(t)\cdot x](v)\,\mathrm{d}v} - 1\right) F^i(\mathrm{d}x) \\
&- \sum_{j=1,\, j\neq i}^{K-1} \lambda_{i,j}(t)\, I\left(T-t, r_t^i, r_t^j\right),
\end{aligned}
$$

*where* $I : \mathbb{R}^+ \times H \times H \mapsto \mathbb{R}$ *is given by*

$$
I(x, u, v) = \exp\left(\int_0^x \big(u(z) - v(z)\big)\,\mathrm{d}z\right)\big(u(x) - v(x)\big). \tag{4.9}
$$

At this point it becomes clear that the drift of any forward rate $A^i$ depends on all other forward rates through the last term. Moreover, it does not only depend on the single value at a certain maturity $T$, but rather on the full curve through the bond prices, respectively integrals in (4.9).

# 5 Simulations

To illustrate the approach with Lévy random fields we present some simulation results. Proposition 2.1 suggests that a Lévy random field can be simulated by choosing a suitable basis of $H$ and simulating one-dimensional Lévy fields. Using the first three elements of the Fourier basis, we can capture typical movements of the interest rate curve. In Figure 5.1 we show a Normal Inverse Gaussian random field as well as a Variance Gamma random field. For an introduction into the simulation of Lévy processes see [25]. We find that the movements of the simulated process very well capture typical movements of the term structure, as parallel shifts, changes in slope and curvature. Furthermore, abrupt changes in the whole interest rate curve are also produced by the presented model.

This simulation can just give a short hint about the usefulness of infinite dimensional Lévy processes in modeling credit risky interest rates. However, a thorough simulation study of such process seems to lead too far away from the focus of this article.

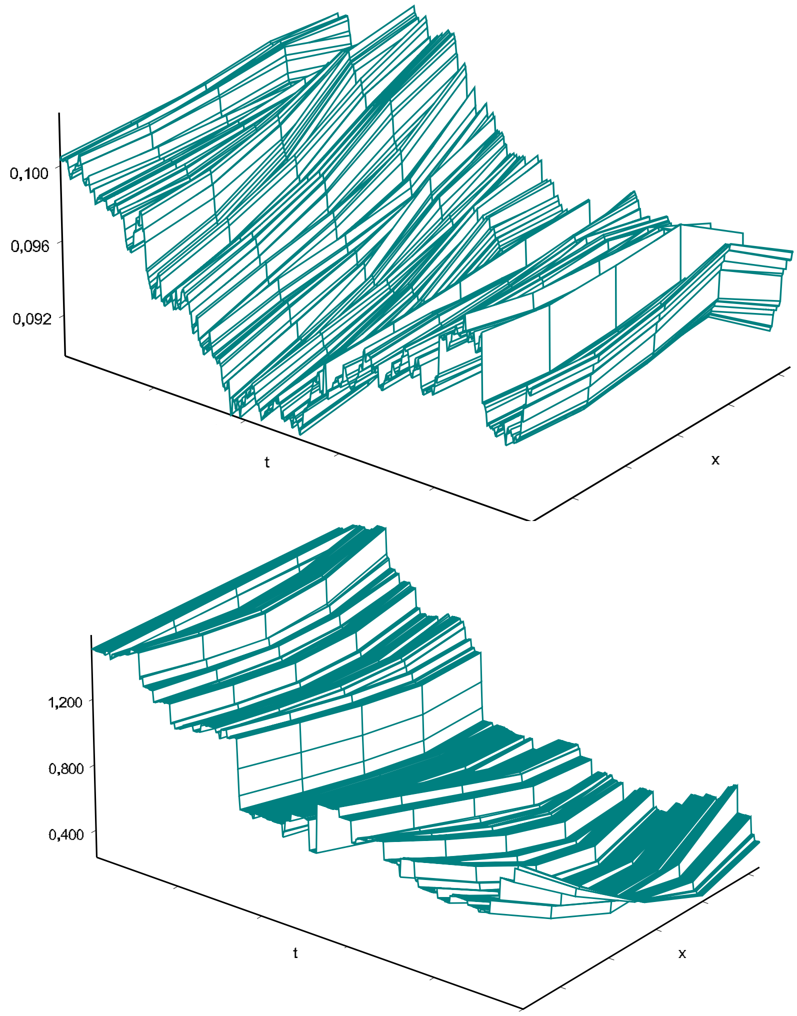


**Figure 5.1** Both graphs show Lévy random fields which were simulated using the decomposition 2.1 and choosing $e_1 = 1, e_2 = \cos(x), e_3 = \cos(2x), x \in [0, \pi]$. The upper graph shows a Normal Inverse Gaussian random field, while the lower graph shows a Variance Gamma random field. Note that in contrast to random fields w.r.t. Wiener processes on $H$, the Lévy random fields admit jumps and therefore show abrupt movements in the whole interest rate curve.

# 6 Conclusion

This article is the starting point for a new class of models in credit risk using Lévy random fields. After revisiting basic facts on Lévy random fields and discussing the default-free case, the main theorem (Theorem 4.2) states the no-arbitrage drift condition for the credit risk framework. This condition is the basis for the risk-neutral valuation formula. The next steps—which are beyond the scope of this article—will be presenting numerical results and the pricing of credit derivatives as well as a calibration to option prices.

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