A Structural Model with Unobserved Default Boundary

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We consider a firm-value model similar to the one proposed by Black and Cox (1976). Instead of assuming a constant and known default boundary, the default boundary is an unobserved stochastic process. This process has a Brownian component, reflecting the influence of uncertain effects on the precise timing of the default, and a jump component, which relates to abrupt changes in the policy of the company, exogenous events or changes in the debt structure. Interestingly, this setup admits a default intensity, so the reduced form methodology can be applied.

Keywords: structural model, equity default swaps, default boundary, jump-diffusion

1 Introduction

The seminal works Black and Scholes (1973) and Merton (1974) introduced the first structural models describing the default risk[3] of companies. This paper belongs to the class of first-passage time models, pioneered by Black and Cox (1976), where default of a company is announced at the first time when the firm-value falls below a certain boundary. It has been shown by Leland and Toft (1996) that under certain assumptions this behaviour is optimal for the company owner. However, in these models it is a fundamental assumption that investors have complete information on the firm's asset value as well as on the default boundary. In fact, usually investors do not have complete information and there are several approaches which deal with this issue. Most researchers concentrate on first-passage time models. Duffie and Lando (2001) consider the case, where investors estimate the firm's asset value from noisy accounting reports. Coculescu, Geman, and Jeanblanc (2006) consider a model where investors observe a correlated index and Frey and Schmidt (2006) filter the asset value from discretely observed news. In contrast to these filtering approaches, there is a different branch of research where the investors have incomplete information of either firm's asset value or default barrier (or both), but no additional information. This results in a class of highly tractable models. For example, Giesecke (2006) considers the case where the firm-value or default barrier (or both) may not be observed, while in Giesecke and Goldberg (2004) the asset value is observed; both papers deal with the case of a time-independent default barrier.

This paper extends to the case where the default barrier is allowed to be a stochastic process. A time-independent default barrier has serious drawbacks: if the firm's asset value is observable, the default boundary necessarily must be smaller than the minimum of the asset value on the considered time interval, say $[0, t]$. If the asset value at t is far above its minimum, this implies credit spreads

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[3]For an introduction into credit risk we refer to the surveys by Giesecke (2004), Schmidt and Stute (2004) or one of the excellent textbooks Lando (2004), Schönbucher (2003), McNeil, Frey, and Embrechts (2005).

which are unrealistically small. Considering a default barrier which is a stochastic process clearly remedies this. On the other side, structural models with a continuous asset value have difficulties in explaining short-term credit spreads. While some models almost overcome this, see for example Fouque, Sircar, and Sølna (2006), the model presented here is clearly able to solve this task as it has a default intensity and hence a positive credit spread for arbitrary small maturities.

The information on the value of the firm's assets is incorporated in two ways: on the one hand we assume that the asset value is observed at discrete time points only. As in practice, investors rely on frequent, but not continuous information this seems to be a reasonable assumption. The main focus in this paper is on this kind of discrete information. On the other hand, we consider the limit case, where the asset value would be monitored continuously and show convergence of the discrete-time results.

The structure of the article is as follows: first, we formulate the problem in a quite general framework and consider several special cases thereafter. One is the case where the firm's asset value and the default barrier follow geometric Brownian motions. Thereafter, we consider a default barrier which incorporates a jump component and show how to handle this setting. Finally, a small simulation study illustrates the results and shows typical credit spread curves implied by the model.

2 The general framework

Consider a structural model, where the firm value is denoted by the process $(V_t)_{t\geq 0}$. Following Black and Cox (1976), it is assumed that company owners declare bankruptcy, if the firm value falls below a certain boundary. This boundary is a stochastic process, denoted by $(D_t)_{t\geq 0}$ with $V_0 > D_0$. A typical interpretation of D is the level of the firm's outstanding debt. As default of the company occurs at the first time where V falls below D, the default time equals

$$\tau = \inf\{t \geq 0 : V_t \leq D_t\}.$$

We always denote the natural filtration of a stochastic process, say V, by $\mathcal{F}^V$, i.e. $\mathcal{F}_t^V := \sigma(0 \leq s \leq t : V_s)$. If $V - D$ is Markovian, the probability of $V - D$ not hitting zero in the interval $(t, T]$ given $\mathcal{F}_t^{V-D}$ can always be written as a function of t, T and $V_t - D_t$ and we set $H(-(V_t - D_t), t, T) := \mathbb{P}(\inf_{s\in(t,T]}(V_s - D_s) > 0|\mathcal{F}_t^{V-D})$. Markovianity of $V - D$ follows for example from Markovianity of V and D and independence. Another example is a two-dimensional Brownian motion with not necessarily independent components. First, we give results for general H and later on show how to compute H in several special cases.

We assume that the firm value is observable, but the default boundary is not. Investors also observe the default state of the company. The information available to investors is therefore represented by the filtration $\mathcal{G}_t := \sigma(s \in [0,t] : V_s, \mathbb{1}_{\{\tau>s\}})$.

Proposition 2.1. *Assume $V - D$ is Markovian and denote the conditional distribution of D_t given $\mathcal{G}_t$ by $\mu_{D_t|\mathcal{G}_t} =: \mu_t^D$. Then,*

$$\mathbb{P}\big(\tau > T \mid \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}} \int_{-\infty}^{V_t} H(x - V_t, t, T)\, \mu_t^D(dx). \tag{1}$$

Proof. As $\mathbb{1}_{\{\tau>t\}}$ is measurable with respect to $\mathcal{G}_t$, it can be taken out. It remains to consider $\mathbb{1}_{\{\inf_{s\in(t,T]}(V_s-D_s)>0\}}$. Set $A_t := \{\inf_{s\in[0,t]}(V_s - D_s) > 0\}$. Then

$$\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s) > 0\big|\mathcal{F}_t^V, A_t\Big) = \mathbb{E}\Big(\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s) > 0\big|\mathcal{F}_t^V \vee \mathcal{F}_t^{V-D}, A_t\Big)\big|\mathcal{F}_t^V, A_t\Big).$$

Because of the Markovian property the inner probability equals

$$\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s-D_s)>0\big|\mathcal{F}_t^V\vee\mathcal{F}_t^{V-D},A_t\Big)=\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s-D_s)>0\big|V_t-D_t\Big)$$
$$=H(D_t-V_t,t,T).$$

We therefore have

$$\mathbb{E}\Big(H(D_t-V_t,t,T)\big|\mathcal{F}_t^V,A_t\Big)=\int_{-\infty}^{V_t}H(x-V_t,t,T)\,\mu_t^D(dx),$$

as the integrand is zero on $\{D_t>V_t\}$. ■

For applications to credit risk it is an important question, if this model admits a default intensity.

Proposition 2.2. *Under the above assumptions the default intensity, if it exists, on $\{\tau>t\}$ is given by*

$$\lambda_t=-\lim_{T\to t}\int_{-\infty}^{V_t}\frac{\partial}{\partial T}H(x-V_t,t,T)\,\mu_t^D(dx). \tag{2}$$

Proof. By results of Aven (1985) the default intensity equals, if it exists,

$$\lambda_t=-\frac{\partial}{\partial T}\Big|_{T=t}\ln\mathbb{P}(\tau>T\mid\mathcal{G}_t).$$

Then, since we are considering the set $\{\tau>t\}$ only and $\mathbb{P}\big(\tau>t\mid\mathcal{G}_t\big)=1$,

$$\begin{aligned}\lambda_t&=-\frac{\partial}{\partial T}\Big|_{T=t}\mathbb{P}\big(\tau>T\mid\mathcal{G}_t\big)\\&=-\lim_{T\to t}\lim_{h\to0}\int_{-\infty}^{V_t}\frac{H(x-V_t,t,T+h)-H(x-V_t,t,T)}{h}\,\mu_t^D(dx).\end{aligned}$$

Recall, that $H(\cdot,t,T)$ is the probability of not hitting in the interval $(t,T]$ and therefore for $h>0$ we have that $H(\cdot,t,t+h)\le H(\cdot,t,t)$. Using monotone convergence we conclude that

$$\lambda_t=-\lim_{T\to t}\int_{-\infty}^{V_t}\frac{\partial}{\partial T}H(x-V_t,t,T)\,\mu_t^D(dx).$$ ■

In the last equation the interchange of limit and integration is typically not allowed, as will be seen in the later examples. In the case where D is time-independent and V is a geometric Brownian motion an intensity does not exist, as shown in Giesecke (2006). This happens due to the fact that a default may occur only when V is at its running minimum. Therefore the compensator of $\mathbb{1}_{\{\tau>t\}}$ is not absolutely continuous and hence its derivative, the default intensity, does exist.

Remark 2.3. Models where D is time-independent have difficulties if the firm's value decreases and thereafter rises substantially, such that V is far above its running minimum. In this case credit spreads for small maturities are too small, as may be seen in Figure 2 in Giesecke (2006). See also Schönbucher (2003), Section 9.6 for a discussion. Letting D be a stochastic process, and in particular one admitting steep upward rises, clearly helps to overcome this drawback. This is also reflected in the existence of a default intensity in the latter models. The simulations in Section 5 illustrate this achievement of the chosen model class.

3 The Brownian case

In the structural model considered by Black and Cox (1976) the firm value follows a geometric Brownian motion. Taking logarithms, one directly arrives at a Brownian motion with drift where default refers to hitting an affine barrier. For simplicity, we consider below the case where D and V are Brownian motions. The case with geometric Brownian motions is a consequence of the following observation: if B^V and B are independent Brownian motions, then

$$\mathbb{P}\Big(\inf_{s\in[0,t]}\big(V_0\exp(B^V_s+\mu^V s)-\exp(B_s+\mu s)\big)>0\Big)=\mathbb{P}\Big(\inf_{s\in[0,t]}\big(B^V_s+B_s+(\mu^V-\mu)s+\ln\frac{V_0}{D_0}\big)>0\Big)$$

This claim follows from

$$\begin{aligned}
&\inf_{s\in[0,t]}\big(V_0\exp(B^V_s+\mu^V s)-D_0\exp(B_s+\mu s)\big)>0\\
\Leftrightarrow\quad & V_0\exp(B^V_s+\mu^V s)>D_0\exp(B_s+\mu s),\qquad \forall s\in[0,t]\\
\Leftrightarrow\quad & \ln\frac{V_0}{D_0}+B^V_s+B_s+(\mu^V-\mu)s>0,\qquad \forall s\in[0,t]\\
\Leftrightarrow\quad & \inf_{s\in[0,t]}\big(B^V_s+B_s+(\mu^V-\mu)s+\ln\frac{V_0}{D_0}\big)>0.
\end{aligned}$$

3.1 Observations at discrete time points

In this section we consider a setting where the firm's value of the company is observed only at discrete time points. On the one hand, this is very much in line with practice, where investors do not have full access to the firm's asset value and rely on frequent reports from analysts or accounting reports. On the other hand, we show in Proposition 3.5 that the conditional distribution based on discretely monitored information converges to the continuously monitored one. Thus the results in this section can also be used as an approximation for a model with continuous information; note that in the continuous case the conditional distribution of V can not be computed in a closed form. The following results give the conditional distribution of V and, as already mentioned, this is sufficient to compute the default probabilities.

In this section we throughout make the following assumption:

Assumption 3.1. *B^V and B are two independent standard Brownian motions. The process V is given by $V_t=v_0+\sigma^V B^V_t$, $\sigma^V>0$ and $v_0>0$ and $D_t=\sigma^D B_t+g(t)$ with $\sigma^D>0$, where g is a twice continuously differentiable function.*

The function g refers to the (expected) level of debt at time t. The debt level, or the default boundary, thus consists of a systematic component g and a random (unobserved) component $\sigma^D B$. The default happens at the first time when V hits D, i.e. $\tau=\inf\{s\geq 0: V_s=D_s\}$.

Furthermore, we fix a current time point t and assume that the investors observe V only at discrete points in time; a default, however, is immediately announced. More precisely, consider $n\in\mathbb{N}$ and let $t_i=it2^{-n}$, $i=0,\dots,2^n$. The investor information is

$$\mathcal{G}^n_t=\sigma\big(V_{t_i}:t_i\leq t,\mathbb{1}_{\{\tau>s\}}:0\leq s\leq t\big).$$

We are then interested in the default probability $\mathbb{P}\big(\tau>T\mid\mathcal{G}^n_t\big)$. As a first step, we analyze $\mathbb{P}\big(\tau>T\mid\mathcal{G}_t\big)$. Since the boundary is no longer affine this probability can not be computed in a closed form.

In the following we need to consider the intervals $[0,t]$ and $(t,t+\Delta]$. t is current time. The first interval represents the history, where information was accumulated. The second interval refers to the future time. We want to estimate the conditional distribution of D at time $t+\Delta$ (required e.g. to price corporate securities). The discrete time points in the first interval are denoted by t_i, while the time points in the future time interval are denoted by $\tilde{t}_i$. Denote the distribution of D_t conditioned on $\mathcal{G}_t^n$ by $\mu_t^{D,n}$.

Proposition 3.2. *For $\Delta, t > 0$, let $\tilde{t}_i := t + i\Delta 2^{-n}$, $i = 0, \dots, 2^n$ and g_n be the piecewise linear function with $g_n(\tilde{t}_i) = g(\tilde{t}_i)$. Then, with $\sigma := \sqrt{(\sigma^V)^2 + (\sigma^D)^2}$, we have on $\{\tau > t\}$*

$$\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - (\sigma^D B_s + g_n(s))) > 0 \mid \mathcal{G}_t^n\Big) = \int_{-\infty}^{V_t} H_n\big(g_n(\cdot) - g(t) + x - V_t)\big)\,\mu_t^{D,n}(dx), \tag{3}$$

where H_n is given by

$$H_n(\hat{g}) := \mathbb{E}\left(\prod_{i=0}^{2^n-1} p_n\Big(\hat{g}(\tilde{t}_i), \hat{g}(\tilde{t}_{i+1}); B_{\tilde{t}_i}, B_{\tilde{t}_{i+1}}\Big)\right),$$

with $p_n(g_1, g_2; x_1, x_2) = 1 - \exp\Big(-\frac{2(x_1 - \frac{g_1}{\sigma})^+(x_2 - \frac{g_2}{\sigma})^+}{2^{-n}}\Big)$.

$H_n(\hat{g})$ is the probability of a Brownian motion with volatility σ staying *above* $\hat{g}$ in $[t, t+\Delta]$. Note that p_n is the probability of a Brownian bridge (in this case a Brownian motion with fixed endpoints) staying above an affine boundary. If g itself is a piecewise linear function, the above formula (3), of course, directly gives the default probability. From the proof (given below) it is obvious that

$$\mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - (\sigma^D B_s + g_n(s))) > 0 \mid \mathcal{G}_t\Big) = \int_{-\infty}^{V_t} H_n(x - V_t, T-t, \sigma)\,\mu_t^D(dx).$$

Proof. First,

$$\begin{aligned}\Big\{&\inf_{s\in(t,T]}(V_s - (\sigma^D B_s + g_n(s))) > 0\Big\}\\ &= \Big\{\inf_{s\in(t,T]}(V_s - V_t - \sigma^D(B_s - B_t) - (g_n(s) - g(t))) > D_t - V_t\Big\}\end{aligned}$$

Note, that $V_s - V_t - \sigma^D(B_s - B_t) = \sigma^V(B_s^V - B_t^V) - \sigma^D(B_s - B_t)$ is a Gaussian process, independent of V_t and D_t, which is equivalent (in distribution) to σB. Hence, we have

$$\begin{aligned}\mathbb{P}\Big(&\inf_{s\in(t,T]}(V_s - V_t - \sigma^D(B_s - B_t) - (g_n(s) - g(t))) > D_t - V_t | D_t = x\Big)\\ &= \mathbb{P}\Big(\inf_{s\in(t,T]}\Big(\sigma B_s - \big(g_n(s) - g(t) + x - V_t\big)\Big) > \Big).\end{aligned}$$

This consideration together with the well-known formula for boundary crossing probabilities (see e.g. Borovkov and Novikov (2005), Equation (8), or Wang and Pötzelberger (1997)) implies Proposition 3.2. ■

Convergence of H_n when g_n converges to g is shown in Borovkov and Novikov (2005) as well as convergence rates are given (as being of order $O(n^{-2})$). To obtain a default probability the next step is a computation of conditional distribution of $V_t - D_t$, i.e. $\mu^{D,n}$.

The conditional distribution. To determine the conditional distribution we study the cumulative distribution function. For $x < V_t$,

$$\mathbb{P}\Big(D_t \le x \mid \mathcal{G}_t^n\Big) = \frac{\mathbb{P}\Big(D_t \le x, \inf_{s\in[0,t]}(V_s - D_s) > 0 \mid \mathcal{F}_t^{V,n}\Big)}{\mathbb{P}\Big(\inf_{s\in[0,t]}(V_s - D_s) > 0 \mid \mathcal{F}_t^{V,n}\Big)} \tag{4}$$

where $\mathcal{F}_t^{V,n} := \sigma(V_{t_i} : i = 0, \dots, 2^{-n})$. First, we consider the numerator, the denominator is obtained for $x \to \infty$.

This time we consider the interval $[0, t]$ and approximate g and the observation therein. As in practice financial data is observed at discrete time points, the results have its own value in this respect. Denote $D_t^n := \sigma^D B_t + g_n(t)$.

Proposition 3.3. *For fixed $t > 0$ and $n \in \mathbb{N}$, set $t_i = it2^{-n}$, $i = 0, \dots, 2^n$. Then*

$$\mathbb{P}\Big(D_t^n \le x, \inf_{s\in[0,t]}(V_s - D_s^n) > 0 \mid V_{t_0} = v_0, \dots, V_{t_{2^n}} = v_{2^n}\Big)$$
$$= \mathbb{E}\left(\mathbb{1}_{\{B_t \le \frac{x-g(t)}{\sigma^D}\}} \prod_{i=0}^{2^n-1} p_n\Big(g(t_i), g(t_{i+1}); \frac{v_i - \sigma^D B_{t_i}}{\sigma}, \frac{v_{i+1} - \sigma^D B_{t_{i+1}}}{\sigma}\Big)\right). \tag{5}$$

Proof. First, by definition, $\{D_t^n \le x\} = \{B_t \le \frac{x-g(t)}{\sigma^D}\}$. Next, observe that

$$\{\inf_{s\in[0,t]}(V_s - D_s^n) > 0\} = \{\inf_{s\in[0,t]}\big(\sigma^V B_s^V - \sigma^D B_s - g_n(s)\big) > 0\}.$$

Using the tower property of conditional expectations, we obtain that for any $A \in \sigma(B_t)$,

$$\mathbb{P}\left(A, \inf_{s\in[0,t]}\big(\sigma^V B_s^V - \sigma^D B_s - g_n(s)\big) > 0 \mid \mathcal{F}_t^n\right)$$
$$= \mathbb{E}\left[\mathbb{P}\left(A, \inf_{s\in[0,t]}\big(\sigma^V B_s^V - \sigma^D B_s - g_n(s)\big) > 0 \mid B_{t_i}, B_{t_i}^V : 0 \le i \le 2^n\right) \mid \mathcal{F}_t^n\right]$$
$$= \mathbb{E}\left[\mathbb{1}_A \prod_{i=0}^{2^n-1} p_n\Big(g(t_i), g(t_{i+1}); \frac{\sigma^V B_{t_i}^V - \sigma^D B_{t_i}}{\sigma}, \frac{\sigma^V B_{t_{i+1}}^V - \sigma^D B_{t_{i+1}}}{\sigma}\Big)\right],$$

where the last equality follows again from Equation (8) in Borovkov and Novikov (2005). Using the independence of B and B^V we arrive at (5). ■

The formula (5) may be evaluated using an 2^n-fold integral over the normal distribution or alternatively, Monte Carlo methods. For the computation as 2^n-fold integral we give a recursive formulation.

Proposition 3.4. *Consider a piecewise linear function g whose discontinuity points are $t_i = it2^{-n} = i\Delta$, $i = 0, \dots, 2^n$. Let $a = a(y) := \frac{2}{\Delta\sigma}\big(V_t - \sigma^D y - g(t)\big)^+$ and*

$$E_{2^n}(x, y, g) := \Phi\Big(\frac{x - g(t) - \sigma^D y}{\sigma^D\sqrt{\Delta}}\Big) - e^{-a(V_t - \sigma^D y - g(t)) + \frac{a^2(\sigma^D)^2\Delta}{2}}\Phi\Big(\frac{x - \sigma^D y - g(t)}{\sigma^D\sqrt{\Delta}} - \frac{a\sigma^D\sqrt{\Delta}}{\sigma}\Big)$$

as well as, for $i = 0, \dots, 2^n - 2$,

$$E_{i+1}(x, y, g) := \int p\Big(g(t_i), g(t_{i+1}); \frac{V_{t_i} - \sigma^D y}{\sigma}, \frac{V_{t_{i+1}} - \sigma^D(y + \sqrt{\Delta}z)}{\sigma}\Big) \cdot E_{i+2}(x, z\sqrt{\Delta} + y, g)\phi(z)dz,$$

where $p(g_1, g_2; x_1, x_2) = 1 - \exp\Big(-\frac{2(x_1 - \frac{g_1}{\sigma})^+(x_2 - \frac{g_2}{\sigma})^+}{\Delta}\Big)$. Then

$$\mathbb{P}\Big(D_0 + \sigma^D B_t + g(t) \le x, \inf_{s\in[0,t]}(D_0 + \sigma^D B_s + g(s) - V_s) > 0 | \mathcal{F}_t^{V,n}\Big) = E_1(x, 0, g + D_0).$$

The proof is given in the appendix.

Convergence. Set

$$P(t,g,x) := \mathbb{P}\Big(D_t \le x,\ \inf_{s\in[0,t]}(V_s - D_s) > 0 \mid \mathcal{F}_t^V\Big).$$

We approximate $P(t,g,x)$ by

$$P_n(t,g_n,x) := \mathbb{P}\Big(\sigma^D B_t + g_n(t) \le x,\ \inf_{s\in[0,t]}(V_s - (\sigma^D B_s + g_n(s))) > 0 \mid \mathcal{F}_t^{V,n}\Big) = (5),$$

where g_n was the piecewise linear approximation of g with $g(t_i) = g_n(t_i)$. The following result gives the desired convergence.

Proposition 3.5. *Assume that Assumption 3.1 is in force and fix $t > 0$. For $t_i = t_i^n := it2^{-n}, i = 0,\dots,2^n$ and the piecewise linear interpolation g_n of g with $g(t_i) = g_n(t_i)$ and for $x \in \mathbb{R}$ we have that*

$$P_n(t,g_n,x) \xrightarrow[n\to\infty]{} P(t,g,x)$$

almost surely.

Proof. The proof consists of two parts. First, we show that $P_n(t,g,x)$ converges to $P(t,g,x)$ almost surely and then we estimate the difference between $P_n(t,g_n,x)$ and $P_n(t,g,x)$.

To this, note that $\mathcal{F}_t^{V,n}$ is an increasing sequence with $\mathcal{F}_t^{V,n} \to \mathcal{F}_t^V$. Then $P_n(t,g,x)$ is a regular martingale. Hence, by Lévy 's theorem[4] $P_n(t,g,x) \to P_\infty(t,g,x) = P(t,g,x)$ with probability one.

For the second part we use the Girsanov theorem to estimate the difference $P_n(t,g_n,x) - P_n(t,g,x)$. For any $A \in \mathcal{F}_t^V$ it holds that

$$\delta_n := \mathbb{P}\big(D_t \le x,\ \inf_{s\in[0,t]}(V_s - D_s) > 0|A\big) - \mathbb{P}\big(\sigma^D B_t + g_n(t) \le x,\ \inf_{s\in[0,t]}\{V_s - (\sigma^D B_s + g_n(s))\} > 0|A\big)$$
$$= \frac{1}{\mathbb{P}(A)}\,\mathbb{E}\left((Z_t - 1)\,\mathbb{1}_{\{\sigma^D B_t + g_n(t)\le x, \inf_{s\in[0,t]}(V_s - (\sigma^D B_s + g_n(s)))>0\}\cap A}\right),$$

where the density $Z_t = \tilde{Z}_t/\mathbb{E}(\tilde{Z}_t)$ with

$$\tilde{Z}_t := \exp\left(\int_0^t \frac{g'(s) - g_n'(s)}{\sigma^D} dB_s\right).$$

The density is used for a change to an equivalent measure $\tilde{\mathbb{P}}$, such that $(\sigma^D B_s + g_n(s))_{0\le s\le t} = (\sigma^D \tilde{B}_s + g(s))_{0\le s\le t}$ and $\tilde{B}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Then

$$|\delta_n| \le \frac{1}{\mathbb{P}(A)}\,\mathbb{E}\Big(\mathbb{1}_A\,|Z_t - 1|\Big) = \mathbb{E}\Big(|Z_t - 1|\Big),$$

as A and Z are independent. An estimate of this expression may be found in Novikov, Frishling, and Kordzakhia (1999) and we obtain

$$\mathbb{E}\Big(|Z_t - 1|\Big) \le \frac{1}{\sqrt{2\pi}}\left(\int_0^t \frac{(g'(s) - g_n'(s))^2}{\sigma_D^2} ds\right)^{1/2}.$$

[4]See, e.g. Shiryaev (1996, Theorem VII.4.3).

For a linear interpolation with step size $\Delta = 2^{-n}$ we have that

$$\sup_{0\le s\le t} |g'(s) - g_n'(s)| \le \sigma_D\, C\Delta^2,$$

where C is a generic constant and we used that g is twice continuously differentiable. This implies that $|P_n(t,g_n,x) - P_n(t,g,x)| \le C\Delta^{3/2} = C2^{-3n/2}$ and, as $P_n(t,g,x) \to P(t,g,x)$ (almost surely) we obtain that with $n\to\infty$

$$P_n(t,g_n,x) \to P(t,g,x)$$

almost surely. ■

4 Including Jumps

Up to now, D was a (geometric) Brownian motion with deterministic drift which excludes abrupt and random changes. In this section we relax the conditions and incorporate a jump-like behavior. Basically, the idea is to have a random drift g. However, g will still be piecewise linear to stay in the up to now developed framework.

Assumption 4.1. *Consider a Poisson process $\tilde N$ with intensity l and jump times $(\tau_i)_{i\ge1}$. Assume that $(J_i)_{i\ge1}$ are independent, identically distributed (i.i.d.) random variables with cumulative distribution function F_J and $\mathbb{E}(J_1)<\infty$. Moreover, (J_i) are independent of $\tilde N$,B and B^V.*

Fix $\epsilon > 0$ and set $h(t) = \frac{\min(t,\epsilon)}{\epsilon}\mathbb{1}_{\{t\ge0\}}$. Then h is piecewise linear and so is the process M defined by

$$M_t := \sum_{i\ge1} J_i h(t-\tau_i) = \sum_{\tau_i\le t} J_i h(t-\tau_i). \tag{6}$$

The process M resembles so-called shot-noise processes, where h is typically of the form $h(t) = \exp(-at)\mathbb{1}_{\{t\ge0\}}$; see Schmidt and Stute (2007) for more details and references. It is straightforward to include a time inhomogeneous intensity or to consider a Cox process instead of the Poisson process, which we do not pursue here for notational simplicity.

4.1 First hitting time distribution

We derive the probability of a Brownian motion B hitting M in the time interval $[0,T]$. Write short $d\mu^k_{J,u}(m^k)$ for $\mu^k_{J,u}(dj1,\dots,dj_k,du_1,\dots,du_k) := F_J(dj_1)\cdots F_J(dj_k)\,du_1\cdots du_k$. $\mu^k_{J,u}$ relates to the distribution of the jumps and the jump times, conditioned on having k jumps.

For a piecewise linear function g denote by $N(g) := \mathbb{P}(\inf_{s\in[0,t]}(B_s - g_s) > 0)$. If the non-differentiable points of g are $t_1 < \cdots < t_n$ then

$$N(g) = \mathbb{E}\Big(\prod_{i=1}^{n-1}\Big(1-\exp\Big(\frac{-2(B_{t_i}-g(t_i))^+(B_{t_{i+1}}-g(t_{i+1}))^+}{t_{i+1}-t_i}\Big)\Big),$$

(compare Borovkov and Novikov (2005)). A numerical scheme for computing N was introduced in Proposition 3.4. Using N, we are able to compute the probability that B stays above M in the interval $[0,t]$.

Proposition 4.2. *Fix $t>0$ and set for $0\le s\le t$*

$$M^k(s,m^k) = M^k(s,j_1,\dots,j_k,u_1,\dots,u_k) := \sum_{i=1}^k j_i h(s-tu_i).$$

Under Assumption 4.1, the following holds for any $x < B_0$

$$\mathbb{P}\big(\inf_{s\in[0,t]}(B_s - M_s) \geq x\big) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k\times\mathbb{R}^k} N\big(M^k(\cdot, m^k) + x\big)\ d\mu^k_{J,u}(m^k).$$

Proof. Note that a Poisson process and a Brownian motion adapted to the same filtration are necessarily independent, so $\tilde{N}$, B and B^V are mutually independent. We condition on the number of jumps in the interval, such that

$$\mathbb{P}\Big(\inf_{s\in[0,t]}(B_s - M_s) \geq x\Big) = \sum_{k=0}^{\infty}\mathbb{P}\Big(\tilde{N}_t = k, \inf_{s\in[0,t]}(B_s - M_s - x) \geq 0\Big). \tag{7}$$

The conditional distribution of the τ_i's can be replaced by an unconditional one[5], because

$$\mathscr{L}\big(\tau_{\tilde{N}_1}, \dots, \tau_{\tilde{N}_t} \big| \tilde{N}_t = k\big) = \mathscr{L}\big(\eta_{1:k}, \dots, \eta_{k:k}\big),$$

where the η_i are i.i.d. $U[0, t]$. Hence,

$$(7) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!}\mathbb{P}\Big(\inf_{s\in[0,t]}\Big(B_s - \sum_{i=1}^{k} J_i h(s - \eta_{i:k}) - x\Big) \geq 0\Big).$$

As the J_i are interchangeable, $\mathscr{L}(\sum_{i=1}^k J_i h(s - \eta_{i:k})) = \mathscr{L}(\sum_{i=1}^k J_i h(s - \eta_i))$. Now we condition on the jump times and the jump sizes. Then

$$\begin{aligned}
&\mathbb{P}\Big(\inf_{s\in[0,t]}\Big(B_s - \sum_{i=1}^{k} J_i h(s - \eta_i) - x\Big) \geq 0\Big) \\
&\qquad = \int\limits_{[0,1]^k\times\mathbb{R}^k} N\Big(\sum_{i=1}^{k} j_i h(\cdot - tu_i) + x\Big)\, F_J(dj_1)\cdots F_J(dj_k)\, du_1 \cdots du_k \\
&\qquad = \int\limits_{[0,1]^k\times\mathbb{R}^k} N\Big(M^k(\cdot, m^k) + x\Big)\ d\mu^k_{J,u}(m^k).
\end{aligned}$$

■

For practical purposes, it is important to note that for small l the series converges very fast, as the integral is bounded by 1.

The next step is to consider the distribution of the first hitting time if only discrete information is available, i.e. to consider the case where we condition on $\mathscr{F}_t^{V,n}$. To this, note that

$$\mathbb{P}\big(\inf_{s\in[0,t]}(V_s - \sigma^D B_s - g(s)) > 0 \mid \mathscr{F}_t^{V,n}\big) = E_1(\infty, \frac{V_0}{\sigma}, g) =: N_n(g). \tag{8}$$

Following the methodology outlined above we directly obtain (recall that g was piecewise linear)

$$\mathbb{P}\big(\inf_{s\in[0,t]}(V_s - \sigma^D B_s - g(s) - M) \geq x\big) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int\limits_{[0,1]^k\times\mathbb{R}^k} N_n\big(g + M^k(\cdot, m^k)\big)\ d\mu^k_{J,u}(m^k).$$

[5]See Rolski, Schmidli, Schmidt, and Teugels (1999), p.502. The $\eta_{i:k}$ denote the order statistics of η_i, that is the η_i are ordered, such that $\eta_{1:k} \leq \eta_{2:k} \leq \cdots \leq \eta_{k:k}$. $\mathscr{L}$ denotes the law of a random variable.

4.2 The conditional distribution

In this section we turn to the question of the conditional distribution of D, when $D_t = \sigma^D B_t + g(t) + M_t$ where $g(t)$ is piecewise linear and M is as in (6). The main task is to compute

$$\mathbb{P}\Big(\sigma^D B_t + g(t) + M_t \leq x, \inf_{s\in[0,t]}(V_s - \sigma^D B_s - g(s) - M_s) > 0 \;\Big|\; \mathcal{F}_t^{V,n}\Big). \tag{9}$$

Define the conditional probability of not hitting the default boundary in the interval $[t,T]$ by

$$N^{t,T}(g) := P\Big(\inf_{s\in[t,T]}(\sigma(B_s - B_t) - (M_s - M_t) - g(s)) > 0\Big).$$

From previous reasoning it is straightforward that

$$N^{t,T}(g) = \sum_{k=0}^{\infty} e^{-l(T-t)} \frac{(l(T-t))^k}{k!} \int_{[0,1]^k\times\mathbb{R}^k} N\Big(\frac{M^k(\cdot,m^k)+g}{\sigma}\Big)\, d\mu^k_{J,u}(m^k).$$

Theorem 4.3. *Under Assumption 4.1 the probability of default is*

$$\mathbb{P}\big(\tau > T \;\big|\; \mathcal{G}_t^n\big) = \frac{\mathbb{1}_{\{\tau>t\}}}{N_n(g)} \int_{-\infty}^{V_t} N^{t,T}\Big(g(\cdot) + x - V_t - g(t)\Big)\, f_{D_t|\mathcal{G}_t^n}(x)\, dx,$$

where the density $f_{D_t|\mathcal{G}_t^n}(x)$ is given in (11).

Proof. First we derive the conditional density $f_{D_t|\mathcal{F}_t^{V,n}}(x)$. Consider a realization of $(M_s)_{0\leq s\leq t}$ with k jumps, denoted by M^k or $\big(\sum_{i=1}^k j_i h(\cdot - tu_i)\big)$, respectively. Then $g + M^k$ is a piecewise linear function and thus the previously obtained results can be applied. Hence, conditional on k jumps we need to consider

$$\begin{aligned}&\mathbb{P}\Big(\sigma^D B_t + M^k(t) + g(t) \leq x, \inf_{s\in[0,t]}(V_s - \sigma^D D_s - M_s^k - g(s)) > 0 \;\Big|\; \mathcal{F}_t^{V,n}, M = M^k\Big)\\ &\quad = E_1(x, \frac{D_0}{\sigma^D}, M^k + g),\end{aligned}$$

where E_1 was introduced in Proposition 3.4. Recall, that $N_n(g) = E_1(\infty, V_0/\sigma, g)$ as introduced in (8). Therefore, we obtain by Bayes' rule

$$\mathbb{P}\Big(\sigma^D B_t + g(t) + M(t) \leq x \;\Big|\; \mathcal{F}_t^{V,n}, A_t\Big) = \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int_{[0,1]^k\times\mathbb{R}^k} \frac{E_1(x, \frac{V_0}{\sigma}, g + M^k)}{N_n(g+M^k)}\, d\mu^k_{J,u}(m^k). \tag{10}$$

Again from Bayes' rule we obtain that

$$f_{\sigma^D B_t + \hat{g}(t) + M_t|\mathcal{G}_t^n}(x) = N_n(g)^{-1} f_{\sigma^D B_t + \hat{g}(t) + M_t|\mathcal{F}_t^{V,n}}(x).$$

Coming to the conditional density we have to derive (10) w.r.t. x. By dominated convergence we obtain that the density equals

$$f_{\sigma^D B_t + \hat{g}(t) + M_t|\mathcal{G}_t^n}(x) = N_n(g)^{-1} \sum_{k=0}^{\infty} e^{-lt}\frac{(lt)^k}{k!} \int_{[0,1]^k\times\mathbb{R}^k} \frac{e_1(x, \frac{V_0}{\sigma}, g + M^k)}{N_n(g+M^k)}\, d\mu^k_{J,u}(m^k), \tag{11}$$

where $e_1(x,y,g)$ is defined through the same iterative procedure as E_1 except that

$$e_N(x,y,\hat{g}) := \frac{1}{\sigma^D\sqrt{\Delta_N}}\Big(\phi\Big(\frac{x-g(t)-\sigma^D y}{\sigma^D\sqrt{\Delta_N}}\Big) \\ - e^{-a_n\frac{v_{i+1}-\sigma^D y-g(t)}{\sigma}+\frac{a_n^2(\sigma^D)^2\Delta_N}{2\sigma^2}}\phi\Big(\frac{x-\sigma^D y-g(t)}{\sigma^D\sqrt{\Delta_N}}-\frac{a_n\sigma^D\sqrt{\Delta_N}}{\sigma}\Big)\Big).$$

Second, we consider the conditional default probability. We have that

$$\mathbb{P}(\tau > T|\mathcal{G}_t^n) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big(\mathbb{P}(\tau > T|\mathcal{G}_t^n, D_t)|\mathcal{G}_t^n\Big)$$

and need to compute the inner probability. As a Brownian motion has independent increments,

$$\begin{aligned}
\mathbb{P}(\tau > T|\mathcal{G}_t^n, D_t) &= \mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - D_s - M_s^k - g(s)) > 0 \mid \mathcal{F}_t^{V,n}, D_t\Big)\\
&= \mathbb{P}\Big(\inf_{s\in(t,T]}(V_s - V_t - \sigma^D(B_s - B_t) - (M_s - M_t) - g(s) + V_t - \sigma^D B_t - M_t) > 0 \mid \mathcal{F}_t^{V,n}, D_t\Big)\\
&= \mathbb{P}\Big(\inf_{s\in(t,T]}(\sigma(B_s - B_t) - (M_s - M_t) - g(s) + V_t - D_t + g(t)) > 0 \mid \mathcal{F}_t^{V,n}, D_t\Big)\\
&= N^{t,T}\Big(g(\cdot) + D_t - V_t - g(t))\Big)
\end{aligned}$$

and we conclude. ■

5 Simulations

In this section we will use some simulations to illustrate the obtained results and to examine typical credit spread curves which are produced by the model. First, we consider the convergence of the boundary crossing probabilities when a continuous boundary is approximated by a piecewise linear one. Second, we analyze the conditional distribution of the default boundary in the setting with observations of the firm's asset value at discrete time points. Finally, we compute credit spreads implied by the proposed model under a number of different specifications.

The boundary crossing probability H_n. As a first step we study the probability that a Brownian motion with volatility σ stays above a given, piecewise linear boundary in the whole interval $[0,1]$, $H_n(\hat{g})$, and its convergence to $H(g) := \mathbb{P}(\inf_{s\in[0,1]}(\sigma B_s - g(s)) > 0)$. The formula used is given in Equation (3). We chose $g(x) = (x-0.5)^2 - 0.75$ and for $n \in \{0,1,2,3,4,5,10\}$ we consider $t_i = i2^{-n}, i = 0,\dots,2^n$; $\hat{g}$ being the piecewise linear function which coincides with g on $t_0,\dots,t_n$. Figure 1 illustrates the setting and the computed values of H_n are:

n	σ	0	1	2	3	4	5	10
$H_n(\hat{g})$	0.3	0.8668	0.6769	0.6307	0.6137	0.6101	0.6116	0.6099
$H_n(\hat{g})$	0.5	0.6332	0.4693	0.4287	0.4177	0.4137	0.4157	0.4162

It is natural that increasing σ decreases the probability to stay above the barrier. Also lifting the barrier would decrease this probability. Note that for $n = 0$, $H_n(\hat{g}) = 1 - 2\Phi(g(0)/\sigma)$ which is 0.8664 for $\sigma = 0.3$ and 0.6319 for $\sigma = 0.5$ such that we have a good match.

The results also show an increasing variance with increasing n. For example, with $\sigma = 0.3$ and 10^5 simulations we have the variances $0.0767, 0.1695, 0.1913, 0.2067, 0.2166, 0.2230, 0.6099$ ($n = 0,1,\dots,5,10$).

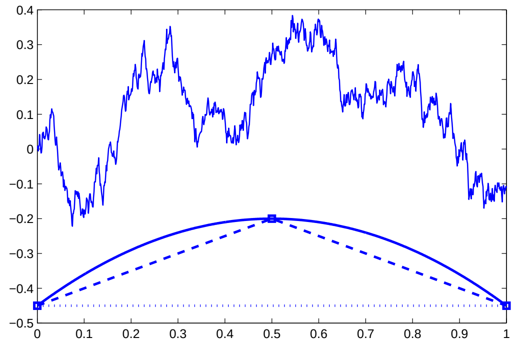

Figure 1: Brownian motion B with volatility $\sigma = 0.5$ staying above the barrier $g(x) = -(x-0.5)^2 - 0.2$. The graph also shows $\hat{g}_0$ and $\hat{g}_1$.

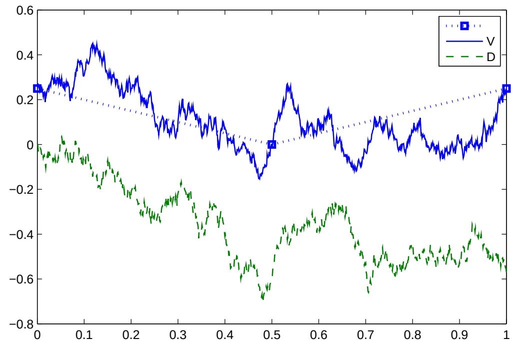

Figure 2: Two Brownian motions, both with volatility $\sigma^D = \sigma^V = 0.3$. The upper Brownian motion, V is conditioned on $V(t_i) = g(t_i)$, in the picture $t_i = i/2, i = 0, 1, 2$ and $g(x) = (x - 0.5)^2 + 0.2$.

The conditional distribution. In this paragraph we discuss the conditional distribution of D given $\mathcal{F}_t^n$, compare Equation (5). We consider the case where $g = 0$ and $V_{t_i} = f(t_i)$ with $f(x) = (x - 0.5)^2 + 0.2$. To retain comparability to the results from the previous paragraph, we chose $f = -g$ with g as above. Also, as previously, $t_i = i2^{-n}, i = 0, \ldots, 2^n$. An illustration of the setting is given in Figure 2. First we consider $\mathbb{P}(\inf_{s\in[0,1]}(V_s - D_s) > 0 | V_{t_i} = f(t_i), i = 0, \ldots, 2^n)$, i.e. the case where $x = \infty$ in (5). Note that this case corresponds exactly to the setting in the previous paragraph. The results of the simulation are

n	σ	0	1	2	3	4	5
$\mathbb{P}(\inf_{s\in[0,1]}(V_s - D_s) > 0 \mid \mathcal{F}_t^n)$	0.3	0.7714	0.5627	0.5376	0.5454	0.5615	0.5734

When we compare the boundary crossing probability $H_n(\hat{g})$ computed in the previous paragraph with this results (giving the probability that D stayed below V, conditioned on $V_{t_i} = f(t_i)$), we observe that the boundary crossing probability $H_n(\hat{g})$ is always larger. This stems from the fact that the Brownian motion V can depart from $f(t_i)$ at points excluding $\{t_i : i = 0, \ldots, 2^n\}$ (in comparison to the fixed function g in $H_n(\hat{g})$) and therefore the likelihood for V and D to meet increases.

A second observation is that for $n = 3$ to $n = 5$ the conditional probability increases stronger than H_n. This is because, for increasing n, V is tightened to more points an therefore has less possibilities to hit D. For large n the conditional probability will become closer and closer to H_n.

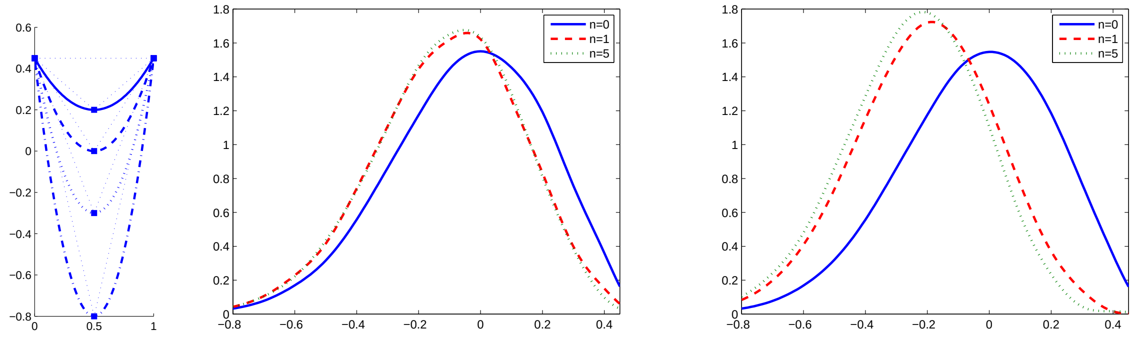

Figure 3: *Left*: Chosen upper barriers for D: $f_j(x) = c_j(x-0.5)^2 + d_j$, where $c_j \in \{1, 1.8, 3, 5\}$ and d_j are s.t. $f_j(0) = 0.45$. The straight line refers to $n = 0$ and the dots mark the information for $n = 1$. *Center/right*: The conditional density according to (5); the density is zero above 0.45. The observation $\mathcal{F}_t^n$ is $V_{t_i} = f_j(t_i), t_i = 0, \dots, 2^n$ with $j = 1$ (center) and $j = 2$ (right).

As a second illustration we compute the conditional density implied by (4) (using, of course, (5)) in the same setting as above. The result is given in Figure 3. To illustrate the effect of different historical information, we condition on $V_{t_i} = f_j(t_i)$ with $f_j(x) = c_j(x-0.5)^2 + d_j$, where $\mathbf{c} = (1, 1.8, 2.5, 3, 4)$ and $d = (0.2, 0, -0.175, -0.3, -0.55)$. We show the resulting densities for $c_j = 1$ and $c_j = 2$ and different n. The most dramatic effect is when changing n from 0 to 1, because this yields a strong restriction on possible paths of D. In particular in the middle graph, it can be spotted that there is a positive probability being close to the right boundary ($V_1 = f(1)$). This corresponds with the existence of a default intensity. The right graph illustrates the strong impact of a past observation of the firm value which was extremely low and thereafter rises substantially.

Credit spreads. Finally, we show possible credit spread curves implied by the model in the case without jumps. First we consider the case where $g = 0$ and later $g(x) = mx$. Of course, a large variety of different curves can be generated using more general g. Additionally, we assume $r = 0$ and consider zero recovery only.

For a defaultable bond $\bar{B}(t,T)$ the credit spread over the default-free bond $B(t,T)$ is given by

$$Y(t,T) := \frac{1}{T-t} \ln \frac{B(t,T)}{\bar{B}(t,T)}.$$

Under zero recovery and zero interest rates $\bar{B}(t,T) = Q(\tau > T | \mathcal{G}_t)$, where Q is a pricing measure. From Equation (1) we easily obtain that

$$\bar{B}(t,T) = \mathbb{1}_{\{\tau > t\}} \int_{-\infty}^{V_t} H(V_t - x, t, T)\, \mu_t^D(dx).$$

As we consider the case where D and V are continuous and $g = 0$, we have $1 - H(x,t,T) = 2\Phi\big(x/\sqrt{\sigma(T-t)}\big)$. For $g(x) = mx$, $1 - H(x,t,T) = \Phi\big(x-m(T-t)/\sigma\sqrt{(T-t)}\big) + e^{2xm}\Phi\big(x+m(T-t)/\sigma\sqrt{(T-t)}\big)$. The conditional distribution was already computed in the previous paragraph.

For the simulations we assume that current time is $t = 1$ and the past observation is of the form $V_{t_i} = f_j(t_i), t_i = 0, \dots, 2^n$ with $f_j(x) = c_j(x-0.5)^2 + d_j$, where $\mathbf{c} = (1, 1.8, 2.5, 3, 4)$ and $d = (0.2, 0, -0.175, -0.3, -0.55)$. The functions f_j are shown in the left of Figure 3. For $n = 0$ the information is the same for all j. Increasing j pulls the observation on V around 0.5 down. In contrast to this, at 1 we always observe $V_1 = 0.45$, such that the observation of V incorporates a steep rise in the firm value. This in turn leaves an increasing freedom for the debt level, such that with higher j the company is less likely to default. This also implies that the company is less likely to default in the near future, which corresponds with the observation that the density is nearly zero around 0.45 in Figure 3 for $n > 1$. In Figure 4 the resulting credit spreads are shown (left: $j = 1$, right: $j = 2$). With increasing j the spreads decrease, which reflects the smaller probability of default. The credit spreads also decrease with n, which stems from the convexity of f_j, compare the left of Figure 3: a higher n leads to observations which restrict the possible paths of D more heavily. The fact that the credit spreads at the short end are closer to zero for higher n is a consequence of the specific choice of f_j. The increasingly pronounced U-shape implies that the default boundary D has typically more distance to V with increasing n. A hat-shaped boundary would of course lead to more and more default risk in the short end.

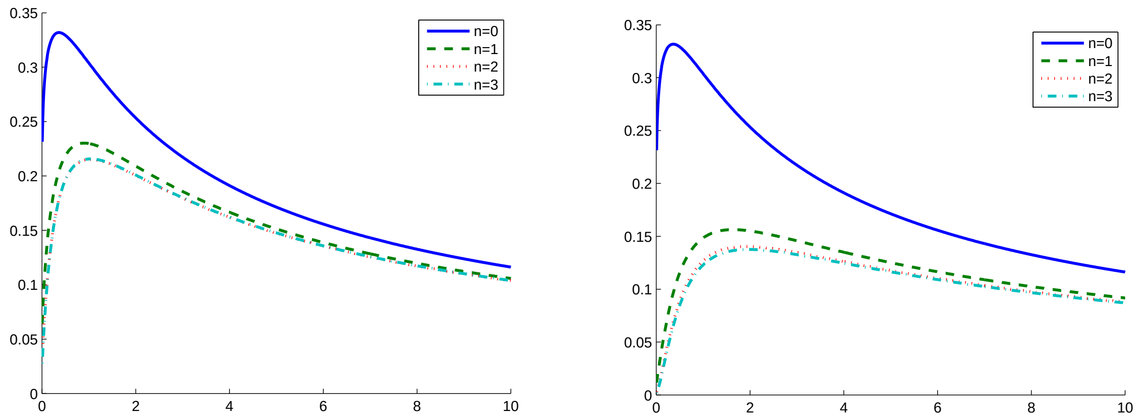

Figure 4: Credit spreads for a zero-recovery bonds (under $r = 0$) according to the information scenarios $V_{t_i} = f_j(t_i), t_i = 0, \ldots, 2^n$ with $j = 1$ (left) and $j = 2$ (right).

In terms of calibration it is important to know which kind of spread curves can be produced by the proposed model. A typical credit spread curve generated by our model is a concave function with a hump in the middle. This leads to three types of spread curves which are also the standard ones observed in the markets: flat (by generating a curve with a very small hump), increasing (by shifting the hump very much to the right) and inverse, i.e. decreasing (by shifting the hump very close to zero, such that the observed maturities just show the decreasing part). This is illustrated in Figure 5. The impact of the volatility of V and D is illustrated in the right of Figure 5, smaller volatility leads to decreasing default risk, i.e. to flattening of the credit spreads.

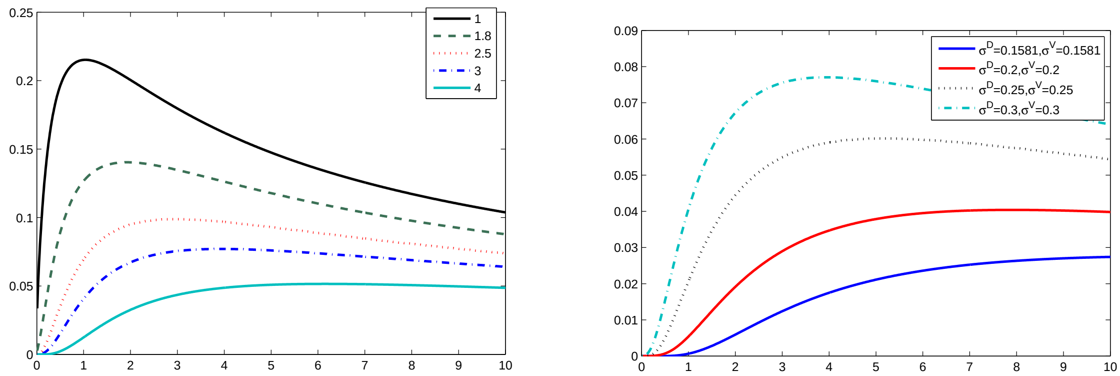

Figure 5: *Left*: credit spreads for a zero-recovery bonds (under $r = 0$) according to the information scenarios as above, with $n = 2$ and $j = 1, 2, 3, 4$. *Right*: credit spreads for different levels of volatility. Here $n = 2$ and $c_j = 3$.

In the left of Figure 6 we consider the case where the observation is $V_0 = V_1 = v$ with different levels of v. A lower level of v increases the credit risk, which is clearly reflected in the curves: they flatten with increasing v. Finally, in the right of Figure 6 we consider the case where g is not zero, but linear: $g(x) = mx$. This introduces an additional degree of freedom in the modelling of credit yield curves. Recall that g is the drift of the default boundary D. Increasing m therefore increases credit risk, as reflected in the curves.

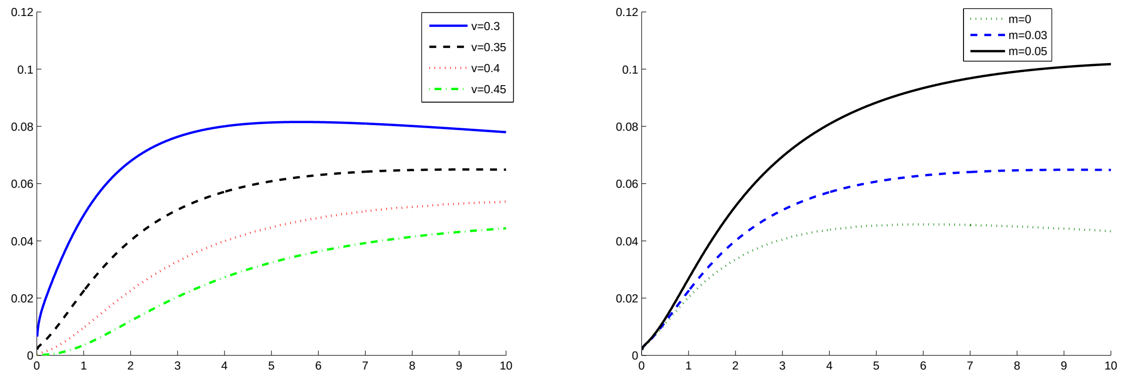

Figure 6: Credit spreads for zero-recovery bonds under $\sigma^D = 0.1$, $\sigma^V = 0.06$, the observation $V_0 = V_1 = v$ and with $g(x) = mx$. *Left*: credit spreads for different levels of v, $0.3, 0.35, 0.4, 0.45$ and $m = 0.03$. *Right*: credit spreads where $v = 0.35$ and $g(x) = mx$ with $m \in \{0, 0.03, 0.05\}$.

Summarizing, the credit spread curves implied by the proposed model (under continuity of D and V) show the classical hump-structure characteristic for first-passage time models. However, through the incomplete information approach additional degrees of freedom are achieved, for example the left endpoint need not necessarily be zero. The model is able to produce increasing, almost flat and decreasing credit spread curves.

6 Conclusion

This article proposes a generalization of several structural models with incomplete information. Default is triggered by the firm value crossing a random barrier, which itself is allowed to be a stochastic process. The default boundary incorporates a jump-like behavior. While the firm value is observed, the default boundary is not. It is shown that under this assumption, generally a default intensity exists and it is discussed how to compute it. This makes use of boundary crossing probabilities for jump-diffusions. A main criticism of incomplete information models with time-independent default boundary is remedied, namely that if the firm's asset value is far above its running minimum credit spreads are to small.

A Proofs

Proof of Proposition 3.4. We start by considering E_{2^n}, i.e. $i = 2^n - 1$. We write $E_i(y)$ for $E_i(x, y, g)$. Assume first that $D_0 = 0$. Our intention is to use iterated conditional expectations, such that for

E_{2^n} we need to consider

$$\mathbb{E}\left(\mathbb{1}_{\{B_t - B_{t_i} \le \frac{x-g(t)}{\sigma^D} - B_{t_i}\}}\, p\Big(g(t_i), g(t); \frac{v_i - \sigma^D B_{t_i}}{\sigma}, \frac{v_{i+1} - \sigma^D B_{t_i} - \sigma^D (B_t - B_{t_i})}{\sigma}\Big) \mid B_{t_i} = y\right) \tag{12}$$
$$= \Phi\Big(\frac{x - g(t) - \sigma^D y}{\sigma^D \sqrt{\Delta}}\Big)$$
$$- \mathbb{E}\left(\mathbb{1}_{\{\xi \le \frac{x-g(t)-\sigma^D y}{\sigma^D\sqrt{\Delta}}\}} \exp\Big(-\frac{2}{\sigma\Delta}\big(v_i - \sigma^D y - g(t_i)\big)^+ \big(v_{i+1} - \sigma^D y - \sigma^D \xi\sqrt{\Delta} - g(t)\big)^+\Big)\right),$$

where ξ is standard normal and we conditioned on $V_{t_i} = v_i, i = 0, \dots, 2^n$. Observe that

$$\frac{1}{\sigma^D\sqrt{\Delta}} \min\Big(x - g(t) - \sigma^D y; v_{i+1} - \sigma^D y - g(t)\Big) = \frac{x - \sigma^D y - g(t)}{\sigma^D \sqrt{\Delta}} =: b.$$

With $\mathbb{E}(\exp(\alpha\xi)\mathbb{1}_{\{\xi \le b\}}) = \exp(\alpha^2/2)\Phi(b - \alpha)$ we obtain

$$(12) = \Phi\Big(\frac{x - g(t) - \sigma^D y}{\sigma^D\sqrt{\Delta}}\Big) - \exp\Big(-a(v_{i+1} - \sigma^D y - g(t)) + \frac{a^2(\sigma^D)^2\Delta}{2}\Big)\,\Phi\Big(b - a\sigma^D\sqrt{\Delta}\Big),$$

which is exactly $E_{2^n}(x, y, g)$. The next step is to consider $E_{i+1}, i < 2^n - 1$, i.e. we need to compute

$$\mathbb{E}\left(\mathbb{1}_{\{B_t \le \frac{x-g(t)}{\sigma^D}\}} \prod_{j=0}^{2^n-1} p\Big(g(t_j), g(t_{j+1}); \frac{v_j - \sigma^D B_{t_j}}{\sigma}, \frac{v_{j+1} - \sigma^D B_{t_{j+1}}}{\sigma}\Big) \mid B_{t_j} : 0 \le j \le i\right). \tag{13}$$

Neglecting the measurable terms we obtain

$$(13) \propto \mathbb{E}\left(E_{2^n}(B_{t_{2^n-1}}) \prod_{j=i}^{2^n-2} p_n\Big(g(t_j), g(t_{j+1}); \frac{v_j - \sigma^D B_{t_j}}{\sigma}, \frac{v_{j+1} - \sigma^D B_{t_{j+1}}}{\sigma}\Big) \mid B_{t_i}\right)$$
$$= \mathbb{E}\left(E_{i+2}(B_{t_{i+1}})\, p_n\Big(g(t_i), g(t_{i+1}); \frac{v_j - \sigma^D B_{t_i}}{\sigma}, \frac{v_{j+1} - \sigma^D B_{t_{i+1}}}{\sigma}\Big) \mid B_{t_i}\right) = E_{i+1}(B_{t_i}).$$

This gives the recursion and as $B_0 = 0$ we arrive at

$$\mathbb{P}\Big(\sigma^D B_t + g(t) \le x, \inf_{s\in[0,t]} (\sigma^D B_s + g(s) - V_s) > 0 | \mathcal{F}_t^{V,n}\Big) = E_1(x, 0, g).$$

If g is piecewise linear, so $D_0 + g$ is. Replacing g by $D_0 + g$ then gives the desired result. ■

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