Modelling Energy Markets with Extreme Spikes

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Summary. This paper suggests a new approach to model spot prices of electricity. It uses a shot-noise model to capture extreme spikes typically arising in electricity markets. Moreover, the model easily accounts for seasonality and mean reversion. We compute futures prices in closed form and show that the resulting shapes capture a large variety of typically observed term structures. For statistical purposes we show how to use the EM-algorithm. An estimation on spot price data from the European Energy Exchange illustrate the applicability of the model.

1 Motivation

It is well-known that as many other commodities electricity prices exhibit strong seasonalities. Besides this, due to the difficulty of storing electricity and inelastic demand, electricity spot prices show extremely strong spikes. The spot price data shown in Figure 1 clearly confirms this. In this paper, we propose a model which naturally captures this spiking behaviour. The model uses a type of shot-noise which is particularly suited for electricity spikes. It is furthermore simple enough to allow for closed-form solutions of futures and other power derivatives.

It is important to mention that electricity markets are young and small markets. For example, in Germany it is possible to trade electricity since 2000 and currently there are about 150 market participants trading at the European Energy Exchange, Leipzig[1]. Electricity prices have a number of featurs which are necessary to capture by a good model.

First, the necessity for using a model incorporating jumps is underlined in Eberlein & Stahl (2004) or Weron (2005). There are two approaches, which are closely related to the model presented here. In Geman & Roncoroni (2006) a model is proposed, where the jump component jumps up until a exogeneously level is reached and thereafter jumps down. The approach of Cartea

[1] 158 participants from 19 countries, cited from the webpage www.eex.de on January 2007.

& Figueroa (2005) is a special case of ours. The authors use a jump-diffusion to capture the spikes and the mean reversion. For an overview of existing literature on electricity models we refer to these papers. An approach using Lévy processes may be found in Benth, Kallsen & Meyer-Brandis (2006). In contrast to the Lévy approach, the shot-noise modelling allows for an easier estimation: an efficient tool for estimating shot-noise models is the EM-algorithm. We derive the necessary densities and apply the model to electricity prices in Section 4.

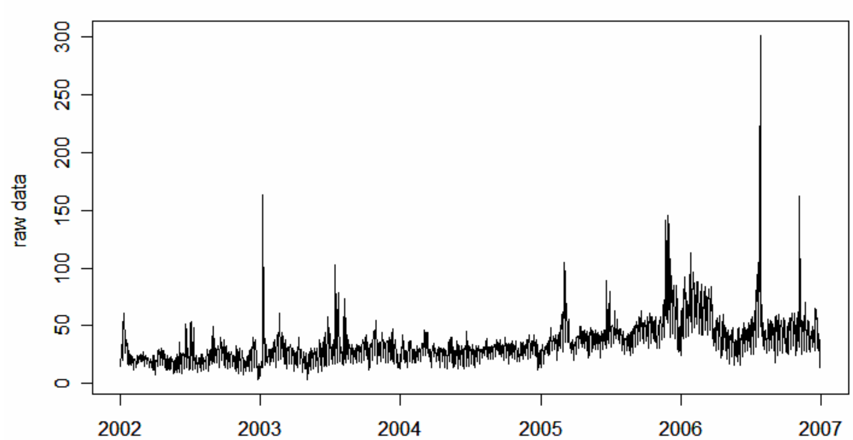


Fig. 1. The spot prices of energy (base load) quoted from the European Energy Exchange (www.eex.de).

The proposed model generalizes Cartea & Figueroa (2005) and offers more flexibility in capturing the statistical properties of the spot price as well as in calibrating to the futures curve. On the other side, the approach to modelling spikes seems more natural as in Geman & Roncoroni (2006), and in contrast to this model, we are able to compute prices of derivatives in closed form.

It seems important to note the specific characteristics of futures traded on electricity markets in contrast to futures, for example, from interest rate markets[2]. Electricity futures offer delivery of electricity over a certain period, typically a month, a quarter or a year. In a certain way this is a practicable approach to insure against extreme price fluctuations, because the payoff

[2] See, for example, the European Energy Exchange (EEX) Contract Specifications, downloadable from `www.eex.de`.

smoothes singular effects like spikes. On the other side, futures with a yearly delivery period also loose the dependence on the seasonalities. We take this into account and derive prices of futures on electricity markets.

2 Setup

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which admits a Brownian motion $(W_t)_{t\geq 0}$, a Poisson process $(N_t)_{t\geq 0}$ and iid rvs $Y_i, i = 1, 2, \dots,$ all independent of each other. We generalize simple shot-noise approaches as eg in Altmann, Schmidt & Stute (2006) in a way suitable for electricity spot prices. A close analysis of electricity prices reveals that the arising spikes either have an up-jump and then a strong decline or a sharp rise followed by a strong decline. The following function h will be able to capture this behaviour. For more general types of shot-noise processes we refer to Schmidt & Stute (2007).

For $a, b > 0$ define $h : (\mathbb{R}^+)^2 \times \mathbb{R} :\mapsto \mathbb{R}^+$ by[3]

$$h(t, \gamma, Y) := Y \cdot \begin{cases} \exp(a(t-\gamma)) & \text{if } 0 \leq t < \gamma, \\ \exp(-b(t-\gamma)) & \text{if } t \geq \gamma. \end{cases}$$

Y is the jump height and typically will be positive, while not necessarily. For $\gamma = 0$ this resembles simple shot-noise as a special case. If $\gamma > 0$, then h jumps at zero to $Y \exp(-a\gamma)$, then rises to Y at γ and thereafter it declines exponentially. For the shot-noise component we propose

$$J_t := \sum_{\tau_i \leq t} h(t - \tau_i, \gamma_i, Y_i). \tag{1}$$

Example 1. A simple example would be to assume that $\gamma_i \in \{0, \tilde{\gamma}\}$ with $p_\gamma := \mathbb{P}(\gamma_1 = 0)$. In this case one has classical shot-noise with probability p_γ and the "steep rise followed by sharp decline" case with probability $1 - p_\gamma$.

The diffusive part is responsible for mean-reversion and seasonalities. As the focus of the paper is mainly on the jump part, we stay quite simple in the assumptions on the diffusion. Assume that D is the strong solution of

$$dD_t = \kappa(\theta(t) - D_t)dt + \sigma dB_t, \tag{2}$$

where B is a standard Brownian motion. Under the above specification we say that

$$S = D + J$$

follows a *Vasicek/shot-noise process* with parameters $(a, b, f_\gamma, f_Y, \lambda, \kappa, \theta(\cdot), \sigma)$.

[3] We set $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$.

The process given in (2) is the well-known dynamics proposed by Vasiček (1977) for interest rate models. This process has a stochastic mean-reversion to the level $\theta(t)$. A different form of mean-reversion is obtained, if $D_t = \theta(t) + \tilde{D}_t$ is chosen, where $\tilde{D}_t$ is mean-reverting to the level 0. This was done in Cartea & Figueroa (2005). Contrary, in the stochastic mean-reversion, as chosen here, the mean reversion speed depends on the distance of D to the mean reversion level. Thus, if $|\theta(t) - D_t|$ is large, the process is pulled strongly back towards (θ), while if this difference is low, the mean-reversion is not so strong.

It is straightforward to extend the given setup to more general dynamics of D. For example, using the formulas obtained in Gaspar & Schmidt (2007), one immediately obtains closed-form solutions for electricity futures using generalized quadratic models for the diffusive part. For example, the well-known CIR-Modell (see Cox, Ingersoll & Ross (1985)) moreover guarantees positivity of D. In contrast to interest rate-models, polynomials of order higher than two can also be considered[4].

2.1 Changing measure

On one side, statistical estimation, as we consider in Section 4, is always done under the real-world measure $\mathbb{P}$ while on the other side pricing of derivatives takes place under the risk-neutral measure $\mathbb{Q}$. There is a vast of literature on specific choices of the risk-neutral measure. However, in this paper we consider a rather pragmatic approach which serves the need of applicability on one side and retains a reasonable amount of flexibility on the other side: we assume that the chosen model retains its structure while changing from $\mathbb{P}$ to $\mathbb{Q}$ although it of course will have different parameter values under $\mathbb{Q}$.

The Girsanov theorem[5] gives all possible changes of measure. For our purposes, we restrict to a sufficiently flexible measure change. Define $L_t := \mathbb{E}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t), t \geq 0$ and assume that L is given by

$$L_t = \prod_{\tau_i \leq t} \Big(\frac{\tilde{\lambda} \tilde{f}_Y(Y_i)}{\lambda f_Y(Y_i)} \Big) \exp \Big(- \int_0^t a(s) dW_s + \int_0^t (b - \frac{1}{2} a^2(s)) ds \Big), \tag{3}$$

where $a(s) = (\theta(t) - \tilde{\theta}(t))\sigma\kappa^{-1}$ for a deterministic function $\tilde{\theta}(t)$ and $b = \int (\tilde{\lambda} \tilde{f}_Y(z) - \lambda f_Y(z)) dz$. The following result precisely states the obtained model under $\mathbb{Q}$.

Proposition 1. *Assume that S is a Vasicek/shot-noise process with parameters $(a, b, f_\gamma, f_Y, \lambda, \kappa, \theta(\cdot), \sigma)$ under $\mathbb{P}$ and the measure change $d\mathbb{Q}/d\mathbb{P}$ is given by the likelihood process in* (3). *Then S is a Vasicek/shot-noise process under $\mathbb{Q}$ with parameters $(a, b, f_\gamma, \tilde{f}_Y, \tilde{\lambda}, \kappa, \tilde{\theta}(\cdot), \sigma)$.*

[4] The degree problem in interest rate models was observed in Filipović (2002).
[5] Compare Protter (2004) for a suitably general version.

Intuitively spoken, this means that under $\mathbb{Q}$, $(W_t + \int_0^t a(s)ds)_{t\geq 0}$ is a standard Brownian motion, N is a Poisson process with intensity $\tilde{\lambda}$. The distribution of Y may be changed in a quite general fashion, provided they are still equivalent. For practical purposes it might be reasonable to choose a parametric family and assume that the parameters change from $\mathbb{P}$ to $\mathbb{Q}$ while the Y stays in the parametric family. We assume the distribution of γ does not change to retain the shot-noise type. However, it is straightforward to also incorporate a change of the distribution of γ.

Proof. The claim follows directly from the Girsanov theorem. First, note that a, b, σ and κ do not change under equivalent measure changes. Second, $(W_t + \int_0^t a(s)ds)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion and hence

$$
\begin{aligned}
dD_t &= \kappa(\theta(t) - \sigma \frac{\theta(t) + \tilde{\theta}(t)}{\sigma} - D_t)dt + \sigma(dW_t + a(t)dt) \\
&= \kappa(\tilde{\theta}(t) - D_t)dt + \sigma(dW_t + a(t)dt).
\end{aligned}
$$

Furthermore, the jump component of L immediately reveals that Y_i, $i \geq 1$ are again i.i.d. under $\mathbb{Q}$ with densities $\tilde{f}_Y$; moreover N is a Poisson process with intensity $\tilde{\lambda}$ (see, for example, Brémaud (1981), Section VIII.3, Theorem T10). This yields the claim. □

It is important to note that there is no kind of no-arbitrage restriction on $\mathbb{Q}$ as the spot price, which is modeled here, is not a traded asset; see also the next section for further details. Any other asset what we consider later on will be an expectation of its discounted payoffs under $\mathbb{Q}$ and hence by definition be in line with no-arbitrage.

Thanks to Proposition 1, we can consider from now on Vasicek/shot-noise processes under $\mathbb{P}$ as well as under $\mathbb{Q}$. Note that, still, estimated parameters are under $\mathbb{P}$ while parameters for pricing as well as calibrated parameters are under $\mathbb{Q}$ throughout and typically do not coincide. A comparison analysis of calibrated prices with estimated parameters could clarify on the market prices or risk chosen by the market and would make a link possible.

2.2 Pricing of electricity futures

To price electricity futures we mainly follow Teichmann (2005), hence we assume that futures are traded for time-to-maturity of at least a small value, say ϵ. As fluctuations in electricity markets are quite large in comparison to interest rate markets, it is reasonable to assume zero interest rates. Then the futures price of a contingent claim $\mathcal{X}$ is $\mathbb{E}^{\mathbb{Q}}(\mathcal{X}|\mathcal{F}_t)$, where the expectation is taken under an equivalent martingale measure $\mathbb{Q}$.

The futures actually traded in electricity markets are not futures on a single spot rate. Instead, they offer electricity for a certain period of length L. More precisely, the future offers delivery of electricity in the period $[T, T + \Delta]$, with the value

$$\sum_{T_i \in [T,T+\Delta]} S_{T_i}$$

where $T_i \in [T, T+\Delta]$ refers to the respective trading days in the period under consideration. We assume that the mesh of the trading days is equidistant, i.e. $T_i - T_{i-1} := \delta$ for all i. In the following, we approximate the sum by an integral, $\sum_{T_i \in [T,T+\Delta]} S_{T_i} \approx 1/\delta \int_T^{T+\Delta} S_u \, du$. This is not necessary and is just used to simplify the formulas. It is an easy exercise to compute the explicit formulas for $\sum_{T_i \in [T,T+\Delta]} S_{T_i}$ instead of the integral.

Using the approximation we consider the following futures price:

$$F(t,T,\Delta) = \frac{1}{\delta} \, \mathbb{E}^{\mathbb{Q}}\Big(\int_T^{T+\Delta} S_u du | \mathcal{F}_t \Big).$$

We take this formula as a starting point and compute futures prices under the proposed shot-noise model. First, notice that as S is a sum of a diffusive and a shot-noise part, for pricing the futures it is sufficient to price the diffusive and the shot-noise part separately. As already mentioned, it is therefore straightforward to incorporate more general dynamics for D. Later on, in Example 2 case (3.) we also show how to consider an exponential model for D. In particular in the german market, spot prices show higher volatilities for higher prices, which can be captured well by an exponential model. This is not the case in the model considered in Benth et al. (2006).

From now on, assume that S is a Vasicek/shot-noise model with parameters $(a, b, f_\gamma, f_Y, \lambda, \kappa, \theta(\cdot), \sigma)$ under $\mathbb{Q}$. First, we give an auxiliary lemma. It basically shows how to compute certain expectations of shot-noise processes on different levels of generality. For an $U[0,1]$-distributed rv, independent of γ_1 and Y_1, define

$$\bar{S}(t) := \mathbb{E}^{\mathbb{Q}}\Big(h\Big(t(1-U_1), \gamma_1, 1\Big)\Big).$$

Furthermore, we set $\bar{Y} := \mathbb{E}(Y_1)$. Throughout we assume $\bar{Y}, \bar{S}(t) < \infty$ for all $t \geq 0$.

Lemma 1. *Consider* $t, \Delta > 0$ *and a function* $h : [0,\infty)^2 \times \mathbb{R} \mapsto \mathbb{R}$. *For the shot-noise process* J*, defined in* (1) *we have that*

$$\mathbb{E}^{\mathbb{Q}}(J_t) = \lambda t \bar{Y} \bar{S}(t), \quad \mathbb{E}^{\mathbb{Q}}\left(\int_t^{t+\Delta} J_u du \right) = \lambda \bar{Y} \int_t^{t+\Delta} u \bar{S}(u) \, du.$$

This small lemma illustrates the typical procedure for computing expectations of shot-noise processes. First, one conditions on the number of jumps in the desired interval. Second, under this condition the jump-times are distributed as order statistics of i.i.d. uniformly distributed random variables U_i. Third, using the i.i.d. property of the other ingredients, on can interchange the order of the U_i and finally ends up with a nice formula.

Proof. We have that

$$\mathbb{E}^{\mathbb{Q}}(J_t) = \sum_{k\geq 0} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=1}^{k} h\Big(t - tU_{i:k}, \gamma_i, Y_i\Big)\Big),$$

where $U_1, U_2, \dots$ are i.i.d. $U[0,1]$. As the random variables γ_i and Y_i are also i.i.d. one can interchange the oder of the second sum and obtains

$$\mathbb{E}^{\mathbb{Q}}(J_t) = \sum_{k\geq 0} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=1}^{k} Y_i\, h\Big(t - tU_i, \gamma_i, 1\Big)\Big).$$

The expectation equals $k\bar{Y}\bar{S}(t)$ and the first result follows. The second assertion follows by interchanging expectation and the integral. □

Denote the Laplace-Transform of γ_1 by $\varphi_\gamma(c) := \mathbb{E}^{\mathbb{Q}}(\exp(-c\gamma_1))$ and assume $\phi_\gamma(c) < \infty$ at least for $c \in \{a, -b\}$.

Theorem 1. *The price of the electricity future offering electricity in the time-period* $[T, T+\varDelta]$ *at* $t \leq T - \epsilon$*, which we denote by* $F(t,T,T+\varDelta)$*, computes according to:*

$$\begin{aligned}\delta \cdot F(t,T,T+\varDelta) &= \tilde{F}(t,T,T+\varDelta)\\ &\quad + \lambda\varDelta\bar{Y}\Big[\varDelta\Big(\frac{1-\varphi_\gamma(a)}{a} + \frac{1}{b}\Big) + \frac{\varphi_\gamma(-b)e^{-b(T-t)}}{b^2}\Big(e^{-b\varDelta} - 1\Big)\Big]\\ &\quad - \frac{D_t}{\kappa}\Big(e^{-\kappa(T+\varDelta)} - e^{-\kappa T}\Big) + \kappa\int_T^{T+\varDelta}\int_t^u e^{\kappa s}\theta(s)\,ds\,du, \qquad (4)\end{aligned}$$

where we denote the $\mathcal{F}_t$*-measurable part of shot-noise component by*

$$\tilde{F}(t,T,T+\varDelta) := \int_T^{T+\varDelta} \sum_{\tau_i \leq t} h(u - \tau_i, \gamma_i, Y_i)du.$$

The term $\tilde{F}$ captures the part of the past shot-noise effects. In practice, if the market at t is not in a extreme spike, $\tilde{F}$ can be safely neglected.

Proof. Following Teichmann (2005), the price of the future is given by an expectation under the risk-neutral martingale measure $\mathbb{Q}$. Hence[6],

$$\delta \cdot F(t,T,T+\varDelta) = \mathbb{E}_t\Big(\int_T^{T+\varDelta} S_u\,du\Big) = \mathbb{E}_t\Big(\int_T^{T+\varDelta} D_u\,du\Big) + \mathbb{E}_t\Big(\int_T^{T+\varDelta} J_u\Big).$$

[6] We use the short notation $\mathbb{E}_t(\cdot)$ for $\mathbb{E}^Q(\cdot|\mathcal{F}_t)$.

We first consider the expectation of the diffusive part and second the expectation of the shot-noise part. It is well-known[7] that (2) has the following explicit solution:

$$D_t = e^{-\kappa t}\Big(D_0 + \kappa \int_0^t e^{\kappa s}\theta(s)ds\Big) + \sigma \int_0^t e^{\kappa(s-t)}dB_s.$$

Then, for $u > t$ we obtain $D_u = e^{-\kappa u}(D_t + \kappa \int_t^u e^{\kappa s}\theta(s)ds) + \sigma \int_t^u e^{\kappa(s-u)}dB_s$, such that

$$\int_T^{T+\Delta} \mathbb{E}_t(D_u)\,du = -\frac{D_t}{\kappa}\Big(e^{-\kappa(T+\Delta)} - e^{-\kappa T}\Big) + \kappa \int_T^{T+\Delta}\int_t^u e^{\kappa s}\theta(s)\,ds\,du.$$

Second, consider the shot-noise part. Observe that

$$\begin{aligned}\mathbb{E}_t\Big(\int_T^{T+\Delta} J_u\,du\Big) &= \int_T^{T+\Delta} \mathbb{E}_t\Big(\sum_{\tau_i>t} h(u-\tau_i,\gamma_i,Y_i)\Big)\,du \\ &+ \int_T^{T+\Delta} \sum_{\tau_i\le t} h(u-\tau_i,\gamma_i,Y_i)du.\end{aligned}$$

As a Poisson process has independent and stationary increments, the expectation on the r.h.s. computes to

$$\begin{aligned}\mathbb{E}_t\Big(\sum_{t<\tau_i\le u} h(u-\tau_i,\gamma_i,Y_i)\Big) &= \mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=N_t+1}^{N_u} h(u-\tau_i,\gamma_i,Y_i)\Big) \\ &= \mathbb{E}^{\mathbb{Q}}\Big(\sum_{i=1}^{N_{u-t}} h(u-t-\tau_i,\gamma_i,Y_i)\Big) = \mathbb{E}(J_{u-t}).\end{aligned}$$

This expectation can be computed using Lemma 1 We therefore compute $\bar{S}$:

$$\begin{aligned}\bar{S}(t) &= \mathbb{E}^{\mathbb{Q}}\bigg(e^{a[t(1-U_1)-\gamma_1]}\mathbb{1}_{\{t(1-U_1)\in[0,\gamma_1]\}} + e^{-b[t(1-U_1)-\gamma_1]}\mathbb{1}_{\{t(1-U_1)>\gamma_1\}}\bigg) \\ &= \int_0^\infty \Big[\int_0^{1-v/t} e^{-b[t(1-u)-v]}du + \int_{1-v/t}^1 e^{a[t(1-u)-v]}du\Big]\,F_\gamma(dv),\end{aligned}$$

where the distribution of γ is denoted by F_γ. Computing the integrals we obtain that

[7] For example, see Schmidt (1997).

$$\bar{S}(t) = \int_0^\infty \Big[\frac{1}{bt}\Big(1 - e^{-b(t-v)}\Big) + \frac{1}{at}\Big(1 - e^{-av}\Big)\Big] F_\gamma(dv)$$
$$= \frac{1}{bt}\Big(1 - e^{-bt}\varphi_\gamma(-b)\Big) + \frac{1}{at}\Big(1 - \varphi_\gamma(a)\Big).$$

Finally, we have to compute the following integrals of $\bar{S}$:

$$\int_{T-t}^{T-t+\Delta} u\bar{S}(u)\,du = \int_{T-t}^{T-t+\Delta} \Big[\frac{1 - e^{-bu}\varphi_\gamma(-b)}{b} + \frac{1-\varphi_\gamma(a)}{a}\Big]\,du$$
$$= \Delta\Big(\frac{1-\varphi(a)}{a} + \frac{1}{b}\Big) + \frac{\varphi_\gamma(-b)e^{-b(T-t)}}{b^2}\Big(e^{-b\Delta} - 1\Big).$$

Using Lemma 1 with the above expressions proves the theorem.

Coming back to the simple case in Example 1 where γ was zero with probability p_γ and $\tilde{\gamma}$ otherwise, we obtain a simple Laplace transform as $\varphi_\gamma(c) = p_\gamma + (1-p_\gamma)\ \exp(-c\tilde{\gamma})$. Of course, there are many other possibilities where the Laplace transform is obtained in closed form (eg. Beta distribution, log-normal distribution or others).

Example 2. There are several interesting special cases or modifications of the above setting:

1. If $\theta(u) = \theta$, then the second line in (4) simplifies considerably to
$$\theta\Delta - \frac{D_t - \theta}{\kappa}\Big(e^{-\kappa(T+\Delta-t)} - e^{-\kappa(T-t)}\Big).$$
2. For incorporating seasonalities one frequently uses a mean-reversion level similar to $\theta(s) = \sin(\omega s)$. In this case we have that
$$\kappa\int_T^{T+\Delta}\int_t^u e^{\kappa s}\theta(s)\,ds\,du = \frac{\kappa}{(\kappa^2+\omega^2)^2}\Big(\omega\Delta\cos(\omega t)e^{\kappa t}(\kappa^2+\omega^2)$$
$$+ e^{\kappa(\Delta+T)}\Big((\kappa^2-\omega^2)\sin((T+\Delta)\omega) - 2\kappa\omega\cos(\omega(T+\Delta))\Big)$$
$$- \kappa\Delta\sin(\omega t)e^{\kappa t}\big(\kappa^2-\omega^2\big)$$
$$+ e^{\kappa T}\Big(2\kappa\omega\cos(\omega T) + \sin(\omega T)(\omega^2-\kappa^2)\Big)\Big).$$
And hence we also obtain a closed-form expression for $\theta(s) = \omega_0 + \sin(\omega_1 s) + \sin(\omega_2 s)$.
3. The chosen Gaussian mean-reverting Diffusion may become negative. If the parameters are suitably chosen this probability might be small, but still positive. To overcome this difficulty one can use $S_t = \exp(D_t) + J_t$.

It is also straightforward to compute the price of the future in this case as then

$$\begin{aligned}\mathbb{E}_t(D_u) &= \exp(D_t)\mathbb{E}_t\big(\exp(D_t - D_u)\big)\\ &= \exp\Big(- \frac{D_t}{\kappa}\Big(e^{-\kappa(T+\Delta)} - e^{-\kappa T}\Big) + \kappa \int_T^{T+\Delta}\int_t^u e^{\kappa s}\theta(s)\, ds\, du\\ &+ \frac{1}{2}\sigma^2 \int_t^{T+\Delta}\Big(\int_{T\vee s}^{T+\Delta} e^{\kappa(s-u)} du\Big)^2 ds\Big),\end{aligned}$$

where the last line computes to

$$\frac{1}{2}\sigma^2\Big\{\frac{4e^{-\kappa(T+\Delta-t)} - 3 - e^{-2\kappa(T+\Delta-t)} + \big(e^{\Delta\kappa}-1\big)^2\big(1-e^{-2\kappa(T+\Delta-t)}\big)}{2\kappa^3} + \frac{T+\Delta-t}{\kappa^2}\Big\}.$$

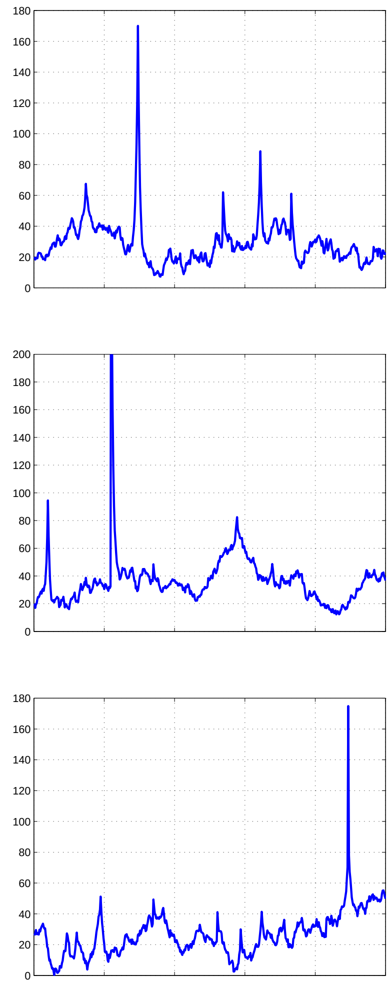
3 Illustration

In this section we give some simulated paths, which illustrate the properties of the model. It should be noted that if the model is used for pricing, the specification under the risk-neutral measure matters. As used in Proposition 1, from a practical viewpoint it is reasonable that the model follows a Vasicek/shot-noise process under $\mathbb{P}$ as well as under $\mathbb{Q}$, of course with different parameters.

We assume constant θ, i.e. no seasonality. The seasonalities have been discussed deeply in the literature, compare for example Lucia & Schwartz (2002), Cartea & Figueroa (2005) or Geman & Roncoroni (2006). As noted in Geman & Roncoroni (2006), it might be profitable to choose a non-constant λ.

In Figure 2 we give several paths of the proposed model under the specifications in (1) and (2). It is clearly seen that the shot-noise model is mean-reverting (in this case to the constant level $\theta = 20$) and the spikes capture the empirically observed up-and-down shape.

In Figure 3 we give examples of computed futures price which illustrate the large variety of shapes which can be captured by the proposed model. Here we use $\theta(t) = \omega_0 + \sin(\omega_1 t)$ which in turn leads to the wavy structure of the futures curves. The left picture gives an example of a decreasing term structure. This is due to the different diffusion levels D_0. Note that the effect of $\tilde{F}$, thus the effect of past spikes declines rapidly due to the fast decay rate of the shot noise. Therefore the influence of a high spot electricity due to a spike on the futures curve is quite low, as it should be. The right plot shows



κ	θ	σ	a, b	λ	ϕ_γ	$Y =$
0.5	20	20	40	4	$0.5(1 + e^{-0.05c})$	$10 + 5 * t^2, t \sim t_3$

Fig. 2. Simulations of the proposed shot-noise process over a horizon of 5 years. The parameters are as given above. The jump height $Y_i \sim 10 + 5 * \tilde{Y}_i^2$, where $\tilde{Y}_i$ are i.i.d. t-distributed with 3 degrees of freedom. Note that this specification does not include any seasonalities.

an example of an increasing term structure. This is due to an increasing mean reversion level, such that the spot prices are expected to increase and therefore also the futures.

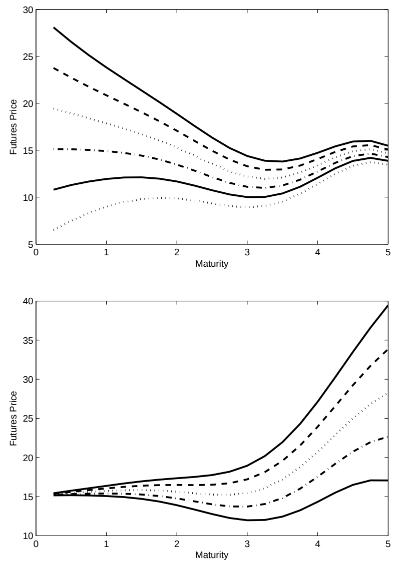


Fig. 3. Computed futures prices $F(0, T, \Delta)$ with $\Delta = 1$ `month` for the model as in Figure 2, but with seasonality of the type $\theta(t) = \omega_0 + \sin(\omega_1 t)$. Maturity T varies from 0 to 5 `years`. Left: futures price for varying $D_0 = 5, 10, \ldots, 30$. Right: futures price for varying $\omega_0 = 0.02, \ldots, 0.1$.

4 Estimation

One of the main points is of course estimation of the proposed model from historical data. This sections explores the use of the EM-algorithm for estimating shot-noise processes. The estimation of seasonalities is quite standard and we refer to Hylleberg (1992) for further reading. It therefore remains to estimate the shot-noise as well as the diffusive part; a plot of the data after removal of seasonalities is given in Figure 4. We first give a short outline of the EM-algorithm in our setting, and provide the estimation results on daily data provided by the EEX[8]. Note that estimation always takes place under the real-world measure $\mathbb{P}$.

4.1 The EM-algorithm

Consider a pair of r.v. $X = (Y, Z)$, $Y \in \mathbb{R}^n$, $Z \in \mathbb{R}^m$. Think of Y as observable quantities, and Z of unobservable quantities. The aim is to estimate the distribution of Y w.r.t. a parametric family $\{f_Y(\cdot; \phi) : \phi \in \Theta \subset \mathbb{R}^d\}$. However, the ML-estimate of Y might not always be at hand, such that we need to

[8] EEX- European Energy Exchange, www.eex.de

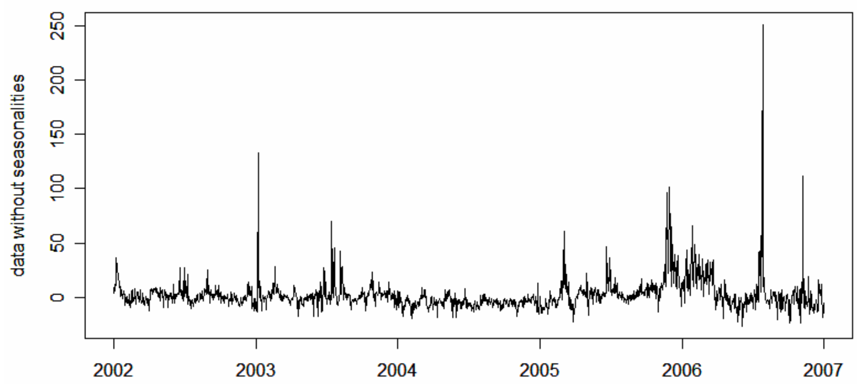


Fig. 4. The spot prices of energy (base load) after removal of seasonalities. Compare also Figure 1.

make use of Z. The EM-algorithm maximizes the density of $X = (Y, Z)$ w.r.t. the distribution of Z which is iteratively improved.

To this, let

$$L(\phi; \tilde{\phi}) := \mathbb{E}_{\tilde{\phi}}\Big(\ln f_X(y, z; \phi) | Y = y \Big)$$
$$= \int \ln f_X(y, z; \phi)\ f_{Z|Y}(z|y; \tilde{\phi}) dz.$$

By Bayes' rule we are able to compute the conditional density of Z given Y:

$$f_{Z|Y}(z|Y; \phi) \propto f_{Y|Z}(y|z; \phi)\, f_Z(z; \phi).$$

With this notation at hand we are able to state the EM-algorithm. Fix an initial value ϕ^0. The iteration $\phi^k \to \phi^{k+1}$ consists of two steps:

E-Step Compute $L(\phi; \phi^k)$
M-Step Choose ϕ^{k+1} as maximizer of $L(\phi; \phi^k)$:

$$\phi^{k+1} := \arg\max_{\phi \in \Theta} L(\phi; \phi^k).$$

These steps are repeated until $|f_Y(\phi^{k+1}) - f_Y(\phi^k)| < \epsilon$.

Of course, the computation of $L(\phi; \phi^k)$ might be far from trivial depending on the considered model. For a large number of examples and applications we refer to McLachlan & Krishnan (1997). The convergence of the EM-algorithm is proved in Wu (1983).

4.2 Application to the proposed model

The application of the EM-algorithm to the proposed model is done as follows. We consider Example 1 and are interested in estimating the parameter vector $\phi = (\kappa, \sigma, a, b, c, \lambda, p_\gamma)^\top$ and we assume that the distribution of Y_1 is described by a parameter c, i.e. $Y_1 \sim f_Y(\cdot; c)$. The observation consists of spot prices[9], for which we write $S = (S_1, \dots, S_n)$. Meanwhile S_i is a sum of a diffusive part and a shot-noise part, s.t. $S_i = D_i + J_i$. In the formulation of the EM-algorithm we therefore consider $X = (S, N, J)$, with $N_t = \sum_{\tau_i \le t}$. Note that S is observable and N, J are not. Clearly, $D = S - J$.

We make the assumption that jumps occur directly at the considered time points and at a specific time point at most one jump occurs. This is reasonable if the chosen time grid is fine enough. Denote the time step by Δ. To compute the likelihood function L it is sufficient to have the common density of S, N and J. Due to the dynamic nature of the processes we compute the density iteratively by

$$f_{X_n} = \prod_{i=1}^{n} f_{X_i|X_{i-1},\dots,X_1}(x_i|x_{i-1}, \dots, x_1).$$

First, observe that the Euler discretisation of (2) immediately gives that

$$D_i|D_{i-1} \sim \mathcal{N}(D_{i-1}(1 - \kappa\Delta), \sigma^2\Delta).$$

Second, as N is a Poisson(λ)-process, we have that

$$\mathbb{P}(N_i = N_{i-1}|N_{i-1}) = \exp(-\lambda\Delta).$$

As the third and last step we give the distribution of J given N. Note that the process is piecewise deterministic. The process J is not Markovian if $a \neq 0$. In the literature techniques for piecewise deterministic Markov processes have been applied to shot-noise processes of this type, compare Dassios & Jang (2003). We treat the two cases separately.

Markovian case

Assume that $a = 0$. Then J is Markovian. Note that J_i is a deterministic function of J_{i-1} if no jump occurs, as in this case $J_i = J_{i-1}\exp(-b\Delta)$. Otherwise, if a jump occurred at time i, which is equivalent to $N_i > N_{i-1}$, then $J_i = J_{i-1}\exp(-b\Delta) + Y_\cdot$, where the $Y_\cdot$ are i.i.d. with density f_Y. We obtain

$$df_{J_i|J_{i-1}}(j_i) = \begin{cases} \delta_{\{j_i = j_{i-1}\exp(-b\Delta)\}}, & \text{if } N_i = N_{i-1} \\ f_Y\big(j_i - j_{i-1}\exp(-b\Delta)\big)dj_i, & \text{otherwise.} \end{cases}$$

[9] Formally, we of course observe data on a certain time scale $t_1, \dots, t_n$ such that the observations are $S_{t_1}, \dots, S_{t_n}$, which we do not consider for expository purposes.

Non-Markovian case

If a is not zero, the case is more complicated. We just consider the case of Example 1, more general cases following similarly. Now we have to distinguish more cases. To begin with, note that there are two kinds of jumps. Jumps, where also $\gamma_\cdot = 0$ (which we call jumps of type 1) and jumps where $\gamma_\cdot = \tilde{\gamma}$ (called jumps of type 2). We additionally assume that $\tilde{\gamma}$ is sufficiently small such that we may neglect two jumps of type 2 in any interval of length up to $\tilde{\gamma}$. This leads to the following cases: first, if no jump of type 1 occurred at i and the last jump of type 2 is before $i - \tilde{\gamma}$. Then $J_i = J_{i-1}\exp(-b\Delta)$. Second, if a jump of type 1 occurred at i, hence $J_i = J_{i-1}\exp(-b\Delta) + Y_\cdot$. Third, if a jump of type 2 occurred at $j \in \{i - \tilde{\gamma}, \dots, i\}$. Then $J_i = J_j \exp(-b\Delta(i-j)) + Y_\cdot \exp(a\Delta(i-j))$. Summarizing we obtain that $df_{J_i|J_{i-1}}(j_i)$ equals

$$\begin{cases} f_Y\big(j_i - j_{i-1}\exp(-b\Delta)\big)dj_i, & \text{if } N_i > N_{i-1} \text{ and } \gamma_{N_i} = 0 \\ f_Y\Big(\big(j_i - j_{i-1}\exp(-b\Delta)\big)\exp(-a\Delta)\Big)dj_i & \text{if } N_i > N_{i-1} \text{ and } \gamma_{N_i} = \tilde{\gamma} \\ \delta_{\{j_i = j_j \exp(-b\Delta(i-j)) + Y_{N_j}\exp(a\Delta(i-j))\}} & \text{if } N_i = N_j > N_{j-1}, \gamma_{N_j} = \tilde{\gamma} \\ \delta_{\{j_i = j_{i-1}\exp(-b\Delta)\}}, & \text{otherwise.} \end{cases}$$

With the above densities at hand the EM-algorithm is easily implemented. In the following section we apply the suggested method to electricity prices obtained from the EEX.

4.3 Estimation of the model on EEX data

We directly work on data where the seasonalities have been removed with standard methods, to illustrate the applicability of the method. A full statistical analysis and comparison with other models is beyond the scope of the article and will be pursued in future work (Reiche & Schmidt (2007)).

Parameter	κ	σ	σ_Y	μ_Y	π_1	π_2	π_3	c
Estimate	0.2865	4.5762	60.34	17.4122	0.838	0.054	0.108	0.95

Table 1. Estimation results for the shot-noise model. See the text for details.

In Figure 5 the analysed data is plotted as well as the filtered shot-noise parts. The graph on the top shows the data decomposed in the diffusive part (lines) as well as the shot-noise part (circles). The graph on the bottom shows the shot-noise part only. As there are negative as well as positive jumps in the time series we assume that the jumps Y_i are normally distributed with mean μ_Y and standard deviation σ_Y.

For simplicity we consider the case with $p_\gamma = 0$ only. A further simplification speeds up the estimation process significantly: assuming that the decay

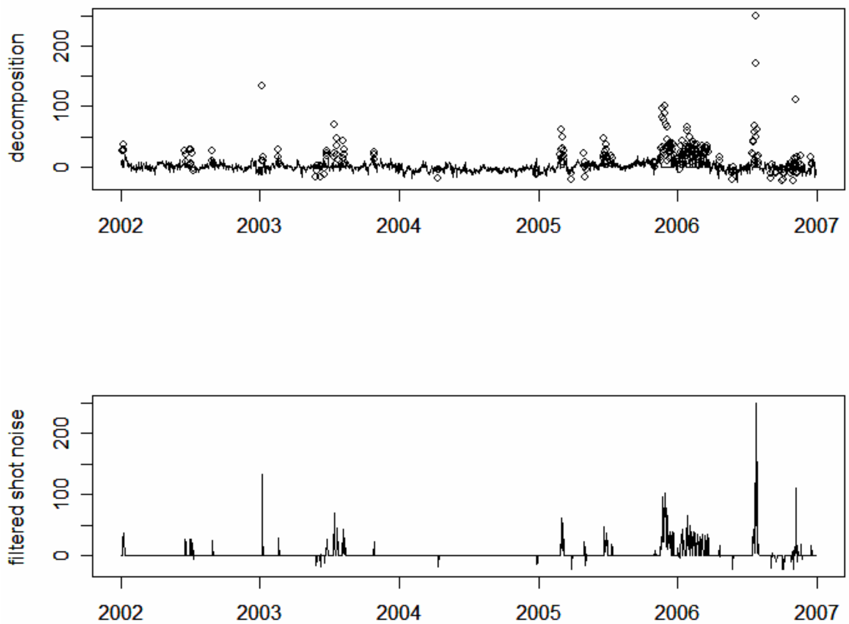


Fig. 5. Filtered shot-noise process from electricity data. The data consists of spot prices for base load from the European Energy Exchange, Leipzig.

rate c is high enough (which is reasonable as the intention is to model extreme spikes by the shot-noise part) the effect of a jump is negligible after a small number of time steps. Then the diffusive and the jump part can be treated in one step distinguishing three cases: first, no jump occurs (exponential decay with rate κ); second, a jump occurred (exponential decay plus jump); third, a small time interval after the jump (exponential decay at rate κ of the diffusive part plus exponential decay at rate c of the jump part). Denote the probabilities to be in either case by π_1, π_2 and π_3, respectively. The estimation results are given in Table 1. $\hat{\pi}_1 = 0.838$, which corresponds to a jump intensity of 0.77, i.e. an average number of 13 jumps per year. In particular in the end of 2006 and the beginning of 2007 a large number of spikes were identified. The volatility of the diffusive part, $\sigma = 4.57$, shows the high variation in the data set. The standard deviation of the jumps, $\sigma_Y = 60.34$, is of course much higher, reflecting the extreme shocks captured by the shot-noise part. Finally, the decay rate of the jump part, $c = 0.95$, shows that the shot-noise part indeed drifts back very fast after occurring jumps.

Of course, the above analysis mainly suffices for an illustration of the concept and shows applicability of the proposed model as well as the estimation procedure. A deeper statistical analysis as well as a comparison to other models will be covered in Reiche & Schmidt (2007).

5 Conclusion

This paper introduces a new model for spot electricity prices which easily captures the typical properties of electricity prices, namely seasonalities, extreme spikes and stochastic mean reversion. Moreover, the model allows for closed-form solutions of futures prices. Due to the flexibility of the model a large variety of shapes for the term structure of futures prices can be captured. It is shown how to use the EM-algorithm for statistical estimation of the model. The model is estimated using data from the European Energy Exchange.

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