PRICING AND CALIBRATION IN MARKET MODELS

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ABSTRACT. The goal of this article is to study in detail the pricing and calibration in market models for credit portfolios. Starting from the framework of market models driven by time-inhomogeneous Lévy processes in a top-down approach proposed in Eberlein, Grbac, and Schmidt (2012) we consider a slightly simplified setup which eases calibration. This leads to a new class of affine models which are highly tractable. Conditions for absence of arbitrage under various types of contagion are given and valuation formulas for single tranche CDOs and options on CDO spreads are obtained. A simple two-factor affine diffusion model is calibrated to iTraxx data using the EM-algorithm together with an extended Kalman filter. The model shows a very good fit to all tranches and all maturities over the full observation period of four years.

1. INTRODUCTION

A credit portfolio consists of a number of different credit names (obligors). Modeling of credit portfolio risk is a challenging task which relies on the adequate quantification of the two main sources of risk. The first one is market risk, which is the risk stemming from the changes in interest rates and changes in the credit quality of the single credit names in the portfolio. The second one is correlation risk (also known as default correlation) among these credit names. A good model for credit portfolio risk should incorporate both sources of risk.

The main purpose of credit portfolio risk modeling is valuation and hedging of various contingent claims on a portfolio. In general, securities whose value and payments depend on a portfolio of underlying assets are termed asset-backed securities. For an overview and detailed descriptions of different types of asset-backed securities we refer to Chapter 1 and Section 5.1 of this book. Credit portfolio risk tranching is discussed in Chapter 3.

In the literature two main approaches can be found for credit portfolio models: the bottom-up approach, where the default intensities of each credit name in the portfolio are modeled, and the top-down approach, where the modeling object is the aggregate loss process of the portfolio. Both approaches have been studied in numerous recent papers; we refer to Lipton and Rennie (2011) and Bielecki, Crépey, and Jeanblanc (2010) for a detailed overview. Since in this article we focus on the top-down approach, we mention some of the recent papers where this approach is studied: Schönbucher (2005), Sidenius, Piterbarg, and Andersen (2008), Ehlers and Schönbucher (2006, 2009), Arnsdorf and Halperin (2008), Longstaff and Rajan (2008), Errais, Giesecke, and Goldberg (2010), Filipović, Overbeck, and Schmidt (2011) and Cont and Minca (2012).

In this article we develop a dynamic market model in the top-down setting, similar in spirit to Eberlein, Grbac, and Schmidt (2012). As discussed in that paper, the market model framework has a number of advantages in comparison to the HJM approaches for credit portfolio modeling. Similarly to Filipović, Overbeck, and Schmidt (2011) we utilize (T, x)-bonds to build an arbitrage-free model. However, we consider only a set of finitely many maturities which are indeed traded in the market. Considering instead a continuum of maturities as in the HJM approaches puts unnecessary restrictions to the model. In particular, this is reflected in the drift

condition which must be satisfied for all maturities. Taking into account only the traded maturities, one gains an additional degree of freedom in the specification of arbitrage-free models. For example, this allows various additional types of contagion as shown in Eberlein, Grbac, and Schmidt (2012). As a consequence, a tractable affine specification of our model which includes contagion can be obtained. It is needless to say that contagion effects are of particular importance in the current crises and a tractable model with contagion is practically highly relevant.

As driving processes for the dynamics of (T_k, x)-forward prices, a wide class of time-inhomogeneous Lévy processes is used. This processes allow for jumps in the forward price dynamics which are not driven by the default dates of the credit names in the portfolio. As pointed out in Cont and Kan (2011), jumps in credit securities can also be triggered by macroeconomic events external to the portfolio. The general specification proposed here allows for both, internal and external, kinds of jumps.

Finally, the practical relevance of the model is illustrated by its ability to provide a good fit to the market data. We use the iTraxx data from August 2006 to August 2010 and calibrate the model to the full dataset of four years. This goes beyond the usual calibration practice, where the models are calibrated to data from one day (cf. Cont, Deguest, and Kan (2010) for an overview and excellent empirical comparison). The calibration is done by applying an EM-algorithm together with an extended Kalman filter to a two-factor affine diffusion specification of our model, as proposed in Eksi and Filipović (2012). Already this simple specification provides a very good performance across *different tranches* and *maturities*.

The paper is organized as follows. In Section 2 we introduce the basic building blocks for credit portfolio market models. In Section 3 the model for th dynamics of the forward (T_k, x)-prices is presented and conditions for the absence of arbitrage are derived. Section 4 is a tractable affine specification of the model. In Section 5 we present valuation formulas for single tranche CDOs and call options on STCDOs. Section 6 is dedicated to the calibration of a two-factor affine diffusion market model to data from the iTraxx series.

2. BASIC NOTIONS

Consider a fixed time horizon $T^* > 0$ and a complete stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$, where $\mathcal{G} = \mathcal{G}_{T^*}$. $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T^*}$ satisfies the usual conditions. The filtration $\mathbb{G}$ represents the full market filtration and all price and interest rate processes in the sequel are adapted to it. We set $\mathbb{Q}^* := \mathbb{Q}_{T^*}$ and denote the expectation with respect to $\mathbb{Q}^*$ by $\mathbb{E}^*$.

Following the market model approach we consider a tenor structure containing finitely many maturities, denoted by $0 = T_0 < T_1 < \ldots < T_n = T^*$. Set $\delta_k := T_{k+1} - T_k$, for $k = 0, \ldots, n-1$.

The studied credit risky market consists of a *pool* of credit risky assets. As laid out in the introduction, we follow the *top-down* approach and directly study the aggregated losses. In this regard, denote by $L = (L_t)_{t \ge 0}$ the non-decreasing *aggregate loss process*. Assume that the total volume is normalized to 1 and denote by $\mathcal{I} := [0, 1]$ the set of loss fractions such that L takes values in $\mathcal{I}$. We assume

(**A1**) $L_t = \sum_{s \le t} \Delta L_s$ is an $\mathcal{I}$-valued, non-decreasing marked point process, which admits an absolutely continuous compensator

$$\nu^L(dt, dy) = F_t^L(dy)dt.$$

As shown in Filipović, Overbeck, and Schmidt (2011, Lemma 3.1), under (**A1**), the indicator process $(\mathbf{1}_{\{L_t \le x\}})_{t \ge 0}$ is càdlàg with intensity process

$$\lambda(t, x) = F_t^L((x - L_t, 1] \cap \mathcal{I}).$$

In particular, this yields that the process M^x given by

$$M_t^x = \mathbf{1}_{\{L_t \leq x\}} + \int_0^t \mathbf{1}_{\{L_s \leq x\}} \lambda(s, x) \mathrm{d}s \tag{1}$$

is a martingale.

The basic instruments are the (T_k, x)-bonds introduced in Filipović, Overbeck, and Schmidt (2011). They are simple securities and prices for more complex products such as CDOs can be derived from them in a model-free way, see Proposition 5.1.

Definition 2.1. A security which pays $\mathbf{1}_{\{L_{T_k} \leq x\}}$ at maturity T_k is called (T_k, x)-*bond*. Its price at time $t \leq T_k$ is denoted by $P(t, T_k, x)$.

If the market is free of arbitrage, $P(t, T_k, x)$ is nondecreasing in x and

$$P(t, T_k, 1) = P(t, T_k), \tag{2}$$

where $P(t, T_k)$ denotes a time-t price of a default-free zero coupon bond with maturity T_k. Moreover, if L already crossed the level x, the (T_k, x)-bond price is zero, i.e. on the set $\{L_t > x\}$ it holds that $P(t, T_k, x) = 0$.

In Filipović, Overbeck, and Schmidt (2011) a forward rate model for (T, x)-bonds has been analyzed under the assumption that (T, x)-bonds are traded for all maturities $T \in [0, T^*]$. This assumption imposes unnecessary restrictions to the model, since in practice the set of traded maturities is only finite. The market model approach takes this fact into account, see Eberlein, Grbac, and Schmidt (2012) for a detailed discussion. Here, we follow the framework introduced in this paper with slight modifications. The main ingredients in market models are the (T_k, x)-forward bond prices defined below.

Definition 2.2. The (T_k, x)-forward price is given by

$$F(t, T_k, x) := \frac{P(t, T_k, x)}{P(t, T_k)} \tag{3}$$

for $0 \leq t \leq T_k$.

The (T_k, x)-forward prices actually give the distribution of L_{T_k} under the $\mathbb{Q}_{T_k}$-forward measure defined later in (8). More precisely, if we take $P(\cdot, T_k)$ as a numeraire we obtain

$$\mathbb{Q}_{T_k}\big(L_{T_k} \leq x | \mathcal{G}_t\big) = \frac{1}{P(t, T_k)} P(t, T_k) \mathbb{E}_{\mathbb{Q}_{T_k}}\big(\mathbf{1}_{\{L_{T_k} \leq x\}} | \mathcal{G}_t\big)$$

$$= \frac{P(t, T_k, x)}{P(t, T_k)} = F(t, T_k, x).$$

3. The model

3.1. **Modeling assumptions.** Let X be an $\mathbb{R}^d$-valued Lévy process on the given stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q}_{T^*})$ with $X_0 = 0$ and Lévy-Itô decomposition

$$X_t = W_t + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(\mathrm{d}x, \mathrm{d}s), \tag{4}$$

where W is a d-dimensional Wiener process with respect to $\mathbb{Q}^*$, μ is the random measure of jumps of X with its $\mathbb{Q}^*$-compensator $\nu(\mathrm{d}x, \mathrm{d}t) = F_t(\mathrm{d}x)\mathrm{d}t$. Note that the canonical representation (4) is justified if X has a finite first moment. This is guaranteed by the following assumption which implies the existence of exponential moments of X, compare Eberlein and Kluge (2006, Lemma 6).

(**A2**) There are constants $\tilde{C}, \varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)\tilde{C}, (1+\varepsilon)\tilde{C}]^d$

$$\sup_{0 \leq t \leq T^*} \left(\int_{|x|>1} \exp\langle u, x\rangle F_t(dy) \right) < \infty.$$

The main ingredient of the approach studied here is the dynamics of the (T_k, x)-forward prices. We assume throughout that

$$F(t, T_k, x) = \mathbf{1}_{\{L_t \leq x\}} G(t, T_k, x), \tag{5}$$

where

$$G(t, T_k, x) = G(0, T_k, x) \exp \left(\int_0^t a(s, T_k, x) ds + \int_0^t b(s, T_k, x) dX_s \right.$$
$$\left. + \int_0^t \int_{\mathcal{I}} c(s, T_k, x; y) \mu^L(ds, dy) \right). \tag{6}$$

Remark 3.1. Specifying the dynamics of G in this way, we allow for two kinds of jumps: the jumps caused by market forces, represented by the Lévy process X, and the jumps caused by defaults in the portfolio, represented through the aggregate loss process L. The latter allows for direct contagion effects; when $\Delta L_t \neq 0$, Assumption (**A5**) below gives that $\Delta G(t, T_k, x) = c(t, T_k, x; \Delta L_t)$.

Remark 3.2. This approach is similar in spirit to Eberlein, Grbac, and Schmidt (2012). However, note that in that paper the forward spreads are modeled, whereas here we decide to model directly the (T_k, x)-forward prices which simplifies calibration of the model. As we shall see later on, the market model framework allows a very general specification for the dynamics of the loss process. We will study an affine special case which includes contagion and provides a highly tractable framework. This is a major advantage of the market approach in contrast to the HJM framework, where the risky short rate is directly connected to the intensity of the loss process. For a detailed discussion of this issue we refer to Eberlein, Grbac, and Schmidt (2012, Section 1 and Remark 5.3).

Additionally, we make the following assumptions.

(**A3**) For all T_k there is an $\mathbb{R}$-valued function $c(s, T_k, x; y)$, which is called the *contagion* parameter and which as a function of $(s, x, y) \mapsto c(s, T_k, x; y)$ is $\mathcal{P} \otimes \mathcal{B}(\mathcal{I}) \otimes \mathcal{B}(\mathcal{I})$-measurable. We also assume

$$\sup_{s \leq T_k, x, y \in \mathcal{I}, \omega \in \Omega} |c(s, T_k, x; y)| < \infty$$

and $c(s, T_k, x; y) = 0$ for $s > T_k$.

(**A4**) For all T_k there is an $\mathbb{R}_+^d$-valued function $b(s, T_k, x)$, which as a function of $(s, x) \mapsto b(s, T_k, x)$ is $\mathcal{P} \otimes \mathcal{B}(\mathcal{I})$-measurable. Moreover,

$$\sup_{s \leq T_k, x \in \mathcal{I}, \omega \in \Omega} b(s, T_k, x) \leq \tilde{C},$$

where $\tilde{C} > 0$ is some constant. If $s > T_k$, then $b(s, T_k, x) = 0$.

(**A5**) $[L, X]_t = 0$ for all $t \geq 0$.

The drift term $a(\cdot, T_k, \cdot)$, for every T_k, is an $\mathbb{R}$-valued, $\mathcal{O} \otimes \mathcal{B}(\mathcal{I})$-measurable process such that $a(s, T_k, x) = 0$, for $s > T_k$, that will be specified later.

The technical assumptions (**A3**) and (**A4**) ensure measurability of the subsequent operations. Assumption (**A5**) states that jumps in L influence F only through c and not via a direct dependence of L and X, which is natural from a modeling point of view.

3.2. **Absence of arbitrage.** It is well known that the market of (T_k, x)-bonds is free of arbitrage, if for each $k, i = 2, \ldots, n$ the process

$$\left(\frac{P(t, T_k, x)}{P(t, T_i)} \right)_{0 \leq t \leq T_i \wedge T_{k-1}} \tag{7}$$

is a $\mathbb{Q}_{T_i}$-local martingale. The forward measure $\mathbb{Q}_{T_k}$ is defined on $(\Omega, \mathcal{G}_{T_k})$ by its Radon-Nikodym derivative with respect to the terminal forward measure $\mathbb{Q}_{T_n} = \mathbb{Q}^*$, i.e.

$$\left. \frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T_n}} \right|_{\mathcal{G}_t} = \frac{P(0, T_n)}{P(0, T_k)} \frac{P(t, T_k)}{P(t, T_n)}. \tag{8}$$

We assume that this density has the following expression as stochastic exponential

$$\left. \frac{\mathrm{d}\mathbb{Q}_{T_k}}{\mathrm{d}\mathbb{Q}_{T_n}} \right|_{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^{\cdot} \alpha(s, T_k) dW_s + \int_0^{\cdot} \int_{\mathbb{R}^d} (\beta(s, T_k, y) - 1)(\mu - \nu)(ds, dy) \right),$$

where $\alpha \in L(W)$ and $\beta \in G_{\mathrm{loc}}(\mu)$. Moreover, under $\mathbb{Q}_{T_k}$ the process

$$W_t^{T_k} := W_t - \int_0^t \alpha(s, T_k) \mathrm{d}s \tag{9}$$

is a d-dimensional standard Brownian motion and

$$\nu^{T_k}(\mathrm{d}s, \mathrm{d}y) := \beta(s, T_k, y)\nu(\mathrm{d}s, \mathrm{d}y) = F_s^{T_k}(\mathrm{d}y)\mathrm{d}s, \tag{10}$$

is the compensator of μ. For a proof in the setting studied here, see Theorems III.3.24,III.5.19, III.5.35 in Jacod and Shiryaev (2003).

The compensator of the random measure μ^L of the jumps in the loss process under $\mathbb{Q}_{T_k}$ is denoted by $\nu^{L,T_k}(\mathrm{d}t, \mathrm{d}x) = F_t^{L,T_k}(\mathrm{d}x)\mathrm{d}t$.

Set

$$D(t, T_k, x) := a(t, T_k, x) + \frac{1}{2} \| b(t, T_k, x) \|^2 \tag{11}$$

$$+ \left\langle b(t, T_k, x), \alpha(t, T_k) \right\rangle$$

$$+ \int_{\mathbb{R}^d} \left(e^{\langle b(t, T_k, x), y \rangle} - 1 - \left\langle b(t, T_k, x), y \right\rangle \beta(t, T_k, y)^{-1} \right) F_t^{T_k}(\mathrm{d}y).$$

The following result gives conditions which provide an arbitrage-free specification of the studied model. More precisely, we obtain necessary and sufficient conditions for (7) to hold, i.e. conditions for the (T_k, x)-forward price process $F(\cdot, T_k, x)$ being a local martingale under the forward measure $\mathbb{Q}_{T_k}$, for $k = 2, \ldots, n$.

Theorem 3.3. *Assume that* (**A1**)*–*(**A5**) *are in force and let* $k \in \{2, \ldots, n\}$, $x \in \mathcal{I}$. *Then the process* $(F(t, T_k, x))_{0 \leq t \leq T_{k-1}}$ *given by* (5) *is a* $\mathbb{Q}_{T_k}$*-local martingale if and only if*

$$D(t, T_k, x) = \lambda^{T_k}(t, x) - \int_{\mathcal{I}} \left(e^{c(t, T_k, x; y)} - 1 \right) \mathbf{1}_{\{L_{t-} + y \leq x\}} F^{L, T_k}(t, dy) \tag{12}$$

on the set $\{L_t \leq x\}$, $\lambda^1 \otimes \mathbb{Q}_{T_k}$*-a.s. where* λ^1 *denotes the Lebesgue measure on* $\mathbb{R}$ *and* $\lambda^{T_k}(t, x) := F^{L, T_k}(t, (x - L_t, 1] \cap \mathcal{I})$.

Proof: The proof follows the same lines as the proof of Theorem 5.2 in Eberlein, Grbac, and Schmidt (2012). Since the specification of the dynamics of the (T_k, x)-forward prices is different in this paper, we include a proof here for the sake of completeness.

Using integration by parts yields

$$\mathrm{d}F(t, T_k, x) = G(t-, T_k, x)\mathrm{d}\mathbf{1}_{\{L_t \leq x\}} + \mathbf{1}_{\{L_{t-} \leq x\}}\mathrm{d}G(t, T_k, x) + \mathrm{d}\left[G(\cdot, T_k, x), \mathbf{1}_{\{L. \leq x\}}\right]_t$$
$$=: (1') + (2') + (3').$$

For the first summand $(1')$ note that analogously to (1),

$$M_t^{x, T_k} := \mathbf{1}_{\{L_t \leq x\}} + \int_0^t \mathbf{1}_{\{L_s \leq x\}}\lambda^{T_k}(s, x)\mathrm{d}s \tag{13}$$

is a $\mathbb{Q}_{T_k}$-martingale with $\lambda^{T_k}(t, x) := F^{L, T_k}(t, (x - L_t, 1] \cap \mathcal{I})$. This yields

$$\mathrm{d}\mathbf{1}_{\{L_t \leq x\}} = \mathrm{d}M_t^{x, T_k} - \mathbf{1}_{\{L_t \leq x\}}\lambda^{T_k}(t, x)\mathrm{d}t$$
$$= \mathbf{1}_{\{L_{t-} \leq x\}}\left(\mathrm{d}M_t^{x, T_k} - \lambda^{T_k}(t, x)\mathrm{d}t\right),$$

since $\mathrm{d}M_t^{x, T_k} = \mathbf{1}_{\{L_{t-} \leq x\}}\mathrm{d}M_t^{x, T_k}$. Hence,

$$(1') = G(t-, T_k, x)\mathbf{1}_{\{L_{t-} \leq x\}}\left(\mathrm{d}M_t^{x, T_k} - \lambda^{T_k}(t, x)\mathrm{d}t\right)$$
$$= F(t-, T_k, x)\left(\mathrm{d}M_t^{x, T_k} - \lambda^{T_k}(t, x)\mathrm{d}t\right).$$

Regarding $(2')$, we have

$$\mathrm{d}G(t, T_k, x) = G(t-, T_k, x)\left(\left(a(t, T_k, x) + \frac{1}{2}\parallel b(t, T_k, x)\parallel^2\right)\mathrm{d}t\right.$$
$$+ \int_{\mathcal{I}}\left(e^{c(t, T_k, x; y)} - 1\right)\mu^L(\mathrm{d}t, \mathrm{d}y)$$
$$+ \int_{\mathbb{R}^d}\left(e^{\langle b(t, T_k, x), \tilde{x}\rangle} - 1 - \langle b(t, T_k, x), \tilde{x}\rangle\right)\mu(\mathrm{d}t, \mathrm{d}\tilde{x})$$
$$\left.+ \int_{\mathbb{R}^d}\langle b(t, T_k, x), \tilde{x}\rangle(\mu - \nu)(\mathrm{d}t, \mathrm{d}\tilde{x}) + b(t, T_k, x)\mathrm{d}W_t\right),$$

where we have used the integration by parts formula, Assumption (**A5**) and Itô's formula for semimartingales. Now we use the representation of X under $\mathbb{Q}_{T_k}$ given via (9) and (10) which

leads to

$$
\begin{aligned}
\mathrm{d}G(t, T_k, x) = G(t-, T_k, x) \Bigg(& \Big(\big(a(t, T_k, x) + \frac{1}{2} \parallel b(t, T_k, x) \parallel^2 + \langle b(t, T_k, x), \alpha(t, T_k) \rangle \big) \mathrm{d}t \\
& + b(t, T_k, x), \mathrm{d}W_t^{T_k} \\
& + \int_{\mathcal{I}} \Big(e^{c(t, T_k, x; y)} - 1 \Big) F_t^{L, T_k}(\mathrm{d}y) \mathrm{d}t \\
& + \int_{\mathcal{I}} \Big(e^{c(t, T_k, x; y)} - 1 \Big) (\mu^L - \nu^{L, T_k})(\mathrm{d}t, \mathrm{d}y) \\
& + \int_{\mathbb{R}^d} (e^{\langle b(t, T_k, x), \tilde{x} \rangle} - 1)(\mu - \nu^{T_k})(\mathrm{d}t, \mathrm{d}\tilde{x}) \\
& + \int_{\mathbb{R}^d} \Big(e^{\langle b(t, T_k, x), \tilde{x} \rangle} - 1 - \langle b(t, T_k, x), \tilde{x} \rangle \Big) \beta^{-1}(t, T_k, \tilde{x}) F_t^{T_k}(\mathrm{d}\tilde{x}) \mathrm{d}t \Bigg).
\end{aligned}
\tag{14}
$$

Finally consider $(3')$. We have

$$
\big[G(\cdot, T_k, x), \mathbf{1}_{\{L \leq x\}} \big]_t = \sum_{s \leq t} \Delta \mathbf{1}_{\{L_s \leq x\}} \Delta G(s, T_k, x).
$$

Since $\mathbf{1}_{\{L \leq x\}}$ drops from 1 to 0 as L crosses the barrier x,

$$
\begin{aligned}
\Delta \mathbf{1}_{\{L_s \leq x\}} &= -\mathbf{1}_{\{L_{s-} \leq x, L_s > x\}} = -\mathbf{1}_{\{L_{s-} \leq x, L_{s-} + \Delta L_s > x\}} \\
&= -\int_{\mathcal{I}} \mathbf{1}_{\{L_{s-} \leq x\}} \mathbf{1}_{\{L_{s-} + y > x\}} \mu^L(\{s\}, \mathrm{d}y).
\end{aligned}
$$

Using this together with (14) and Assumption (**A5**) leads to

$$
\mathrm{d} \big[G(\cdot, T_k, x), \mathbf{1}_{\{L \leq x\}} \big]_t = -G(t-, T_k, x) \int_{\mathcal{I}} \mathbf{1}_{\{L_{t-} \leq x\}} \mathbf{1}_{\{L_{t-} + y > x\}} \Big(e^{c(t, T_k, x; y)} - 1 \Big) \mu^L(\mathrm{d}t, \mathrm{d}y),
$$

which concludes the proof. $\qquad\square$

4. AN AFFINE SPECIFICATION

Up to now, the presented framework was very general. In this section, an affine factor model is studied in more detail. Affine factor models present a subclass of Markovian factor models which are used frequently in practice because of their high degree of tractability. The following section on calibration will show that a simple two-factor affine model shows an excellent fit to market data.

For simplicity, we study affine diffusion models only, i.e. affine models driven by a Brownian motion. The extension to Lévy processes as drivers can be done following the path laid out here. Consider a d-dimensional Brownian motion W and let μ and σ be the functions from $\mathcal{Z}$ to $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$ satisfying

$$
\mu(z) = \mu_0 + \sum_{i=1}^{d} \mu_i z_i, \quad \frac{1}{2}\sigma(z)^{\top}\sigma(z) = \nu_0 + \sum_{i=1}^{d} \nu_i z_i
$$

for some vectors $\mu_i \in \mathbb{R}^d$ and matrices $\nu_i \in \mathbb{R}^{d \times d}, i = 0, \ldots, d$. Note that with $\sigma(z)^{\top}\sigma(z)$ also ν_i, $i = 0, \ldots, d$ are symmetric matrices. We assume that for all $z \in \mathcal{Z}$, $Z = Z^z$ is the continuous, unique strong solution of

$$
dZ_t = \mu(Z_t)dt + \sigma(Z_t)^{\top}dW_t, \quad Z_0 = z \in \mathcal{Z}.
$$

We specify G in the following variant of an affine specification. Let

$$G(t, T_k, x) = \exp\left(A(t, T_k, x) + B(t, T_k, x)^\top Z_t + \int_0^t \int_{\mathcal{I}} c(s, T_k, x, L_{s-}; y) \mu^L(ds, dy) \right. \tag{15}$$
$$\left. + \int_0^t d(s, T_k, x, L_{s-}) ds \right),$$

for $t \leq T_k$. Here A, B, c and d are deterministic functions which have to be specified in an appropriate way to guarantee absence of arbitrage. Note that while G has an (exponential)-affine dependence on Z its dependence on the loss process via the function c is much more general. It is this extension of the affine framework which allows to introduce contagion in an arbitrage-free model as we will show in the sequel.

Finally, we assume that the compensator of the loss process has the following affine structure: assume that $F_t^L(dy) = m(t, L_{t-}, Z_t, dy)$ where

$$m(t, l, z, dy) := m_0(t, l, dy) + \sum_{i=1}^d m_i(t, l, dy) z_i. \tag{16}$$

We assume that $m(t, l, z, dy)$ is a Borel-measure, in particular $m(\cdot, A) \geq 0$. This gives a restriction on m_i depending on the state space: consider, for example, the state space $\mathcal{Z} = \mathbb{R}^{d_1} \times (\mathbb{R}_{\geq 0})^{d_2}$. This implies that $m_1 = \cdots = m_{d_1} \equiv 0$, as otherwise m would attain negative values. We assume that $m_i(t, l, z, \mathcal{I}) < \infty$, $i = 1, \ldots, d$ (finite activity).

All appearing functions are assumed to be càdlàg in each variable. Furthermore we assume a flat interest rate structure, i.e. $P(t, T_k) = 1$ for all $0 \leq t \leq T_k$, so that the $\mathbb{Q}_{T_k}$-forward measures coincide. This can be extended in a straightforward manner to the setup where risk-free bond prices are independent of X and L, as we only use that the semimartingale characteristics of the driving processes coincide under all forward measures.

Proposition 4.1. *Assume G is given by (15). Moreover, assume that*

$$d_i(t, T_k, x, l) = m_i(t, l, \mathcal{I}) - \int_{\mathcal{I}} e^{c(t, T_k, x, l; y)} \mathbf{1}_{\{y \leq x+l\}} m_i(t, l, dy), \tag{17}$$

$i = 0, \ldots, d$. *If A and B satisfy the following system of differential equations*

$$-\partial_t A(t, T_k, x) = B(t, T_k, x)^\top \mu_0 + \frac{1}{2} B(t, T_k, x)^\top \nu_0 B(t, T_k, x) \tag{18}$$

$$-\partial_t B(t, T_k, x)_j = B(t, T_k, x)^\top \mu_j + \frac{1}{2} B(t, T_k, x)^\top \nu_j B(t, T_k, x) \tag{19}$$

then the model given by (5) is free of arbitrage.

Proof: In order to verify absence of arbitrage we analyze the drift condition (12). As in the proof of Theorem 3.3 we obtain the dynamics of the forward rates

$$\frac{dF(t, T_k, x)}{F(t-, T_k, x)} = \left(-\lambda(t, x) + D(t, T_k, x) + \int_{\mathcal{I}} \left(e^{c(t, T_k, x, L_{t-}; y)} - 1 \right) F_t^L(dy) \right) dt$$
$$- \int_{\mathcal{I}} \left(e^{c(t, T_k, x, L_{t-}; y)} - 1 \right) \mathbf{1}_{\{L_t + y > x\}} F_t^L(dy) dt + d\tilde{M}_t$$

where $D(t, T_k, x)$ is given by

$$D(t, T_k, x) = \partial_t A(t, T_k, x) + \partial_t B(t, T_k, x) Z_t$$
$$+ \langle B(t, T_k, x), \mu(Z_t) \rangle + \frac{1}{2} \parallel B(t, T_k, x) \sigma(Z_t) \parallel^2 + d(t, T_k, x, L_{t-})$$

and $\tilde{M}$ is a local martingale. In order to obtain absence of arbitrage, forward price processes need to be local martingales and hence the drift term needs to vanish. Note that

$$\lambda(t, x) - \int_{\mathcal{I}} \left(e^{c(t, T_k, x, L_{t-}; y)} - 1 \right) \mathbf{1}_{\{y \leq x + L_{t-}\}} F_t^L(\mathrm{d}y)$$

$$= \lambda(t, 0) - \sum_{i=0}^d (Z_t)_i \int_{\mathcal{I}} e^{c(t, T_k, x, L_{t-}; y)} \mathbf{1}_{\{y \leq x + L_{t-}\}} m_i(t, L_{t-}, \mathrm{d}y)$$

$$= -\sum_{i=0}^d (Z_t)_i \left(\int_{\mathcal{I}} e^{c(t, T_k, x, L_{t-}; y)} \mathbf{1}_{\{y \leq x + L_{t-}\}} - 1 \right) m_i(t, L_{t-}, dy) \qquad (20)$$

where we used $(Z_t)_0 = 1$ to simplify the notation. In the following we consider the drift for all possible values $L_{t-} = l \in \mathcal{I}$ and $Z_t = z \in \mathcal{Z}$. Observe that (20) at values $L_{t-} = l$ and $Z_t = z$ reads

$$-\sum_{i=0}^d z_i \left(\int_{\mathcal{I}} e^{c(t, T_k, x, l; y)} \mathbf{1}_{\{y \leq x + l\}} m_i(t, l, dy) - m_i(t, l, \mathcal{I}) \right) = \sum_{i=0}^d z_i d_i(t, T_k, x, l)$$

and the equality is implied by the assumption (17). On the other hand, the remaining terms of $D(t, T_k, x)$, considered at values $L_{t-} = l$ and $Z_t = z$ are given by

$$\partial_t A(t, T_k, x) + \partial_t B(t, T_k, x)^\top z + \sum_{i=0}^d B(t, T_k, x)^\top \mu_i z_i$$

$$+ \frac{1}{2} \sum_{i=0}^d z_i B(t, T_k, x)^\top \nu_i B(t, T_k, x).$$

Observe that this term is zero if the following two equations are satisfied:

$$-\partial_t A(t, T_k, x) = B(t, T_k, x)^\top \mu_0 + \frac{1}{2} B(t, T_k, x)^\top \nu_0 B(t, T_k, x)$$

$$-\partial_t B(t, T_k, x)_i = B(t, T_k, x)^\top \mu_i + \frac{1}{2} B(t, T_k, x)^\top \nu_i B(t, T_k, x),$$

and hence, the drift term in the dynamics of the forward price vanishes. $\qquad \square$

It is important to note that, in spirit of the market model approach, we do not have to satisfy boundary conditions for the Riccati equations. Of course one typically would nevertheless choose $B(T_k, T_k, x)_i = A(T_k, T_k, x) = 0$.

If the dependence on the loss process is such that d_i does no longer depend on l, then we may study the following, simpler model. This is in particular suitable for the calibration exercise where no default appears in the data. With no observed default it would be unreasonable to extrapolate a specific functional dependence on the losses, such that it seems reasonable to work in this simplified setup.

Corollary 4.2. *Assume G is given by (15). Moreover, assume that*

$$d_i(t, T_k, x) = \int_{\mathcal{I}} e^{c(t, T_k, x; y)} \mathbf{1}_{\{y \leq x + l\}} m_i(t, l, dy) - m_i(t, l, \mathcal{I}),$$

$i = 0, \ldots, d$. *If A and B satisfy the following system of differential equations*

$$-\partial_t A(t, T_k, x) = B(t, T_k, x)^\top \mu_0 + \frac{1}{2} B(t, T_k, x)^\top \nu_0 B(t, T_k, x)$$
$$+ d_0(t, T_k, x),$$

$$-\partial_t B(t, T_k, x)_j = B(t, T_k, x)^\top \mu_j + \frac{1}{2} B(t, T_k, x)^\top \nu_j B(t, T_k, x)$$
$$+ d_i(t, T_k, x),$$

then the model given by (5) is free of arbitrage.

5. Pricing

The aim of this section is to discuss the pricing of credit portfolio derivatives in the market model framework. A single tranche CDO (STCDO) is a typical example of such a derivative and it is a standard market instrument for investment in a pool of credits. For a detailed overview on credit portfolio risk tranching we refer to Chapter 3 of this book.

We consider here a STCDO which is specified as follows: $0 < T_1 < \cdots < \cdots T_m$ denotes a collection of future payment dates and $x_1 < x_2$ in $[0, 1]$ are a so-called lower and upper detachment points. The fixed spread is denoted by S. An investor in the STCDO receives premium payments in exchange for payments at defaults: the *premium leg* consists of a series of payments equal to

$$S[(x_2 - L_{T_k})^+ - (x_1 - L_{T_k})^+] =: Sf(L_{T_k}), \tag{21}$$

received at T_k, $k = 1, \ldots, m - 1$. The function f is defined by

$$f(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{x_1}^{x_2} \mathbf{1}_{\{x \leq y\}} \mathrm{d}y. \tag{22}$$

The *default leg* consists of a series of payments at tenor dates T_{k+1}, $k = 1, \ldots, m - 1$, given by

$$f(L_{T_k}) - f(L_{T_{k+1}}). \tag{23}$$

This payment is non-zero only if $\Delta L_t \neq 0$ for some $t \in (T_k, T_{k+1}]$. In the literature alternative payment schemes can be found as well (see Filipović, Overbeck, and Schmidt (2011), for example). Note that

$$(23) = \int_{x_1}^{x_2} \Big[\mathbf{1}_{\{L_{T_k} \leq y\}} - \mathbf{1}_{\{L_{T_{k+1}} \leq y\}} \Big] \mathrm{d}y = \int_{x_1}^{x_2} \mathbf{1}_{\{L_{T_k} \leq y, L_{T_{k+1}} > y\}} \mathrm{d}y.$$

Similarly to Eberlein, Grbac, and Schmidt (2012, Section 8.1), it is convenient to replace the forward measures $\mathbb{Q}_{T_k}$ by so-called (T_k, x)-forward measures. In order to do so, we assume henceforth that the processes $(F(t, T_k, x))_{0 \leq t \leq T_{k-1}}$, are *true* $\mathbb{Q}_{T_k}$-martingales for every $k = 2, \ldots, n$ and $x \in \mathcal{I}$. Moreover, $(F(t, T_1, x))_{0 \leq t \leq T_1}$ is a *true* $\mathbb{Q}_{T_1}$-martingale. For $x \in [0, 1]$ and $k \in \{1, \ldots, m - 1\}$, the (T_k, x)-*forward measure* $\mathbb{Q}_{T_k, x}$ on $(\Omega, \mathcal{G}_{T_k})$ is defined by its Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}_{T_k, x}}{\mathrm{d}\mathbb{Q}_{T_k}} := \frac{F(T_{k-1}, T_k, x)}{\mathbb{E}_{\mathbb{Q}_{T_k}}[F(T_{k-1}, T_k, x)]} = \frac{F(T_{k-1}, T_k, x)}{F(0, T_k, x)}$$

and the corresponding density process is given by

$$\frac{\mathrm{d}\mathbb{Q}_{T_k, x}}{\mathrm{d}\mathbb{Q}_{T_k}}\bigg|_{\mathcal{G}_t} = \frac{F(t, T_k, x)}{F(0, T_k, x)}.$$

Note that $\mathbb{Q}_{T_k, x}$ is not equivalent to $\mathbb{Q}_{T_k}$ if $\mathbb{Q}_{T_k}(L_{T_{k-1}} > x) > 0$, but it is absolutely continuous with respect to $\mathbb{Q}_{T_k}$. Similar measure changes – yielding so-called defaultable forward measures

– have been introduced and applied in the pricing of credit derivatives in Schönbucher (2000) and Eberlein, Kluge, and Schönbucher (2006).

Proposition 5.1. *The value of the STCDO at time $t \le T_1$ is*

$$\pi^{STCDO}(t,S) = \int_{x_1}^{x_2} \Big(\sum_{k=1}^{m} c_k P(t,T_k,y) - \sum_{k=1}^{m-1} P(t,T_{k+1},y) v(t,T_{k+1},y) \Big) dy, \qquad (24)$$

where $c_1 = S$, $c_k = 1 + S$, for $2 \le k \le m-1$, $c_m = 1$ and

$$v(t,T_{k+1},y) := \mathbb{E}_{\mathbb{Q}_{T_{k+1},x}} \big(G(T_k,T_{k+1},x)^{-1} | \mathcal{G}_t \big) \qquad (25)$$

with $G(\cdot, T_{k+1}, x)$ specified in (6). The STCDO spread S_t^* at time t, i. e. the spread which makes the value of the STCDO at time t equal to zero, is given by

$$S_t^* = \frac{\sum_{k=1}^{m-1} \int_{x_1}^{x_2} P(t,T_{k+1},y) \, (v(t,T_{k+1},y) - 1) \, dy}{\sum_{k=1}^{m-1} \int_{x_1}^{x_2} P(t,T_k,y) dy}. \qquad (26)$$

Proof: The premium $Sf(L_{T_k})$ is paid at times $T_1, \ldots, T_{m-1}$ and thus, the value of the premium leg at time t equals

$$\sum_{k=1}^{m-1} P(t,T_k) \mathbb{E}_{\mathbb{Q}_{T_k}} (Sf(L_{T_k}) | \mathcal{G}_t) = \sum_{k=1}^{m-1} S P(t,T_k) \int_{x_1}^{x_2} \mathbb{E}_{\mathbb{Q}_{T_k}} (\mathbf{1}_{\{L_{T_k} \le y\}} | \mathcal{G}_t) dy$$

$$= S \sum_{k=1}^{m-1} \int_{x_1}^{x_2} P(t,T_k,y) dy,$$

where we have used

$$P(t,T_{k+1},x) = P(t,T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \big(\mathbf{1}_{\{L_{T_{k+1}} \le x\}} | \mathcal{G}_t \big) \qquad (27)$$

for the last equality.

On the other side, the default payments are given by $f(L_{T_k}) - f(L_{T_{k+1}})$ at tenor dates T_{k+1}, $k = 1, \ldots, m-1$. For each k the value at time t of this payment is

$$P(t,T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} (f(L_{T_k}) - f(L_{T_{k+1}}) | \mathcal{G}_t) \qquad (28)$$

$$= P(t,T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \Big(\int_{x_1}^{x_2} \big(\mathbf{1}_{\{L_{T_k} \le y\}} - \mathbf{1}_{\{L_{T_{k+1}} \le y\}} \big) dy \, | \, \mathcal{G}_t \Big)$$

$$= \int_{x_1}^{x_2} P(t,T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \big(\mathbf{1}_{\{L_{T_k} \le y\}} - \mathbf{1}_{\{L_{T_{k+1}} \le y\}} \, | \, \mathcal{G}_t \big) dy$$

$$= \int_{x_1}^{x_2} \Big(P(t,T_{k+1}) \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \big(\mathbf{1}_{\{L_{T_k} \le y\}} \, | \, \mathcal{G}_t \big) - P(t,T_{k+1},y) \Big) dy.$$

It remains to calculate the conditional expectation $\mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \big(\mathbf{1}_{\{L_{T_k} \le y\}} \, | \, \mathcal{G}_t \big)$. We have

$$\mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \big(\mathbf{1}_{\{L_{T_k} \le y\}} | \mathcal{G}_t \big)$$

$$= \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \Big(\mathbf{1}_{\{L_{T_k} \le y\}} G(T_k,T_{k+1},y) G(T_k,T_{k+1},y)^{-1} \Big| \mathcal{G}_t \Big)$$

$$= \mathbb{E}_{\mathbb{Q}_{T_{k+1}}} \Big(F(T_k,T_{k+1},y) G(T_k,T_{k+1},y)^{-1} \Big| \mathcal{G}_t \Big)$$

$$= F(t,T_{k+1},y) \mathbb{E}_{\mathbb{Q}_{T_{k+1},y}} \Big(G(T_k,T_{k+1},y)^{-1} \Big| \mathcal{G}_t \Big),$$

where the last equality follows by changing the measure to $\mathbb{Q}_{T_{k+1},x}$. Denoting

$$v(t, T_{k+1}, y) := \mathbb{E}_{\mathbb{Q}_{T_{k+1},x}}\left(G(T_k, T_{k+1}, y)^{-1}\big|\mathcal{G}_t\right), \tag{29}$$

the value of the default leg at time t is given by

$$\sum_{k=1}^{m-1} \int_{x_1}^{x_2} (P(t, T_{k+1}, y)v(t, T_{k+1}, y) - P(t, T_{k+1}, y))\, dy. \tag{30}$$

Finally, the value of the STCDO is the difference of the time-t values of the payment leg and of the default leg. Thus, we obtain (24). Solving $\pi^{STCDO}(t, S) = 0$ in S yields the spread S_t^*. $\qquad\square$

In Section 4 the constant risk-free term structure assumption is imposed, i.e. it is assumed $P(t, T_k) = 1$, for every T_k and $t \leq T_k$. In this case the previous result takes the following form.

Corollary 5.2. *Under the constant risk-free term structure assumption, the value at time $t \leq T_1$ of the STCDO is given by*

$$\pi^{STCDO}(t, S) = \int_{x_1}^{x_2} \left(\sum_{k=1}^{m-1} SF(t, T_k, y) + F(t, T_m, y) - F(t, T_1, y)\right) dy. \tag{31}$$

The STCDO spread S_t^* at time t is equal to

$$S_t^* = \frac{\int_{x_1}^{x_2}(F(t, T_1, y) - F(t, T_m, y))\,dy}{\sum_{k=1}^{m-1}\int_{x_1}^{x_2} F(t, T_k, y)\,dy}.$$

Proof: The result follows by inspection of the previous proof whilst noting that

$$\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}\left(\mathbf{1}_{\{L_{T_k}\leq y\}}|\mathcal{G}_t\right) = \mathbb{E}_{\mathbb{Q}^*}\left(\mathbf{1}_{\{L_{T_k}\leq y\}}|\mathcal{G}_t\right) = P(t, T_k, y) = F(t, T_k, y),$$

since $P(t, T_k) = 1$ and all forward measures coincide. Hence,

$$v(t, T_{k+1}, y) = \frac{F(t, T_k, y)}{F(t, T_{k+1}, y)}$$

and the corollary is proved. $\qquad\square$

We conclude this section by studying an option on the STCDO defined above. This option gives the right to enter into such a contract at time T_1 at a pre-specified spread S. This is equivalent to an European call on the STCDO with payoff

$$\left(\pi^{STCDO}(T_1, S)\right)^+$$

at T_1.

The assumption of the constant risk-free term structure is still in force. We further assume $G(0, T_k, y)$, $a(t, T_k, y)$, $b(t, T_k, y)$ and $c(t, T_k, y; z)$ are constant in y between x_1 and x_2. For simplicity we denote $a(t, T_k, y) = a(t, T_k, x_1)$ by $a(t, T_k)$ and similarly for the other quantities.

Proposition 5.3. *The value of the option $\pi^{call}(t, S)$ at time $t \leq T_1$ is*

$$\pi^{call}(t, S) = \mathbb{E}_{\mathbb{Q}^*}\left(f(L_{T_1})\left(\tilde{d}_1 + \sum_{k=2}^{m} \tilde{d}_k \exp\left(\int_0^{T_1} a(t, T_k)dt\right.\right.\right.$$

$$\left.\left.\left. + \int_0^{T_1} b(t, T_k)dX_t + \int_0^{T_1}\int_{\mathcal{I}} c(t, T_k; z)\mu^L(dt, dz)\right)\right)^+\bigg|\mathcal{G}_t\right),$$

where $\tilde{d}_1 = (S-1)G(0,T_1)$, $\tilde{d}_k = SG(0,T_k)$, for $2 \leq k \leq m-1$, and $\tilde{d}_m = G(0,T_m)$. If in addition X, $a(t,T_k)$ and $b(t,T_k)$ are conditionally independent of L given $\mathcal{G}_t$, then

$$\pi^{call}(t,S) = \mathbb{E}_{\mathbb{Q}^*}\left(f(L_{T_1})\right) \mathbb{E}_{\mathbb{Q}^*}\left(\left(\left(\tilde{d}_1 + \sum_{k=2}^{m} \tilde{d}_k \exp\left(\int_0^{T_1} a(t,T_k)\mathrm{d}t\right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \int_0^{T_1} b(t,T_k)dX_t + \int_0^{T_1}\int_{\mathcal{I}} c(t,T_k;z)\mu^L(dt,dz)\right)\right)^+ \Big| \mathcal{G}_t\right),$$

where

$$\mathbb{E}_{\mathbb{Q}^*}\left(f(L_{T_1})|\mathcal{G}_t\right) = x_2\mathbb{Q}^*\left(L_{T_1} \leq x_2|\mathcal{G}_t\right) - x_1\mathbb{Q}^*\left(L_{T_1} \leq x_1|\mathcal{G}_t\right)$$
$$- \mathbb{E}_{\mathbb{Q}^*}\left(L_{T_1}\mathbf{1}_{\{x_1 < L_{T_1} \leq x_2\}}|\mathcal{G}_t\right).$$

Proof: The value of the option at time $t \leq T_1$ is given by the conditional expectation

$$\pi^{call}(t,S) = \mathbb{E}_{\mathbb{Q}^*}\left(\left(\pi^{STCDO}(T_1,S)\right)^+|\mathcal{G}_t\right)$$
$$= \mathbb{E}_{\mathbb{Q}^*}\left(\left(\int_{x_1}^{x_2}\left(\sum_{k=1}^{m-1} SF(T_1,T_k,y) + F(T_1,T_m,y) - F(T_1,T_1,y)\right)\mathrm{d}y\right)^+ \Big| \mathcal{G}_t\right),$$

where we have used Corollary 5.2. Now the result follows by inserting (5) and (6).

The second result is obvious by conditional independence and definition of f. $\qquad\square$

6. CALIBRATION

This section is devoted to the detailed description of the calibration of a two-factor affine diffusion model. The method which turned out to provide the best results utilizes the EM-algorithm together with an unscented Kalman filter. In contrast to typical calibration approaches, see Cont, Deguest, and Kan (2010) where the models are fit to one or two single days, we calibrate the model to the full observation period which encompasses four years of observed data from the iTraxx Europe. The model is able to provide a very good fit throughout all tranches and maturities. In the calibration methodology we follow the scheme suggested in Eksi and Filipović (2012). As discussed previously, the considered affine market model entails a direct contagion effects which improves the calibration results.

The data consists of implied *zero-coupon spreads* of the iTraxx Europe from 30 August 2006 to 3 August 2010. In contrast to Eberlein, Grbac, and Schmidt (2012) we incorporate also data from 2006 and 2007, which is characterized by steady spread movements at an extremely low level. It will turn out, that the fit to this time period is not as good as the fit to the more volatile period starting in the year 2008. This suggests a structural break starting from the credit crisis, which is very reasonable.

The implied zero-coupon spreads are observed at detachment points $\{x_1,\ldots,x_J\}$ which equal $\{0, 0.03, 0.06, 0.09, 0.12, 0.22, 1\}$. They are obtained by computing the spread of quoted STCDO premiums over the risk-free interest yield over the same period, in our notation given by

$$R(t,\tau,j) := -\frac{1}{\tau}\log\left(\frac{1}{x_{j+1}-x_j}\int_{x_j}^{x_{j+1}} F(t,t+\tau,x)dx\right). \tag{32}$$

Here $\tau := T - t$ denotes time to maturity and the observed values are $\{3,5,7,10\}$. It is important to remark that there was no default in the underlying pool in the observation period. The realized index spreads are shown in Figure 1. With the beginning of the credit crisis volatility as well as
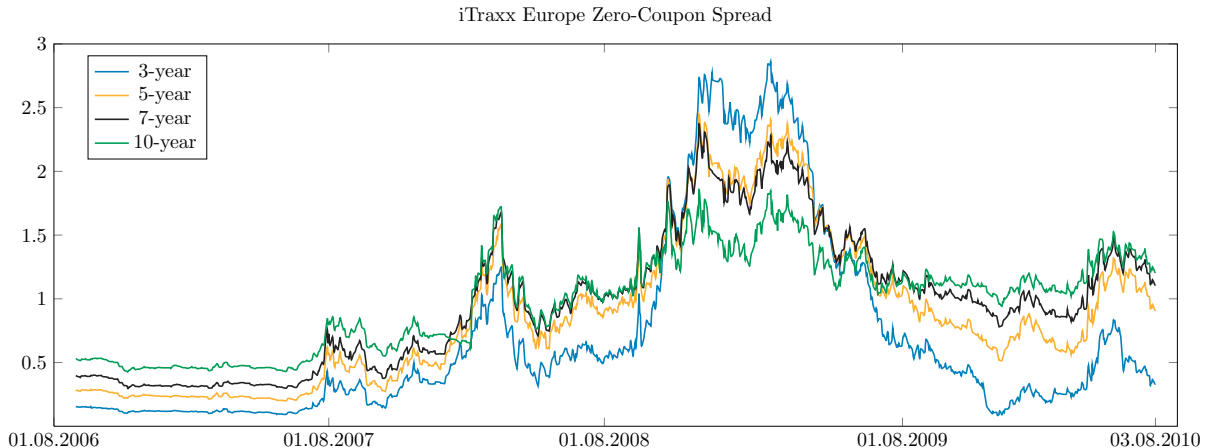
FIGURE 1. The iTraxx Europe zero-coupon index spread for the period August 2006 to August 2010. The different graphs refer to the time to maturity of 3,5,7 and 10 years.

levels of credit spreads raised to levels never seen before. After a period of stabilization, from early 2010 on the spread levels started again to rise to higher levels when the European debt crises evolved. This heterogeneous dataset makes the calibration very difficult.

Figure 2 shows STCDOs spread premiums over different maturities and tranches. It is remarkable that curves for different maturities look quite similar, which makes it plausible to capture the observed dynamics with a low number of factors. The principal component analysis performed in Eksi and Filipović (2012) reveals that two factors already explain 88.30% of the realized variance and we therefore consider a two-factor affine model.

More precisely, we consider the following two-dimensional affine diffusion Z with values in the state space $\mathcal{Z} := \mathbb{R}^+ \times \mathbb{R}^+$. We assume that Z solves the SDEs

$$dZ_t^1 = \kappa_1(Z_t^2 - Z_t^1)dt + \sigma_1\sqrt{Z_t^1}dW_t^1 \tag{33}$$

$$dZ_t^2 = \kappa_2(\theta_2 - Z_t^2)dt + \sigma_2\sqrt{Z_t^2}dW_t^2, \tag{34}$$

with $Z_0 = (z_1, z_2)^\top \in \mathcal{Z}$. Here $\kappa_1, \kappa_2, \theta_2, \sigma_1, \sigma_2$ are positive constants and W^1 and W^2 are two independent standard Brownian motions under the objective probability measure $\mathbb{P}$. In this formulation, Z^2 is a Feller square-root process and Z^1 is a non-negative process with stochastic mean reversion level Z^2.

Pricing is done under a risk-neutral measure $\mathbb{Q}^*$ and we chose a class of equivalent measure which keep the affine structure of Z. This is done by considering the market prices of risk λ^1 and λ^2, given by

$$\lambda_t^i = \frac{\lambda_i\sqrt{Z_t^i}}{\sigma_i},$$

with some constants $\lambda_1, \lambda_2 \in \mathbb{R}$. Applying Girsanov's theorem, we change to the equivalent probability measure $\mathbb{Q}^*$ where $\tilde{W}_t^i = W_t^i + \int_0^t \lambda_s^i ds$, $i = 1, 2$ are independent standard $\mathbb{Q}^*$-Brownian motions. Then, under $\mathbb{Q}^*$, Z is again affine and satisfies the following SDEs; see
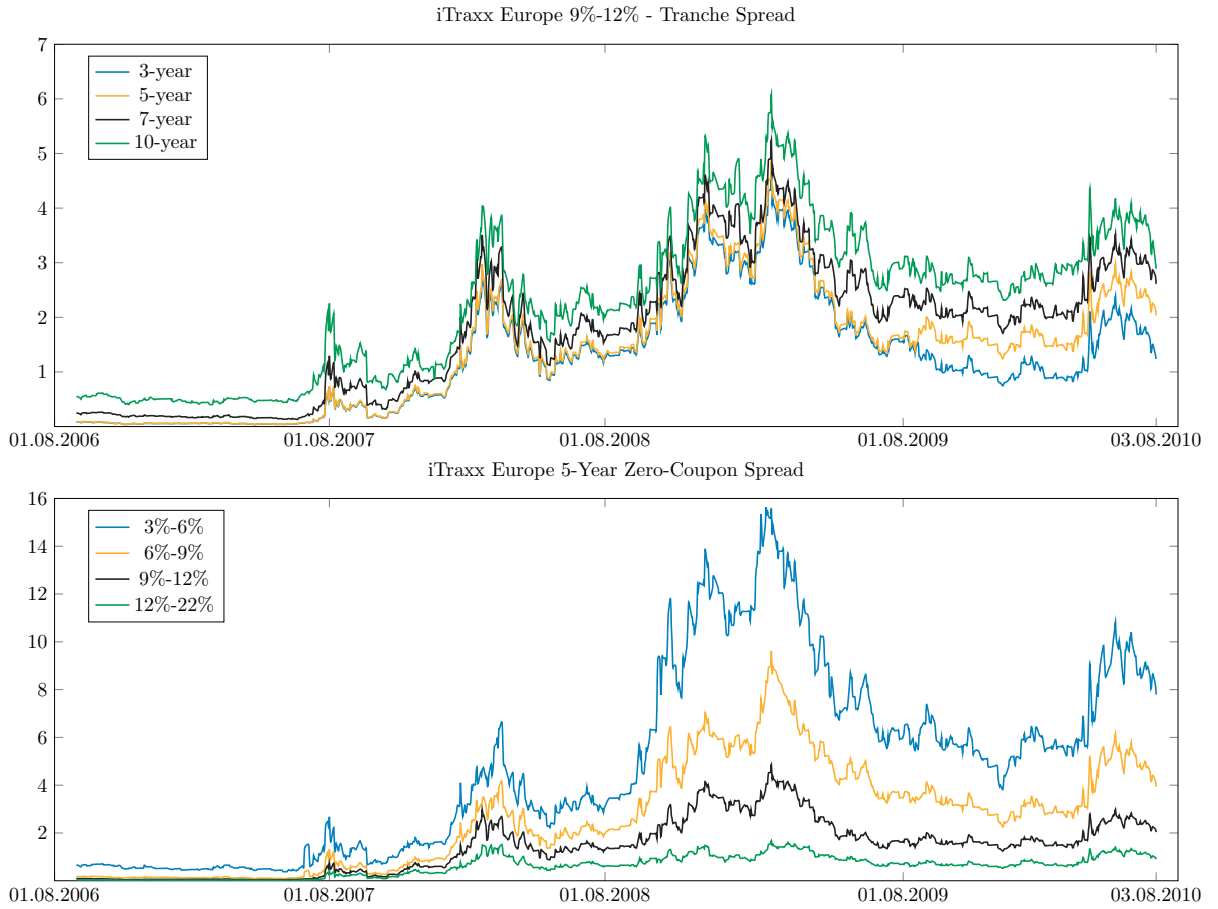
FIGURE 2. The upper graph shows the iTraxx Europe 9%-12% tranche spread from August 2006 to August 2010 for different maturities. The lower graph illustrates the iTraxx Europe tranche spreads from August 2006 to August 2010 for a fixed maturity of five years.

Cheridito, Filipović, and Kimmel (2010):

$$dZ_t^1 = (\kappa_1 + \lambda_1) \left(\frac{\kappa_1}{\kappa_1 + \lambda_1} Z_t^2 - Z_t^1 \right) dt + \sigma_1 \sqrt{Z_t^1} d\tilde{W}_t^1, \tag{35}$$

$$dZ_t^2 = (\kappa_2 + \lambda_2) \left(\frac{\kappa_2}{\kappa_2 + \lambda_2} \theta_2 - Z_t^2 \right) dt + \sigma_2 \sqrt{Z_t^2} d\tilde{W}_t^2. \tag{36}$$

This is the starting point to apply the results from Section 4. Additionally to the factor process, we need to specify the compensator of the loss process. We chose the following affine specification

$$m(t, l, z, dy) = m_0(t, l, dy) + m_1(t, l, dy)z_1,$$

where the jump distributions $m_i(t, l, \cdot)$ are chosen from the Beta family. This in turn gives

$$m(t, l, z, dy) = \frac{1}{B(a_1, b_1)} y^{a_1 - 1} (1 - y)^{b_1 - 1} dy + \frac{z_1}{B(a_2, b_2)} y^{a_2 - 1} (1 - y)^{b_2 - 1} dy,$$

where all coefficients are positive. Finally, the contagion term is assumed to be linear in the loss level, i.e.

$$c(t, T_k, x, L_{t-}; y) = cy(T_k - t). \tag{37}$$

One reason for this choice is that due to a lack of defaults in the observation period, a precise estimation of a nonlinear relation does not seem reasonable. Finally, we consider F as in (5). Together with (18) and (19) this yields an arbitrage-free model.

6.1. **Calibration procedure.** For the calibration of the model we follow the ideas in Eksi and Filipović (2012) and utilize the EM-algorithm together with Kalman filter techniques for the estimation of the (unobserved) factor process Z from the observed STCDO prices.

For the calibration procedure itself we make the following two assumptions: first, we assume that tranche spreads are piecewise constant in between the detachment points, i.e.

$$G(t, T_k, x) = G(t, T_k, x_{i+1}), \quad \text{for } x \in [x_i, x_{i+1}). \tag{38}$$

As previously, $F(t, T_k, x) = \mathbf{1}_{\{L_t \le x\}} G(t, T_k, x)$. Second, we assume that observed prices are model prices plus an additive measurement error. More precisely, we consider observation times $0 = t_0, t_1, t_2, \ldots$ and assume that

$$R(t_k, \tau, j) = -\frac{1}{\tau} \log \left(\frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} G(t_k, t_k + \tau, x) \, dx \right) + \varepsilon(k, \tau, j+1)$$

The error terms $\varepsilon(k, \tau, j)$ are assumed to be independent and normally distributed random variables with zero mean and tranche dependent variance σ_j^2. Moreover, they are independent of Z and L.

Considering equation (18), our two-factor affine specification (35) and (36) yields that $\nu_0 = 0$ as well as $\mu_0 = (0, \kappa_2 \theta_2)^\top$. Letting $\tau = T_k - t$ we obtain that

$$A(\tau, x) = \kappa_2 \theta_2 \int_0^\tau B(s, x)_1 \mathrm{d}s + \int_0^\tau C_0(s, x; l) \mathrm{d}s - \tau.$$

Integrating this terms w.r.t. x gives, using (38), that

$$\int_{x_j}^{x_{j+1}} \mathrm{e}^{A(\tau, x)} \mathrm{d}x = \mathrm{e}^{\kappa_2 \theta_2 \int_0^\tau B(s, x_{j+1})_1 \mathrm{d}s - \tau} \cdot \int_{x_j}^{x_{j+1}} \mathrm{e}^{\int_0^\tau C_0(s, x; l) \mathrm{d}s} \mathrm{d}x.$$

We obtain the following representation for the observed spreads:

$$R(t_k, \tau, j) = \frac{1}{\tau} \log(x_{j+1} - x_j) - \frac{1}{\tau} \kappa_2 \theta_2 \int_0^\tau B(s, x_{j+1})_1 \mathrm{d}s + 1 - \frac{1}{\tau} B(\tau, x_{j+1})^\top Z_t$$

$$- \frac{1}{\tau} \log \left(\int_{x_j}^{x_{j+1}} \mathrm{e}^{\int_0^\tau C_0(s, x; l) \mathrm{d}s} \mathrm{d}x \right) + \varepsilon(t_k, \tau, j)$$

$$=: K(\tau, x_{j+1}) - \frac{1}{\tau} B(\tau, x_{j+1})^\top Z_t + \varepsilon(t_k, \tau, j) \tag{39}$$

for all tranches $j \in \{1, \ldots, J-1\}$ and maturities $\tau \in \{3, 5, 7, 10\}$. As R is a linear function of Z, it may be represented by

$$R_{t_k} := (R(t_k, \tau_1, 1), \ldots, R(t_k, \tau_4, J-1))^\top = f(Z_{t_k}) + \epsilon_k$$

with error vector $\epsilon_k := (\epsilon(t_k, \tau_1, 1), \ldots, \epsilon(t_k, \tau_4, J-1))^\top$.

Following the quasi-maximum likelihood procedure, we approximate the transition density of this equation by a normal density where we match first and second conditional moments. Taking into account the dependence structure of the process, we approximate the conditional distribution of Z_{t_k} given $Z_{t_{k-1}}$ by a normal distribution with mean $g(Z_{t_{k-1}})$ and covariance matrix Q_{k-1}. The computations of g and Q are relegated to the appendix, see Proposition A.1.

Essentially, the affine structure allows to derive the conditional Fourier transform in tractable form which gives the conditional moments.

The EM-algorithm basically requires two iterative steps, namely filtering (expectation step) and the maximization of the likelihood (maximization step). The most difficult step in our setup is the expectation step which we approach with an extended Kalman filter.

The linearity of the functions $f(z) =: f_0 + f_1^\top z$ and $g(z) = g_0 + g_1^\top z$ makes it straightforward to compute the moments required for the Kalman filter as we show now. Let $\mathcal{F}_t^R := \sigma(R_s : s \leq t)$ Denote $\mathbf{m}_k := \mathbb{E}(Z_{t_k}|\mathcal{F}_{t_k}^R)$ and $\mathbf{m}_k^- := \mathbb{E}(Z_{t_k}|\mathcal{F}_{t_{k-1}}^R)$. Analogously, denote by $\mathbf{P}_k$ and $\mathbf{P}_k^-$ the conditional variance of Z_{t_k} given $\mathcal{F}_{t_k}^R$ and $\mathcal{F}_{t_{k-1}}^R$, respectively.

In the *prediction step*, we compute $\mathbf{m}_k^-$ and $\mathbf{P}_k^-$ which gives

$$\mathbf{m}_k^- = \mathbb{E}(Z_{t_k}|\mathcal{F}_{t_{k-1}}^R) = f_0 + f_1^\top \mathbf{m}_{k-1}$$
$$\mathbf{P}_k^- = f_1 \mathbf{P}_{k-1} f_1^\top + Q_{k-1}.$$

The *updating step* incorporates the new information given by R_{t_k}. By $\mathrm{Cov}(X,Y)$ for two random vectors X and Y we denote the variance-covariance matrix and $\mathrm{Var}\,(X) := \mathrm{Cov}(X,X)$. We obtain

$$\mathbf{r}_k^- = \mathbb{E}(R_k|\mathcal{F}_{t_{k-1}}^R) = g(\mathbf{m}_k^-) = g_0 + g_1^\top \mathbf{m}_k^-$$
$$\mathbf{F}_k = \mathrm{Var}\,(R_k|\mathcal{F}_{t_{k-1}}^R) = g_1 \mathbf{P}_k^- g_1^\top + \Sigma$$
$$\mathbf{S}_k = \mathrm{Cov}(Z_{t_k}, R_k|\mathcal{F}_{t_{k-1}}^R) = \mathbf{P}_k^- g_1;$$

here Σ denotes the diagonal matrix with entries $\sigma_1^2, \ldots, \sigma_{J-1}^2$. Furthermore, we set

$$\mathbf{K}_k := \mathbf{S}_k \mathbf{F}_k^{-1}$$
$$\mathbf{m}_k := \mathbf{m}_k^- + \mathbf{K}_k\,(R_k - \mathbf{r}_k^-)$$
$$\mathbf{P}_k := \mathbf{P}_k^- - \mathbf{K}_k \mathbf{F}_k \mathbf{K}_k^\top.$$

Here, $\mathbf{K}_k$ is the so-called Kalman gain and $R_k - \mathbf{r}_k^-$ the innovation. For further details on Kalman filtering we refer to the book of Grewal and Andrews (1993).

To initialize the filter we use the unconditional moment and the unconditional covariance matrix given in Corollary A.2. Starting from $\xi = (\kappa_1, \kappa_2, \theta_2, \sigma_1, \sigma_2, \lambda_1, \lambda_2, c, a_1, b_1, a_2, b_2)$, an initial parameter vector, the Kalman filter computes recursively an estimation of the unobserved factor process with approximate likelihood function given by

$$L(R_1, \ldots, R_n; \xi) = -\frac{n}{2} \log 2\pi - \frac{1}{2} \sum_{k=1}^n \log |\mathbf{F}_k| - \frac{1}{2} \sum_{k=1}^n (\mathbf{r}_k^- - R_k)^\top \mathbf{F}_k^{-1} (\mathbf{r}_k^- - R_k).$$

The EM-algorithm proceeds iteratively between filtering and maximization until a prescribed precision is obtained.

6.2. Calibration results.

Using the calibration methodology described above we fit the model to the full dataset from August 2006 to August 2010. Table 1 gives the parameter values obtained by the calibration.

θ_2	κ_1	κ_2	λ_1	λ_2	σ_1^2	σ_2^2	c	a_1	b_1	a_2	b_2
1.8178	2.0639	6.6046	-0.6444	-6.7376	0.7734	0.2685	-0.0690	0.7318	6.1632	0.2938	23.1966

TABLE 1. Estimated parameter values.

The contagion parameter c is negative, implying that (upward) jumps in the loss process lead to downward jumps in the forward price, i.e. a loss in (T,x)-bond prices, which is very

intuitive. Note that in Eksi and Filipović (2012) a two-factor affine model together with a catastrophic component was used to obtain a good fit even to super-senior spreads. The tow-factor affine market model considered here, however, is able to obtain a fairly good fit without this catastrophic component. Figures 3 and 3 illustrate the results of our calibration example. As can be seen, the proposed model is able to capture the market dynamics across all tranches and maturities even though there is a structural break in the observed spread data.

Finally, it needs to be remarked, that a good fit of the calibration per se is not yet implying a good hedging performance. Therefore a detailed analysis of the model for further applications is required. Hedging in an affine models can be studied along the lines of Filipović and Schmidt (2010). This, however, is beyond the scope of this article. Nevertheless, the hedging analysis in Eksi and Filipović (2012) suggests that affine factor models which show a good fit over a long observation period perform very well for hedging purposes.

Appendix A. Computations

As shown in Keller-Ressel, Schachermayer, and Teichmann (2011) stochastically continuous, time-homogeneous affine processes on the canonical state space are always regular. Thus the process Z given by (33) and (34) possesses an exponentially-affine characteristic function

$$\mathbb{E}(e^{\langle u, Z_T \rangle} | \mathcal{G}_t) = \exp(\phi(T - t, u) + \psi(T - t, u) Z_t) \quad \forall t \leq T,$$

where $\phi(t, u)$, $\psi(t, u)$ are sufficiently differentiable, $\mathbb{C}$ respectively $\mathbb{C}^d$ valued functions. The moments, as derivatives of the characteristic function evaluated at zero, are of polynomial order. We write

$$m_k(T - t, z_1, z_2) := \mathbb{E}\left((Z_T^1)^{k_1} (Z_T^2)^{k_2} | Z_t^1 = z_1, Z_t^2 = z_2 \right), \quad k_1, k_2 \in \mathbb{N}, k_1 + k_2 = k,$$

for the k-th conditional moment. Observe that $m_k(\tau, z_1, z_2)$ has to be a local martingale thus an application of Itô's formula yields

$$\frac{\partial}{\partial \tau} m_k(\tau, z_1, z_2) = \kappa_1 (z_2 - z_1) \frac{\partial m_k}{\partial z_1} + \kappa_2 (\theta_2 - z_2) \frac{\partial m_k}{\partial z_2} + \frac{1}{2} \sigma_1^2 z_1 \frac{\partial^2 m_k}{\partial z_1^2} + \frac{1}{2} \sigma_2^2 z_2 \frac{\partial^2 m_k}{\partial z_2^2}$$

$$m_k(0, z_1, z_2) = z_1^{k_1} z_2^{k_2}.$$

Proposition A.1. *Assume Z is given by (33) and (34). Then the $\mathbb{P}$-conditional first moments of Z are given by*

$$\mathbb{E}(Z_t^1 | Z_0^1 = z_1, Z_0^2 = z_2) = -\frac{\left(-\kappa_1 + \kappa_1 e^{-\kappa_2 t} - e^{-\kappa_1 t} \kappa_2 + \kappa_2 \right) \theta_2}{\kappa_1 - \kappa_2} + e^{-\kappa_1 t} z_1$$

$$- \frac{\kappa_1 \left(e^{-t(\kappa_1 - \kappa_2)} - 1 \right) e^{-\kappa_2 t}}{\kappa_1 - \kappa_2} z_2. \tag{40}$$

$$\mathbb{E}(Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2) = -\left(e^{-\kappa_2 t} - 1 \right) \theta_2 + e^{-\kappa_2 t} z_2. \tag{41}$$
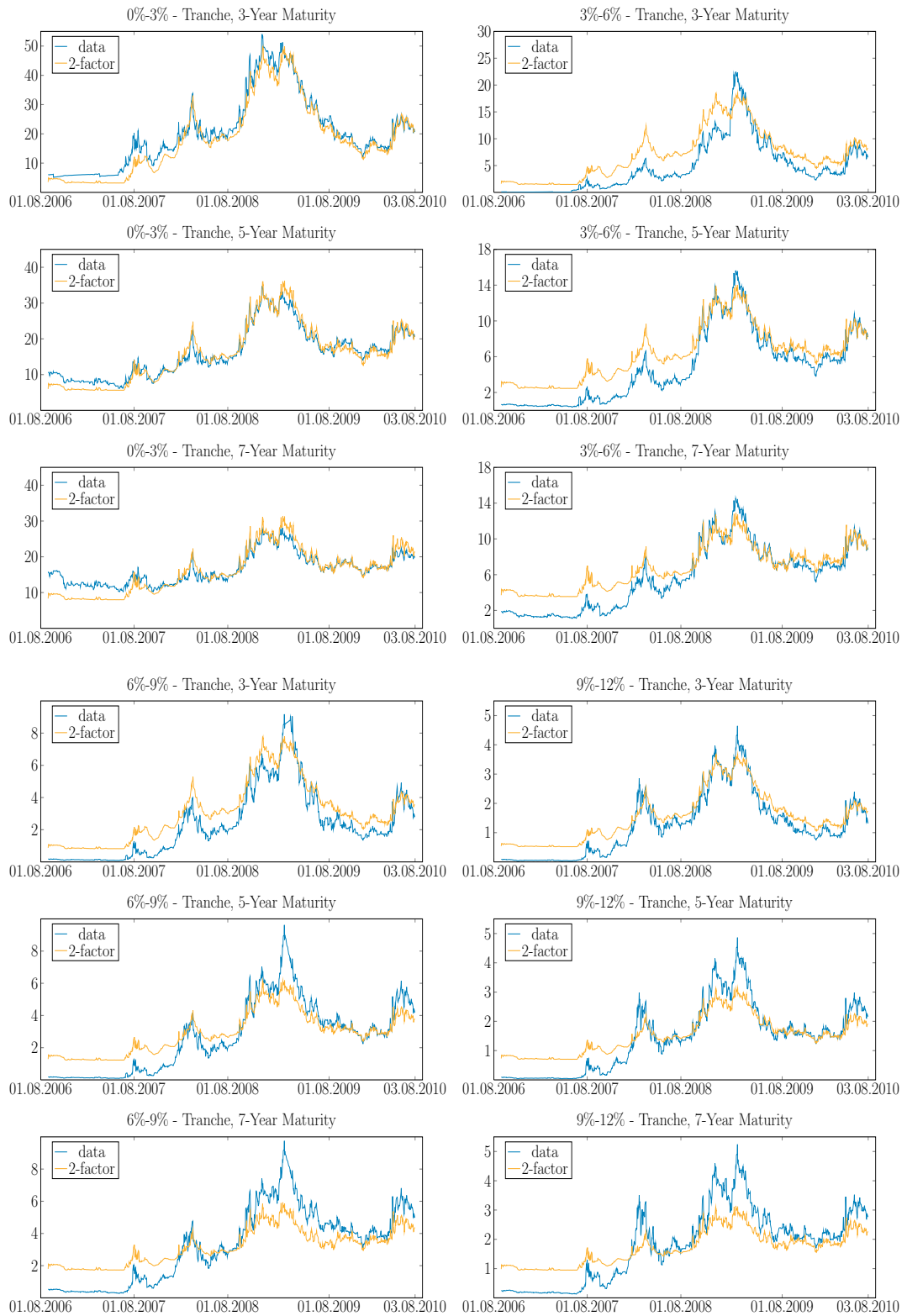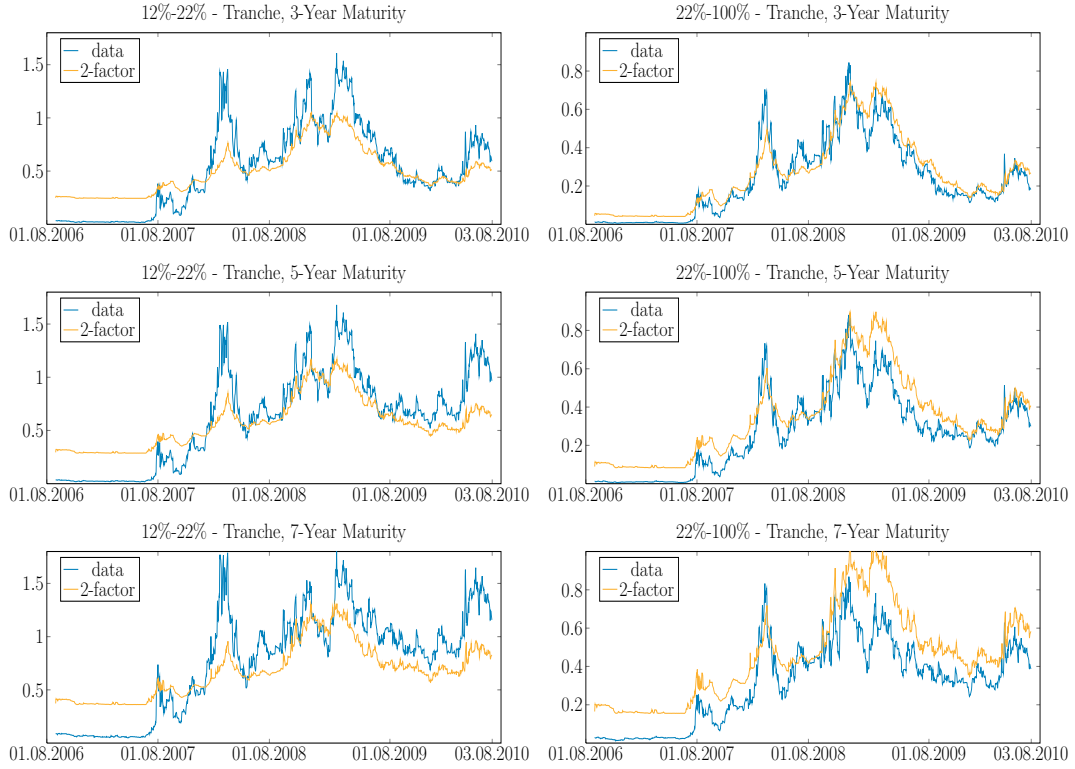
FIGURE 3. Estimated and observed spreads - part 1

FIGURE 4. Estimated and observed spreads - part 2

Furthermore,

$$
\begin{aligned}
\mathrm{Var}\left(Z_t^1 | Z_0^1 = z_1, Z_0^2 = z_2\right) = {} & \frac{1}{2} \frac{1}{\left(\kappa_1 - \kappa_2\right)^2 \kappa_2 \kappa_1 \left(2\kappa_1 - \kappa_2\right)\left(\kappa_1 + \kappa_2\right)} \\
& \Big(4\,\kappa_1^2 \kappa_2 \sigma_2^2 \left(2\kappa_1 - \kappa_2\right)\left(\kappa_1 z - \theta_2 \kappa_2 + \kappa_2 z\right) \mathrm{e}^{-(\kappa_1 + \kappa_2)t} \\
& - \kappa_2 \left(\kappa_1 + \kappa_2\right)\big(4\,\kappa_1^3 y \sigma_1^2 + 2\,\kappa_1^3 z \sigma_2^2 - 2\,\kappa_1^3 z \sigma_1^2 + \kappa_1^2 \kappa_2 \sigma_1^2 \theta_2 \\
& + 4\,\kappa_1^2 z \kappa_2 \sigma_1^2 - \kappa_1^2 \kappa_2 \sigma_2^2 \theta_2 - 10\,\kappa_1^2 \kappa_2 y \sigma_1^2 - 2\,\kappa_1 z \kappa_2^2 \sigma_1^2 \\
& - 2\,\kappa_1 \kappa_2^2 \sigma_1^2 \theta_2 + 8\,\kappa_1 \kappa_2^2 y \sigma_1^2 + \kappa_2^{\,3} \sigma_1^2 \theta_2 - 2\,\kappa_2^{\,3} y \sigma_1^2 \big) \mathrm{e}^{-2\,\kappa_1 t} \\
& + 2\,\kappa_1^2 \left(\kappa_1 - \kappa_2\right)\left(\kappa_1 + \kappa_2\right)\big(2\,\kappa_2 \kappa_1 + 2\,\sigma_2^2 \kappa_1 - 2\,\sigma_2^2 \kappa_2 \\
& - \kappa_2^2 + \kappa_2 \sigma_1^2 \big)\left(z - \theta_2\right) \mathrm{e}^{-\kappa_2 t} + 2\,\kappa_2 \left(\kappa_1 + \sigma_1^2\right) \\
& \left(\kappa_1 - \kappa_2\right)\left(2\kappa_1 - \kappa_2\right)\left(\kappa_1 + \kappa_2\right)\left(-z\kappa_1 + y\kappa_1 + \theta_2 \kappa_2 - \kappa_2 y\right) \mathrm{e}^{-\kappa_1 t} \\
& + \theta_2 \left(2\kappa_1 - \kappa_2\right)\left(\kappa_1 - \kappa_2\right)^2 \\
& \big(2\kappa_2 \kappa_1^2 + \kappa_1^2 \sigma_2^2 + 2\,\kappa_2^2 \kappa_1 + \kappa_1 \kappa_2 \sigma_1^2 + \kappa_2^2 \sigma_1^2 \big)\Big)
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\mathrm{Var}\left(Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2\right) = {} & \frac{1}{2} \frac{1}{\left(\kappa_1 - \kappa_2\right)\kappa_2} \big(2\,\kappa_2 \left(-z\kappa_1 + y\kappa_1 + \theta_2 \kappa_2 - \kappa_2 y\right) \mathrm{e}^{-\kappa_1 t} \\
& - \sigma_2^2 \left(2z - \theta_2\right)\left(\kappa_1 - \kappa_2\right) \mathrm{e}^{-2\,\kappa_2 t} \\
& + 2\left(-\theta_2 + z\right)\left(\sigma_2^2 \kappa_1 - \sigma_2^2 \kappa_2 + \kappa_2 \kappa_1\right) \mathrm{e}^{-\kappa_2 t} \\
& + \theta_2 \left(\sigma_2^2 + 2\,\kappa_2\right)\left(\kappa_1 - \kappa_2\right)\big)
\end{aligned}
\tag{43}
$$

$$\text{Cov}(Z_t^1 Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2) = \frac{1}{2\kappa_2 \left(\kappa_1^2 - \kappa_2^2 \right)} \left(2\,\kappa_2 \sigma_2^2 \left(\kappa_2 z - \theta_2 \kappa_2 + z\kappa_1 \right) \mathrm{e}^{-(\kappa_1 + \kappa_2)t} \right.$$
$$- \kappa_1 \sigma_2^2 \left(2\,z - \theta_2 \right) \left(\kappa_1 + \kappa_2 \right) \mathrm{e}^{-2\,\kappa_2 t}$$
$$+ 2 \left(\kappa_1 + \kappa_2 \right) \left(\sigma_2^2 \kappa_1 + \kappa_2 \kappa_1 - \sigma_2^2 \kappa_2 \right) \left(z - \theta_2 \right) \mathrm{e}^{-\kappa_2 t}$$
$$+ 2\,\kappa_2 \left(\kappa_1 + \kappa_2 \right) \left(-z\kappa_1 + y\kappa_1 + \theta_2 \kappa_2 - \kappa_2 y \right) \mathrm{e}^{-\kappa_1 t}$$
$$\left. + \theta_2 \left(\kappa_1 - \kappa_2 \right) \left(2\,\kappa_2 \kappa_1 + \sigma_2^2 \kappa_1 + 2\,\kappa_2^2 \right) \right) \tag{44}$$

such that the conditional covariance is given by

$$Q_{t_k} = \begin{pmatrix} \text{Var}\left(Z_{t_k}^1 | Z_{t_{k-1}}^1, Z_{t_{k-1}}^2 \right) & \text{Cov}(Z_{t_k}^1 Z_{t_k}^2 | Z_{t_{k-1}}^1, Z_{t_{k-1}}^2) \\ \text{Cov}(Z_{t_k}^1 Z_{t_k}^2 | Z_{t_{k-1}}^1, Z_{t_{k-1}}^2) & \text{Var}\left(Z_{t_k}^2 | Z_{t_{k-1}}^1, Z_{t_{k-1}}^2 \right) \end{pmatrix}.$$

Proof: Denote $\mathbb{E}(Z_t^1 | Z_0^1 = z_1, Z_0^2 = z_2) =: h(t, z_1, z_2)$. The Kolmogorov backward equation implies that

$$\partial_t h = \kappa_1 (z_2 - z_1) \partial_{z_1} h + \kappa_2 (\theta_2 - z_2) \partial_{z_2} h + \frac{1}{2} \sigma_1^2 z_1 \partial_{z_1 z_1} h + \frac{1}{2} \sigma_2^2 z_2 \partial_{z_2 z_2} h$$

Thus inserting the polynomial property of moments yields

$$\frac{\mathrm{d}}{\mathrm{d}t} h_0 + \frac{\mathrm{d}}{\mathrm{d}t} h_{z_1} z_1 + \frac{\mathrm{d}}{\mathrm{d}t} h_{z_2} z_2 = \kappa_1 (z_2 - z_1) h_{z_1} + \kappa_2 (\theta_2 - z_2) h_{z_2}, \tag{45}$$

for some function h_0, h_{z_1} and h_{z_2} which fulfill the following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} h_0 = \kappa_2 \theta_2 h_{z_2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} h_{z_1} = -\kappa_1 h_{z_1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} h_{z_2} = \kappa_1 h_{z_1} - \kappa_2 h_{z_2}$$

with respect to the boundary conditions $h_0(0) = h_{z_2}(0) = 0, h_{z_1}(0) = 1$. The solution to this system is given by

$$h_0(t) = -\frac{\left(-\kappa_1 + \kappa_1 \mathrm{e}^{-\kappa_2 t} - \mathrm{e}^{-\kappa_1 t} \kappa_2 + \kappa_2 \right) \theta_2}{\kappa_1 - \kappa_2}$$
$$h_{z_1}(t) = \mathrm{e}^{-\kappa_1 t}$$
$$h_{z_2}(t) = -\frac{\kappa_1 \left(\mathrm{e}^{-t(\kappa_1 - \kappa_2)} - 1 \right) \mathrm{e}^{-\kappa_2 t}}{\kappa_1 - \kappa_2}$$

which yields the first assertion (40). Regarding $\mathbb{E}(Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2) =: h(t, z_1, z_1)$, we get a system of ordinary differential equations with the modified boundary condition $h_0(0) = h_{z_1}(0) = 0, h_{z_2}(0) = 1$ in an analogous way. Its solution is given by

$$h_0(t) = -\left(\mathrm{e}^{-\kappa_2 t} - 1 \right) \theta_2$$
$$h_{z_1}(t) = 0$$
$$h_{z_2}(t) = \mathrm{e}^{-\kappa_2 t}$$

which yields (41).

Consider $\mathbb{E}(Z_t^1 Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2) =: h(t, z_1, z_2)$ we consider terms of second order, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}h_0 + \frac{\mathrm{d}}{\mathrm{d}t}h_{z_1}z_1 + \frac{\mathrm{d}}{\mathrm{d}t}h_{z_2}z_2 + \frac{\mathrm{d}}{\mathrm{d}t}h_{z_1 z_2}z_1 z_2 + \frac{\mathrm{d}}{\mathrm{d}t}h_{z_1^2}z_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}h_{z_2^2}z_2^2 =$$
$$\kappa_1(z_2 - z_1)(h_{z_1} + 2h_{z_1^2}z_1 + h_{z_1 z_2}z_2) + \kappa_2(\theta_2 - z_2)(h_{z_2} + h_{z_1 z_2}z_1 + 2h_{z_2^2}z_2) + \sigma_1^2 z_1 h_{z_1^2} + \sigma_2^2 z_2 h_{z_2^2}$$

for some functions $h_0 \ldots, h_{z_2^2}$ which solves following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}h_0 = \kappa_2 \theta_2 h_{z_2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h_{z_1} = -\kappa_1 h_{z_1} + \kappa_2 \theta_2 h_{z_1 z_2} + \sigma_1^2 h_{z_1^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h_{z_2} = \kappa_1 h_{z_1} - \kappa_2 h_{z_2} + (2\kappa_2 \theta_2 + \sigma_2^2)h_{z_2^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h_{z_1 z_2} = 2\kappa_1 h_{z_1^2} - (\kappa_1 + \kappa_2)h_{z_1 z_2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h_{z_1^2} = -2\kappa_1 h_{z_1^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}h_{z_2^2} = \kappa_1 h_{z_1 z_2} - 2\kappa_2 h_{z_2^2}$$

with respect to the boundary condition $h_0(0) = h_{z_1}(0) = h_{z_2}(0) = h_{z_1^2}(0) = h_{z_2^2}(0) = 0, h_{z_1 z_2}(0) = 1$. The solution to this system is given by

$$\begin{aligned}
h_0(t) &= \frac{1}{2}\frac{\theta_2 e^{-t(2\kappa_2 + \kappa_1)}}{(\kappa_1^2 - \kappa_2^2)\kappa_2}\left(e^{\kappa_1 t}\kappa_1\left(2\theta_2 \kappa_2 + \sigma_2^2\right)(\kappa_1 + \kappa_2)\right. \\
&\quad - 2 e^{\kappa_1 t}\kappa_2^2\left(\theta_2 \kappa_1 + \sigma_2^2 + \theta_2 \kappa_2\right) + 2 e^{2\kappa_2 t}\kappa_2^2\left(\theta_2 + 1\right)(\kappa_1 + \kappa_2) \\
&\quad - \left(2\left(\kappa_1 + \kappa_2\right)\left(2\theta_2 \kappa_2 \kappa_1 + \kappa_2 \kappa_1 + \sigma_2^2 \kappa_1 - \sigma_2^2 \kappa_2 - \theta_2 \kappa_2^2\right)e^{(\kappa_1 + \kappa_2)t}\right. \\
&\quad \left.\left. + (\kappa_1 - \kappa_2)\left(\sigma_2^2 \kappa_1 + 2\kappa_2 \kappa_1 + 2\theta_2 \kappa_2 \kappa_1 + 2\kappa_2^2 + 2\theta_2 \kappa_2^2\right)e^{t(2\kappa_2 + \kappa_1)}\right)\right) \\
h_{z_1}(t) &= -e^{-\kappa_1 t}\left(\theta_2 e^{-\kappa_2 t} - \theta_2 - 1\right) \\
h_{z_2}(t) &= -\frac{\left(2\theta_2 \kappa_2 \kappa_1 + \sigma_2^2 \kappa_1\right)e^{-2\kappa_2 t}}{(\kappa_1 - \kappa_2)\kappa_2} - \frac{e^{-\kappa_2 t}}{(\kappa_1 - \kappa_2)\kappa_2}\left(\kappa_2\left(\theta_2 \kappa_1 + \sigma_2^2 + \theta_2 \kappa_2\right)e^{-\kappa_1 t}\right. \\
&\quad \left. - \kappa_1 \kappa_2\left(\theta_2 + 1\right)e^{-t(\kappa_1 - \kappa_2)} + 2\theta_2 \kappa_2 \kappa_1 - \kappa_2 \kappa_1 - \sigma_2^2 \kappa_1 + \sigma_2^2 \kappa_2 + \theta_2 \kappa_2^2\right) \\
h_{z_1 z_2}(t) &= e^{-(\kappa_1 + \kappa_2)t} \\
h_{z_1^2}(t) &= 0 \\
h_{z_2^2}(t) &= \frac{\left(e^{-2\kappa_2 t} - e^{-(\kappa_1 + \kappa_2)t}\right)\kappa_1}{\kappa_1 - \kappa_2}
\end{aligned}$$

which yields

$$\mathbb{E}(Z_t^1 Z_t^2 | Z_0^1 = z_1, Z_0^2 = z_2) = h_0(t) + h_{z_1}(t)z_1 + h_{z_2}(t)z_2 + h_{z_1 z_2}(t)z_1 z_2 + h_{z_1^2}(t)z_1^2 + h_{z_2^2}(t)z_2^2.$$

Analogously we compute $\mathbb{E}\left((Z_t^1)^2 | Z_0^1 = z_1, Z_0^2 = z_2\right) =: h(t, z_1, z_2)$ with boundary condition $h_0(0) = h_{z_1}(0) = h_{z_2}(0) = h_{z_1 z_2}(0) = h_{z_2^2}(0) = 0, h_{z_1^2}(0) = 1$.

The solution is given by

$$
\begin{aligned}
h_0(t) = {} & \frac{1}{2}\frac{\mathrm{e}^{-2\,(\kappa_1+\kappa_2)t}\theta_2^2}{(\kappa_1+\kappa_2)}\Big(2\,\mathrm{e}^{2\,\kappa_2 t}\kappa_1\kappa_2{}^3\,(\kappa_1+\kappa_2)\,(2\,\kappa_1-\kappa_2)+2\,\kappa_1^3\kappa_2\,(\kappa_1+\kappa_2)\,(2\,\kappa_1-\kappa_2)\,\mathrm{e}^{2\,\kappa_1 t}\\
& -4\,\kappa_1^2\kappa_2^2\,(\kappa_1+\kappa_2)\,(2\,\kappa_1-\kappa_2)\,\mathrm{e}^{(\kappa_1+\kappa_2)t}-4\,\kappa_2\kappa_1^2\,(\kappa_1-\kappa_2)\,(2\,\kappa_1-\kappa_2)\,(\kappa_1+\kappa_2)\,\mathrm{e}^{t(2\,\kappa_1+\kappa_2)}\\
& +4\,\kappa_2^2\kappa_1\,(\kappa_1-\kappa_2)\,(2\,\kappa_1-\kappa_2)\,(\kappa_1+\kappa_2)\,\mathrm{e}^{t(2\,\kappa_2+\kappa_1)}\\
& +2\,\kappa_1\kappa_2\,(\kappa_1+\kappa_2)\,(2\,\kappa_1-\kappa_2)\,(\kappa_1-\kappa_2)^2\,\mathrm{e}^{2\,(\kappa_1+\kappa_2)t}\Big)\\
& +\frac{1}{2}\frac{\mathrm{e}^{-2\,(\kappa_1+\kappa_2)t}\theta_2}{(\kappa_1+\kappa_2)}\Big(-\kappa_2^2\,(\kappa_1+\kappa_2)\,(-\kappa_1\sigma_2+\kappa_1\sigma_1-\kappa_2\sigma_1)\,(\kappa_1\sigma_2+\kappa_1\sigma_1-\kappa_2\sigma_1)\,\mathrm{e}^{2\,\kappa_2 t}\\
& +\kappa_1^3\sigma_2^2\,(\kappa_1+\kappa_2)\,(2\,\kappa_1-\kappa_2)\,\mathrm{e}^{2\,\kappa_1 t}-4\,\kappa_1^2\kappa_2^2\sigma_2^2\,(2\,\kappa_1-\kappa_2)\,\mathrm{e}^{(\kappa_1+\kappa_2)t}\\
& -2\,\kappa_1^2\,(\kappa_1-\kappa_2)\,(\kappa_1+\kappa_2)\,\big(2\,\kappa_2\kappa_1+2\,\sigma_2^2\kappa_1-\kappa_2^2+\kappa_2\sigma_1^2-2\,\sigma_2^2\kappa_2\big)\,\mathrm{e}^{t(2\,\kappa_1+\kappa_2)}\\
& +2\,\kappa_2^2\,(\kappa_1+\sigma_1^2)\,(\kappa_1-\kappa_2)\,(2\,\kappa_1-\kappa_2)\,(\kappa_1+\kappa_2)\,\mathrm{e}^{t(2\,\kappa_2+\kappa_1)}\\
& +(2\,\kappa_1-\kappa_2)\,(\kappa_1-\kappa_2)^2\,\big(\kappa_1^2\sigma_2^2+2\,\kappa_2\kappa_1^2+\kappa_2\sigma_1^2\kappa_1+2\,\kappa_2^2\kappa_1+\kappa_2^2\sigma_1^2\big)\,\mathrm{e}^{2\,(\kappa_1+\kappa_2)t}\Big)
\end{aligned}
$$

$$
\begin{aligned}
h_{z_1}(t) = {} & \frac{\big(-\kappa_1\sigma_1^2+2\,\theta_2\kappa_2\kappa_1+\kappa_2\sigma_1^2\big)\,\big(\mathrm{e}^{-\kappa_1 t}\big)^2}{(\kappa_1-\kappa_2)\,\kappa_1}\\
& +\frac{\big(\kappa_1^2+2\,\kappa_1^2\theta_2-2\,\kappa_1^2\theta_2\mathrm{e}^{-\kappa_2 t}-2\,\theta_2\kappa_2\kappa_1-\kappa_2\kappa_1+\kappa_1\sigma_1^2-\kappa_2\sigma_1^2\big)\,\mathrm{e}^{-\kappa_1 t}}{(\kappa_1-\kappa_2)\,\kappa_1}
\end{aligned}
$$

$$
\begin{aligned}
h_{z_2}(t) = {} & \frac{\mathrm{e}^{-\kappa_2 t}}{\kappa_2\,(\kappa_1-\kappa_2)^2\,(2\,\kappa_1-\kappa_2)}\Big(2\,\kappa_2\kappa_1\,(2\,\kappa_1-\kappa_2)\,\big(\kappa_1\theta_2+\sigma_2^2+\theta_2\kappa_2\big)\,\mathrm{e}^{-\kappa_1 t}\\
& -\kappa_2\,\big(4\,\kappa_2\kappa_1^2\theta_2-\kappa_1^2\sigma_1^2+\kappa_1^2\sigma_2{}^2-2\,\kappa_2^2\kappa_1\theta_2+2\,\kappa_2\sigma_1{}^2\kappa_1-\kappa_2^2\sigma_1^2\big)\,\mathrm{e}^{-t(2\,\kappa_1-\kappa_2)}\\
& -\kappa_1^2\,\big(2\,\theta_2\kappa_2+\sigma_2^2\big)\,(2\,\kappa_1-\kappa_2)\,\mathrm{e}^{-\kappa_2 t}\\
& -\kappa_2\,(2\,\kappa_1-\kappa_2)\,(\kappa_1-\kappa_2)\,\big(\kappa_1+2\,\kappa_1\theta_2+\sigma_1^2\big)\,\mathrm{e}^{-t(\kappa_1-\kappa_2)}\\
& +\kappa_1\,(\kappa_1-\kappa_2)\,\big(4\,\theta_2\kappa_2\kappa_1+2\,\kappa_2\kappa_1+2\,\sigma_2{}^2\kappa_1-2\,\theta_2\kappa_2^2-\kappa_2^2+\kappa_2\sigma_1^2-2\,\sigma_2^2\kappa_2\big)\Big)
\end{aligned}
$$

$$
h_{z_1^2}(t) = \mathrm{e}^{-2\,\kappa_1 t}
$$

$$
h_{z_1 z_2}(t) = 2\,\frac{\kappa_1\,\big(\mathrm{e}^{-(\kappa_1+\kappa_2)t}-\mathrm{e}^{-2\,\kappa_1 t}\big)}{\kappa_1-\kappa_2}
$$

$$
h_{z_2^2}(t) = \frac{\big(\mathrm{e}^{-2\,\kappa_1 t}-2\,\mathrm{e}^{-(\kappa_1+\kappa_2)t}+\mathrm{e}^{-2\,\kappa_2 t}\big)\,\kappa_1^2}{(\kappa_1-\kappa_2)^2}
$$

Finally, observe that for $\mathbb{E}\big((Z_t^2)^2|Z_0^1=z_1, Z_0^2=z_2\big) =: h(t, z_1, z_2)$ with boundary condition $h_0(0) = h_{z_1}(0) = h_{z_2}(0) = h_{z_1 z_2}(0) = h_{z_1^2}(0) = 0, h_{z_2^2}(0) = 1$. We obtain the solution

$$
\begin{aligned}
h_0(t) = {} & \frac{1}{2}\frac{\theta_2\mathrm{e}^{-t(2\,\kappa_2+\kappa_1)}}{(\kappa_1-\kappa_2)\,\kappa_2}\Big(\big(2\,\theta_2\kappa_2+\sigma_2^2\big)\,(\kappa_1-\kappa_2)\,\mathrm{e}^{\kappa_1 t}\\
& +\big(2\,\theta_2\kappa_2+2\,\kappa_2+\sigma_2^2\big)\,(\kappa_1-\kappa_2)\,\mathrm{e}^{t(2\,\kappa_2+\kappa_1)}\\
& +\big(-2\,\sigma_2^2\kappa_1-4\,\theta_2\kappa_2\kappa_1-2\,\kappa_2\kappa_1+2\,\sigma_2^2\kappa_2+4\,\theta_2\kappa_2^2\big)\,\mathrm{e}^{(\kappa_1+\kappa_2)t}+\kappa_2^2\mathrm{e}^{2\,\kappa_2 t}\Big)
\end{aligned}
$$

$$
h_{z_1}(t) = \mathrm{e}^{-\kappa_1 t}
$$

$$
\begin{aligned}
h_{z_2}(t) = {} & \frac{\mathrm{e}^{-\kappa_2 t}}{\kappa_2(\kappa_1-\kappa_2)}\Big(\sigma_2^2\kappa_1+2\,\theta_2\kappa_2\kappa_1+\kappa_2\kappa_1-\sigma_2^2\kappa_2-2\,\theta_2\kappa_2^2-\kappa_1\kappa_2\mathrm{e}^{-t(\kappa_1-\kappa_2)}\Big)\\
& +\frac{\big(2\,\theta_2\kappa_2+\sigma_2^2\big)\,\mathrm{e}^{-2\,\kappa_2 t}}{\kappa_2}
\end{aligned}
$$

$$h_{z_1 z_2}(t) = 0$$
$$h_{z_1^2}(t) = 0$$
$$h_{z_2^2}(t) = \mathrm{e}^{-2\kappa_2 t}$$

$\square$

It is an easy exercise to derive the unconditional moments from the above result by taking the limit $t \to \infty$, and we give the result in the next corollary.

Corollary A.2. *Assume Z is given by (33) and (34). The unconditional moments of Z up to order two are given by*

$$\mathbb{E}(Z_t^1) = \theta_2$$
$$\mathbb{E}(Z_t^2) = \theta_2$$
$$\mathrm{Var}\,(Z_t^1) = \frac{1}{2} \frac{\theta_2 \left(2\kappa_2\kappa_1^2 + \kappa_1^2\sigma_2^2 + 2\kappa_2^2\kappa_1 + \kappa_1\kappa_2\sigma_1^2 + \kappa_2^2\sigma_1^2\right)}{\kappa_2\kappa_1\left(\kappa_1 + \kappa_2\right)}$$
$$\mathrm{Var}\,(Z_t^2) = \frac{1}{2} \frac{\theta_2\left(\sigma_2^2 + 2\kappa_2\right)}{\kappa_2}$$
$$\mathrm{Cov}(Z_t^1 Z_t^2) = \frac{1}{2} \frac{\left(2\kappa_2\kappa_1 + \sigma_2^2\kappa_1 + 2\kappa_2^2\right)\theta_2}{\left(\kappa_1 + \kappa_2\right)\kappa_2}$$

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