

CDOs in the light of the Current Crisis

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ABSTRACT

This paper proposes a top-down model for pricing Collateralized Debt Obligation (CDOs). Our proposal is both treatable and realistic, in the sense we are able to obtain closed-form solutions to single tranche CDOs and capturing extreme credit events. We use as key ingredients the so-called (T, x) -bonds, as proposed in Filipović, Overbeck, and Schmidt (2008), but generalize their affine specification by including shot-noise processes.

Our claim is that affine diffusions combined with shot-noise processes lead to an improved modeling of CDO spreads in comparison to existing affine jump-diffusion models. The proposed approach allows in particular for better capturing the possibility of extreme events, like the ones underlying the current crisis. We illustrate our results with a very concrete (simple) instance of our class of models. Finally, we identify the connections between the top-down and bottom-up approaches for modeling credit risk, within our class of models. Concretely, we show that even when taking a bottom-up approach the aggregate loss process would be a process of affine shot-noise type.

1. INTRODUCTION

A collateralized debt obligation (CDO) is a security backed by a pool of reference entities such as bonds, loans or credit default swaps (CDSs). The reference entities form the *asset side* of the CDO. Issued notes on tranches of different seniority build the *liability side* of the CDO. A CDO is, therefore, a derivative on a portfolio of N credit risky instruments. CDO markets have experienced great growth during the last decade. Nowadays the most liquidly traded CDOs are based upon so-called credit indices that were created in 2004 – the CDX for North-America and the Itraxx for Europe. For more details and references on credit risk products and markets we refer to the respective chapters in McNeil, Frey, and Embrechts (2005).

The valuation of CDOs tranches is far from trivial and relies considerably on the default correlation between the various reference entities in the pool. Copula models have emerged as a way to deal with such correlations. The one-factor Gaussian copula approach is, still nowadays, the industry standard. See e.g. Laurent and Gregory (2005). Copula models, however, have serious limitations (mainly due to their non-dynamic nature) and have been quite heavily criticized both in the academic world and, more recently in connection to the credit risk crisis, also in the industry. For more details on copula models we refer to the appropriate chapter in Schönbucher (2003).

Reduced-form models of credit risk have been popular for pricing of single-name credit products. The question of how to extend them to the multi-name case is of great interest. Two alternative approaches have emerged: the bottom-up approach and the top-down approach.

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In the bottom-up approach the credit risk of each reference entity is modeled as well as all default correlations. To arrive to the loss process associated with the CDO we need to aggregate all this information. This approach has great flexibility, as it allows to consider the specificities of each entity separately as well as their connections, but lead to non parsimonious models that can be hard to calibrate to market data. The top-down approach models the credit risk of the entire pool at once. See Gieseke and Goldberg (2007) or Gieseke (2008). If we are interested only in the behavior of the entire pool this is sufficient and lead to dynamic and parsimonious models. The need of parsimonious alternatives to the copula models and the development of credit risk markets made the top-down approach more and more popular. Specially with the appearance of standardized credit indices and liquidly traded derivative products written on those indices. Despite their philosophical differences, it is quite straightforward to apply mathematical methods used in one approach to develop a model belonging to the other approach.

Here we take a top-down approach and extend the well-known class of affine models (see e.g. Duffie, Filipović, and Schachermayer (2003), Singleton and Umantsev (2002) or Filipović, Overbeck, and Schmidt (2008)) by a shot-noise process as proposed by Gaspar and Schmidt (2007). The shot-noise effect is particularly important to guarantee extreme movements like the ones recently observed in credit markets.

The remaining of the paper is organized as follows. In Section 2 we present the mathematical formalism proposed by Filipović, Overbeck, and Schmidt (2008), useful when dealing with CDOs. In Section 3 we present the affine shot-noise setup and show how to obtain the distribution of the loss process, prices of (T, x) -bonds and, thus, CDO tranche spreads in closed-form. We study the link between the top-down and bottom-up approach in our setup and show that affine shot-noise intensities of individual names lead to affine shot-noise loss processes. Finally we discuss measure changes and calibration issues. In Section 4 we compare a concrete instance of our class of models with the model proposed by Duffie and Gârleanu (2001) and argue ours is more flexible and can capture better extreme events like the ones recently experienced in credit risk markets.

2. COLLATERALIZED DEBT OBLIGATIONS

Consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. We directly work under an equivalent martingale measure \mathbb{Q} . The measure change from the real-world measure \mathbb{P} to \mathbb{Q} is quite standard and we give more details in Section 3.5.

The total nominal of the CDO is normalized to one. Let $L = (L_t)_{t \geq 0}$ be the process of accumulated losses over time. This is a pure-jump process which jumps at default of entities in the pool by the size of the occurred loss. A special case, quite often considered in practice, is where the loss on each default is a constant. By $\mathcal{I} = [0, 1]$ we denote the set of attainable loss fractions³.

As in Filipović, Overbeck, and Schmidt (2008) we consider defaultable (T, x) -bonds, which pay one if the aggregated CDO loss process has not exceeded x at maturity T and zero else, as basic constituents. A (T, x) -bond has a payoff of $\mathbf{1}_{\{L_t \leq x\}}$ at maturity T , for $x \in \mathcal{I}$. We denote its price, at time $t \leq T$, by $p(t, T, x)$. Note that $p(t, T, x)$ is decreasing in T and increasing in x . For $x = 1$ we obtain $p(t, T, 1) =: p(t, T)$, i.e. the price of a default-free bond.

³The case where \mathcal{I} is finite may be considered, see Filipović, Overbeck, and Schmidt (2008).

Knowledge of prices of all (T, x) -bonds is sufficient for pricing derivatives on the loss process. Indeed, consider the payoff

$$F(L_T) = F(1) - \int_0^1 F'(y) \mathbf{1}_{\{L_T \leq y\}} dy$$

for some bounded measurable function F' . A Fubini argument shows that the price of a derivative offering $F(L_T)$ at T is given by

$$F(1)p(t, T) - \int_0^1 F'(y)p(t, T, y) dy.$$

Investing in CDOs is done via a so-called single-tranche CDO (STCDO), sometimes also called tranche credit default swap. A STCDO is specified by a number of future dates $T_0 < T_1 < \dots < T_n$, its lower and upper detachment points, x_1 and x_2 , with $0 \leq x_1 < x_2 \leq 1$ and a fixed spread S . $T_0 \geq t$ is the starting date of the swap. The investor receives coupon payments at times T_1, \dots, T_n . In exchange, the investor covers a certain part of the occurring losses. Set

$$G(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1, x_2]} \mathbf{1}_{\{x \leq y\}} dy.$$

Then, investing in the STCDO with swap rate S is equivalent to the following payment stream:

- (1) *Payment leg.* The investor receives $S G(L_{T_i})$ at T_i , $i = 1, \dots, n$.
- (2) *Default leg.* The investor pays $-dG(L_t) = G(L_{t-}) - G(L_t)$ at default times (any time where $\Delta L_t \neq 0$).

In Filipović, Overbeck, and Schmidt (2008) it is shown that the value of the STCDO at time $t \leq T_0$ can be derived solely on the basis of (T, x) -bonds, and is given by

$$V(t, S) = \int_{(x_1, x_2]} \mathbf{1}_{\{L_t \leq y\}} \left(S \sum_{i=1}^n p(t, T_i, y) + p(t, T_n, y) - p(t, T_0, y) + \gamma(t, y) \right) dy,$$

where $\gamma(t, y) = \int_{T_0}^{T_n} \mathbb{E}^{\mathbb{Q}} \left[r_u e^{-\int_t^u r_s ds} \mathbf{1}_{\{L_u \leq y\}} | \mathcal{F}_t \right] du.$ (1)

In the case where default-free interest rates and the loss process are conditionally independent, $\gamma(t, T)$ simplifies to $\gamma(t, y) = \int_{T_0}^{T_n} f(t, u) p(t, u, y) du$ where $f(t, u)$ is defined for $t \leq u$ and denotes the risk-free forward rate. Setting $V = 0$ and solving for S gives the par-spread for this investment. Under this assumption we also have

$$p(t, T, x) = p(t, T) \mathbb{E}^{\mathbb{Q}} \left(\mathbf{1}_{\{L_T \leq x\}} | \mathcal{F}_t \right). \quad (2)$$

3. THE AFFINE SHOT-NOISE SETTING

The current crisis shows a dramatic behavior of spreads of credit indices like iTraxx or CDX. In Figure 1 we show data from the iTraxx. The purpose of this paper is to propose a simple model which is able to capture the dramatic increase in spreads.

Typically, in credit indices like iTraxx, the underlying entities have zero recovery for each defaultable instrument. We assume that there are N constituents of the credit pool and so the loss process L jumps by $\delta = N^{-1}$ at each default. We assume that L is a conditional Markov process that, in our framework, is nothing but δ times a Cox process with a given intensity, say λ , under \mathbb{Q} . As outlined above it is sufficient to price (T, x) -bonds for pricing STCDOs. We assume the risk-free rate



FIGURE 1. A graph of rolled-over iTraxx spreads. A diffusive movement as well as high price peaks are a typical characteristics of the spreads.

of interest is independent of λ . This is a typical assumption in credit risk⁴. Also note that, since at each default time the nominal amount reduces by δ , we can easily keep track of the numbers of defaults occurred, up to any time t as they are given by the ratio L_t/δ and the expectation in (2), simplify to

$$\mathbb{Q}(L_T \leq x | \mathcal{F}_t) = \mathbf{1}_{\{L_t \leq x\}} \sum_{k=0}^{(x-L_t)/\delta} \mathbb{Q}(L_T - L_t = k\delta | \mathcal{F}_t) \quad (3)$$

where n gives us the number of jumps (defaults) in the interval $(t, T]$. So, it suffices to compute the risk-neutral probability for all possible k defaults occurring in the interval $(t, T]$, i.e. $k = 0, \dots, (x - L_t)/\delta$.

3.1. Setup. The class of models here proposed results from a combination of continuous affine processes with shot-noise processes. Concretely, we assume that the intensity of the Cox process driving the defaults is given by

$$\lambda_t = \eta_t + J_t \quad (4)$$

where η and J are some continuous and pure jump processes, respectively.

(A1): Assume J is an exponentially declining shot-noise process, i.e.

$$J_t = \sum_{\tilde{\tau}_i \leq t} Y_i h(t - \tilde{\tau}_i). \quad (5)$$

Here $Y_i, i = 1, 2, \dots$ are i.i.d., independent of everything else and $\tilde{\tau}_i$ are the jump times of a Poisson process with intensity l and we take $h(x) = \exp(-cx)$. Phrased in different words, J is a mean-reverting compound Poisson process.

(A2): Assume η is a continuous affine process, i.e.

$$\eta_t = g(t)^\top Z_t + f(t) \quad (6)$$

for $Z \geq 0$ the m -dimensional vector of state variables that uniquely solves

$$dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dW_t. \quad (7)$$

⁴All formulas can be derived without these assumptions, most expectations would then need to be considered under forward measures and formulas would become more cumbersome. For details we refer to Gaspar and Schmidt (2007).

Here W is a n -dimensional \mathbb{Q} -Wiener process. $\alpha : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \mapsto \mathbb{R}^{n \times n}$ are such that

$$\alpha(t, z) = d(t) + E(t)z \quad (8)$$

$$\sigma(t, z)\sigma^\top(t, z) = k_0(t) + \sum_{i=1}^m k_i(t)z_i \quad (9)$$

f, g, d and E, k_0, k_i (for $i = 1, \dots, m$) are smooth functions mapping \mathbb{R}_+ to \mathbb{R}, \mathbb{R}^m and to $\mathbb{R}^{m \times m}$, respectively.

A typical example, at which we look more closely in Section 4, satisfying (A2) is the one-factor CIR-process where $\eta = Z$ and it has \mathbb{Q} -dynamics given by

$$d\eta_t = \kappa(b - \eta_t)dt + \sigma\sqrt{\eta_t}dW_t.$$

More general dynamics, for example to include various factors, can be easily incorporated in the setting given by (A2).

Comparing to Figure 1, assumptions (A1) and (A2) seem very reasonable. A diffusive part which captures the risk of smaller movements in λ . The shot-noise part captures the heavy reactions of the market to news like in the current crisis. General choices of h allow to capture different modeling aspects but, for simplicity, we concentrate on exponentially declining h .⁵

3.2. Loss Process distribution. The following proposition gives a closed-form expression for the conditional (an unconditional) risk-neutral distribution function of loss process L . The proof is provided in the appendix. From this result we immediately obtain prices of (T, x) -bonds and hence of STCDOs (equations (2) and (1), respectively).

Proposition 3.1. *Consider the affine shot-noise setting. Then*

$$\mathbb{Q}(L_T \leq x | \mathcal{F}_t) = \mathbf{1}_{\{L_t \leq x\}} \sum_{k=0}^{(x-L_t)/\delta} \frac{1}{k!} \frac{\partial^k S(\theta, t, T)}{\partial \theta^k} \Big|_{\theta=-1} \quad (10)$$

where $S(\theta, t, T) = \exp \{A(\theta, t, T) + B^\top(\theta, t, T)Z_t + C(\theta, t, T)J_t + D(\theta, t, T)\}$.

A and B on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ are deterministic in \mathbb{R} and \mathbb{R}^m , respectively, solving system of Riccati equations, for all $\theta \in \mathbb{R}$ and $t \leq T$

$$\frac{\partial A}{\partial t} + d^\top(t)B + \frac{1}{2}B^\top k_0(t)B = -\theta f(t) \quad (11)$$

$$\frac{\partial B}{\partial t} + E^\top(t)B + \frac{1}{2}\tilde{B}^\top K(t)B = -\theta g(t) \quad (12)$$

subject to the boundary conditions $A(\theta, T, T) = 0, B(\theta, T, T) = 0$. A and B should always be evaluated at (θ, t, T) . f, g are as in (6) and E, d, k_0 , are as in (8)-(9), while \tilde{B} and $K(t) \in \mathbb{R}^{m^2 \times m}$ are simply

$$\tilde{B} := \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & B \end{pmatrix}, \quad K(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_m(t) \end{pmatrix}.$$

⁵It is possible to be quite general for the choice of h . However, if h is not of the exponential form, Markovianity of J is lost and computations become more involved. See Gaspar and Schmidt (2007) for details.

C and D on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ are also deterministic in \mathbb{R} and given by

$$C(\theta, t, T) = \frac{\theta}{c} \left(1 - e^{-c(T-t)} \right) \quad (13)$$

$$D(\theta, t, T) = l \cdot \left(\int_t^T \varphi \left(\frac{\theta}{c} (e^{-cu} - e^{-cT}) \right) du - (T - t) \right) \quad (14)$$

with $\varphi(u) := \mathbb{E}(\exp(uY_1))$ the Laplace transform of Y_1 .

3.3. Calibration issues. It is far beyond the scope of this paper to provide a calibration of the model to observed prices of credit indices as this is indeed a computationally difficult task. However, recently there emerged some calibration results of top-down approaches in the literature. One of the most important contributions in this direction is the work of Cont and Minca (2008). They introduce the so-called effective intensity $\lambda_{\text{eff}}(t, L_t) = \mathbb{E}(\lambda_t | L_t)$ and calibrate it in a non parametric way to observed index prices. They show that qualitatively, the market implied λ_{eff} is increasing for small L , stays on a plateau for intermediate L and finally decrease. We show how to compute λ_{eff} in our case and give an intuition how this behavior can be replicated in our setup. In the next proposition we compute λ_{eff} in our setting and show that this form can indeed be achieved. A proof is given in the appendix.

Proposition 3.2. *In the affine shot-noise setting*

$$\lambda_{\text{eff}}(t, k) = \mathbb{E}(\lambda_t | L_t = k\delta) = \frac{\left. \frac{\partial^k}{\partial \theta^k} \right|_{\theta=-1} \frac{\partial}{\partial t} \frac{1}{\theta} S(\theta, 0, t)}{k! \left. \frac{\partial^k}{\partial \theta^k} \right|_{\theta=-1} S(\theta, 0, t)}$$

Qualitatively, it is quite intuitive, that λ_{eff} will first increase sharply and then stay on a medium level in our setup. This is because λ_t is not observed and L_t is used as a statistic to estimate it (in the conditional expectation, λ_{eff}). If L_t increases this comes as a surprise if λ_t were small. The conditional distribution therefore will be updated strongly which explains a first sharp rise. If L increases further, this is no longer such a big surprise and this effect vanishes.

3.4. Relation to the bottom-up approach. In this section we show that a bottom-up approach based on conditionally independent defaults with a factor structure, where the factors consists of affine and shot-noise processes lead to a jump intensity λ of the loss process as proposed in Assumptions (A1)-(A2).

We say a process $\mu = (\eta, J)$ satisfying (A1) and (A2) is *shot-noise linear* with parameters (g, f, c, l, F) where F denotes the distribution of Y_1 . A special case of Proposition 4.3 in Gaspar and Schmidt (2007) tells that the sum of two shot-noise linear processes is itself shot-noise linear. Thus, by induction, the sum of a finite number of such processes will also be shot-noise linear.

Proposition 3.3. *Consider two independent processes μ^1 and μ^2 , both shot-noise linear with parameters (g^1, f^1, c, l^1, F^1) and (g^2, f^2, c, l^2, F^2) , respectively. Set $q := l_1(l_1 + l_2)^{-1}$. Then $\mu^1 + \mu^2$ is shot-noise linear with parameters $(g^1 + g^2, f^1 + f^2, l^1 + l^2, c, qF^1 + (1 - q)F^2)$.*

For bottom-up modeling with the affine shot-noise setting, we propose the following approach. Consider m companies. Assume that their default times are conditionally independent with default intensities $\lambda^1, \dots, \lambda^m$.

For each company i take

$$\lambda_t^i = [g^i(t) + \epsilon_1^i g(t)]^\top Z_t + f^i(t) + \epsilon_2^i J(t) . \quad (15)$$

Note that in the above expression we suppose the specific default risk of each firm is only affected by the affine diffusion processes Z (through the functions g^i and f^i)

while the common (systematic) risk is affected by both affine processes and a shot-noise processes (through f , g and J). The parameters $\epsilon_1^i, \epsilon_2^i$ allow us to control for different sensitivities of the companies to the common factors. Clearly this setting can be generalized to include shot-noise processes also in the specific part of credit risk, however, we believe the shot-noise component is really what explain default *correlation*, and therefore is mostly important in the common part.

By setting $\epsilon_j := \sum_{i=1}^m \epsilon_j^i$, it is easy to show that in this case the jump intensity of the loss process is $\lambda_t = \sum_{i=1}^m \lambda_t^i$. Due to Proposition 3.3 λ is the shot-noise linear and given the specification in (15) its parameters are $(\sum_{i=1}^m g^i + \epsilon_1 g, \sum_{i=1}^m f^i, c, l, F^*)$ where F^* is the distribution of $\epsilon_2 Y$.⁶

3.5. Measure Changes. Up to now we always assumed that \mathbb{Q} was a risk-neutral measure. It is of course possible to consider the proposed model with respect to some objective probability measure $\mathbb{P} \sim \mathbb{Q}$. In particular this is necessary for the computation of risk measures and statistical analysis of the model. In the considered framework it is reasonable to assume that the model is also shot-noise linear under \mathbb{P} , which restricts the class of equivalent probability measures. A measure change from \mathbb{Q} to \mathbb{P} will then have the following impact:

$$\begin{aligned} d(\cdot), E(\cdot) &\rightsquigarrow \tilde{d}(\cdot), \tilde{E}(\cdot) \\ l, F_Y &\rightsquigarrow \tilde{l}, \tilde{F}_Y \end{aligned}$$

for some appropriate deterministic functions \tilde{d}, \tilde{E} , constant $\tilde{l} > 0$ and distribution function of Y under \mathbb{P} , \tilde{F}_Y . Note that this kind of measure change leaves Y_1, Y_2, \dots i.i.d. under \mathbb{P} . For a general reference see Theorem III.3.24 in Jacod and Shiryaev (1987).

4. A CONCRETE (SIMPLE) EXAMPLE

In this section we use a concrete instance of our class of models to illustrate the results previously derived. We present only the main results but more details about the computations can be found in the appendix. We then compare our concrete instance with a model proposed by Duffie and Gârleanu (2001).

Recall that $\lambda_t = \eta_t + J_t$. Consider the following instance of our class of models:

- (1) $Z = \eta$ with the following dynamics

$$d\eta_t = \kappa(b - \eta_t)dt + \sigma\sqrt{\eta_t}dW_t$$

- (2) $J_t = \sum_{\tilde{\tau}_i \leq t} Y_i \exp(-c(t - \tilde{\tau}_i))$ where Y_1, Y_2, \dots are i.i.d. $\chi^2(2)$ -distributed.

Under (i)-(ii), $A(\theta, t, T)$, $B(\theta, t, T)$ and $D(\theta, t, T)$ with $\theta < \frac{\kappa^2}{2\sigma^2}$ are given by⁷

$$\begin{aligned} A(\theta, t, T) &= \frac{2\kappa b}{\sigma^2} \ln \left(\frac{(\psi - \kappa) [e^{\psi(T-t)} - 1] + 2\psi}{2\psi e^{(\psi - \kappa)\frac{(T-t)}{2}}} \right) \\ B(\theta, t, T) &= \frac{2\theta [e^{\psi(T-t)} - 1]}{(\psi - \kappa) [e^{\psi(T-t)} - 1] + 2\psi} \\ D(\theta, t, T) &= l \left(\frac{c(T-t) - \ln \left(1 + 2\frac{\theta}{c} (e^{-c(T-t)} - 1) \right)}{c - 2\theta} - (T-t) \right) \end{aligned}$$

where $\psi = \sqrt{\kappa^2 - 2\sigma^2\theta}$.

We take this simple example to illustrate the applicability of our approach. The full implementation and comparison to data is far beyond the scope of this paper.

⁶Note that we did not need to use the result of Proposition 3.3 for the sum of the shot-noise parts as we considered only one shot-noise process in the common part.

⁷For the detailed computations we refer to the appendix.

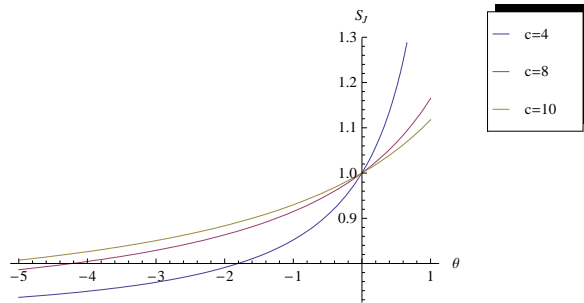


FIGURE 2. The plot shows $S_D(\theta, t, T) = \exp(D(\theta, t, T))$ which is, for $\theta = -1$, the probability of no loss in $(t, T]$. l has the value 0.4.

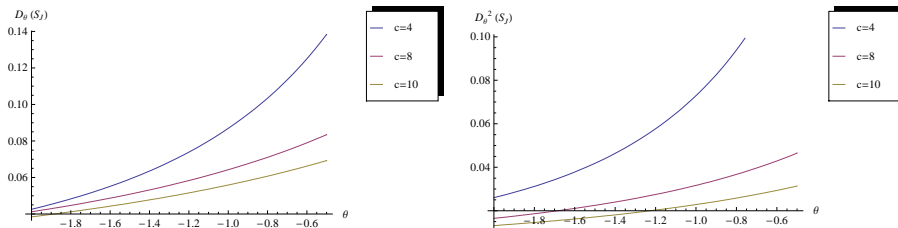


FIGURE 3. The plot shows the first and second derivative of $S_D(\theta, t, T)$

However, the expressions obtained up to now relate on sums of certain derivatives and the question may be raised, how to compute these in practise. As our main goal is to emphasize the importance of shot-noise processes, we concentrate on the terms regarding the shot-noise part. Without loss of generality we assume that $J_t = 0$, i.e. there were no past jumps. Then to obtain the shot-noise impact on the loss distribution given in (10), we have to compute

$$\sum_{k=0}^{(x-L_t)/\delta} \frac{1}{k!} \frac{\partial^k \exp(D(\theta, t, T))}{\partial \theta^k} \Big|_{\theta=-1}$$

where $S_D(\theta, t, T) := \exp(D(\theta, t, T))$. For $\theta = -1$, S_D denotes the probability that in $(t, T]$ no jump happens. In the single-name case this would be the survival probability. As by intuition, S_D has the following properties: first, it is increasing in c : if $c = 0$, jumps persist forever and J is strictly increasing. If c is large, the effect of a jump vanishes very fast and J is pulled back to zero, so for large c we will see less defaults and so the probability of having no default in $(t, T]$ clearly increases. Second, it is decreasing in l : if the intensity of jumps in J increases, this immediately decreases the probability of no default in $(t, T]$. For $l = 0.4$ we plot the function and its first two derivatives in Figure 2 and Figure 3, respectively. Note that the derivatives are quite small, in particular at $\theta = -1$. This together with the factor $(k!)^{-1}$ implies that the crucial impact is reasonably measured by considering a few terms of the above sum.

4.1. Relation to the Duffie-Garleanu model. The frequently used Duffie and Garleanu (2001) (DG) model is a bottom-up model which uses affine technology to obtain pricing formulas for CDOs. Our approach shows qualitatively a similar behavior when, as in our concrete example, we take $\alpha(t, z) = \kappa(b - z)$, $h(t) = \exp(-ct)$ and we impose the additional *restriction* $\kappa = c$. Note, however, that the freedom, that exist only in shot-noise models, to choose κ different from c is

essential: this allows to have reasonable parameters for the diffusive part of the dynamics (κ not too big) and on the other side a high default correlation (large jumps in the shot-noise part with a high c).

Of course, the DG model considered from a bottom-up approach leads to a model for λ which has an inherent factor structure. This approach has in principle the same tractability as the approach outlined here, but formulas get much more involved. How to achieve this is described in full detail in Gaspar and Schmidt (2007).

APPENDIX A. PROOFS

Proof of Proposition 3.1. The proof heavily relies on the independence of η and J and uses the techniques set out in Gaspar and Schmidt (2007). Given (3) it suffices to show that

$$\mathbb{Q}(L_T - L_t = k\delta | \mathcal{F}_t) = \frac{1}{k!} \frac{\partial^k S(\theta, t, T)}{\partial \theta^k} \Big|_{\theta=-1} \quad (16)$$

Let $\int_t^T \lambda_s ds =: \Lambda(t, T)$. Note $\mathbb{Q}(L_T - L_t = k\delta | \mathcal{F}_t) = \mathbb{Q}\left(\frac{L_T}{\delta} - \frac{L_t}{\delta} = k | \mathcal{F}_t\right)$ and $(L_T - L_t)/\delta$ is simply the number of defaults in the time interval $(t, T]$. From the Cox process properties, namely that under the conditionally independence assumption, $(L_T - L_t)/\delta$ is Poisson $(\Lambda(t, T))$, we get

$$\begin{aligned} \mathbb{Q}\left(\frac{L_T}{\delta} - \frac{L_t}{\delta} = k | \mathcal{F}_t\right) &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}} \left(\mathbf{1}_{\{(L_T - L_t)/\delta = k\}} | \mathcal{F}_t, \Lambda(t, T) \right) | \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\exp(-\Lambda(t, T)) \frac{(\Lambda(t, T))^k}{k!} | \mathcal{F}_t \right]. \end{aligned}$$

In the following we use $\mathbb{E}(A^k e^{\theta A}) = \frac{\partial^k}{\partial \theta^k} \mathbb{E}(e^{\theta A})$, for $\theta \in \mathbb{R}$, such that the expectation and their derivatives exist. Thus,

$$\mathbb{E}^{\mathbb{Q}} \left(\exp(-\Lambda(t, T)) (\Lambda(t, T))^k | \mathcal{F}_t \right) = \frac{\partial^k}{\partial \theta^k} \Big|_{\theta=-1} \mathbb{E}^{\mathbb{Q}} \left(\exp(\theta \Lambda(t, T)) | \mathcal{F}_t \right) \quad (17)$$

Moreover, $\Lambda(t, T) = \int_t^T (\eta_s + J_s) ds$, with η and J being independent. Hence,

$$(17) = \frac{\partial^k}{\partial \theta^k} \Big|_{\theta=-1} \left(S_{\eta}(\theta, t, T) S_J(\theta, t, T) \right)$$

with $S_{\eta}(\theta, t, T) := \mathbb{E}^{\mathbb{Q}} \left(e^{-\theta \int_t^T \eta_s ds} | \mathcal{F}_t \right)$ and $S_J(\theta, t, T) := \mathbb{E}^{\mathbb{Q}} \left(e^{-\theta \int_t^T J_s ds} | \mathcal{F}_t \right)$.

First, using standard results from affine term structures (see, e.g. Bingham and Kiesel (1998)) and the fact that η is of the form (4) with a state variable Z with dynamics (7)-(9), we get

$$S_{\eta}(\theta, t, T) = \exp \left\{ A(\theta, t, T) + B^{\top}(\theta, t, T) Z_t \right\}$$

where A, B solve (11)-(12).

Second,

$$\begin{aligned} \int_t^T J_u du &= \sum_{\tau_i \leq t} Y_i \int_t^T e^{-c(u-\tau_i)} du + \sum_{\tau_i \in (t, T]} Y_i \int_{\tau_i}^T e^{-c(u-\tau_i)} du \\ &= \sum_{\tau_i \leq t} Y_i e^{c\tau_i} \frac{e^{-ct} - e^{-cT}}{c} + \sum_{\tau_i \in (t, T]} Y_i \frac{1 - e^{-c(T-\tau_i)}}{c}. \end{aligned}$$

Note that the first term is measurable w.r.t. \mathcal{F}_t and equals $C(1, t, T) J_t$ with C as in (13). To compute the expectation containing the second part, we condition on the number of jumps in the interval $(t, T]$ and use the well-known fact that then jump

times have the same distribution as order statistics from i.i.d. uniform $(t, T]$ -random variables. Hence, π_1, π_2, \dots being uniform $(t, T]$,

$$\begin{aligned} & \mathbb{E} \left(\exp \left(\sum_{\tau_i \in (t, T)} \frac{\theta Y_i}{c} (1 - e^{-c(T-\tau_i)}) \right) \right) \\ &= e^{-l(T-t)} \sum_{k \geq 0} \frac{(l(T-t))^k}{k!} \mathbb{E} \left(\exp \left(\sum_{i=1}^k \frac{\theta Y_i}{c} (1 - e^{-c(T-\pi_i)}) \right) \right). \end{aligned}$$

Observe that with π_1 uniform on $(t, T]$,

$$\mathbb{E}^{\mathbb{Q}} \left[\exp \left(\frac{\theta Y_1}{c} (1 - e^{-c(T-\pi_1)}) \right) \right] = \frac{1}{T-t} \int_t^T \varphi \left(\frac{\theta}{c} (1 - e^{-c(T-u)}) \right) du$$

and we find

$$\begin{aligned} S_J(\theta, t, T) &= \mathbb{E}^{\mathbb{Q}} \left(e^{\int_t^T \theta J_s ds} | \mathcal{F}_t^J \right) \\ &= \exp \left(C(\theta, t, T) J_t + l(T-t) \left[-1 + \frac{1}{T-t} \int_t^T \varphi \left(\frac{\theta}{c} (1 - e^{-c(T-u)}) \right) \right] \right) \\ &= \exp \left(C(\theta, t, T) J_t + D(\theta, t, T) \right). \quad \blacksquare \end{aligned}$$

Proof of Proposition 3.2. We have to compute the conditional distribution of λ_t given only the number of jumps. Using Bayes' rule we have that

$$\mathbb{E}(\lambda_t | L_t = k\delta) = \frac{\mathbb{E}(\lambda_t \mathbf{1}_{\{L_t = k\delta\}})}{\mathbb{Q}(L_t = k\delta)}.$$

Recall that under the conditional independence assumption L_t is Poisson $(\Lambda(0, t))$. Then, using (16),

$$\mathbb{Q}(L_t = k\delta) = \frac{\partial^k}{\partial \theta^k} \Big|_{\theta=-1} S(\theta, 0, t).$$

Let us denote $\Lambda(0, t) = \int_0^t \lambda_s ds = \Lambda(t)$. To compute $\mathbb{E}(\lambda_t \mathbf{1}_{\{L_t = k\delta\}})$ note that

$$\begin{aligned} \mathbb{E}(\lambda_t \mathbf{1}_{\{L_t = k\delta\}}) &= \mathbb{E} \left(\mathbb{E} \left[\lambda_t \mathbf{1}_{\{L_t/\delta = k\}} | \Lambda(t) \right] \right) = \mathbb{E} \left(\lambda_t \exp(-\Lambda(t)) \frac{\Lambda(t)^k}{k!} \right) \\ &= \frac{1}{k!} \frac{\partial^k}{\partial \theta^k} \Big|_{\theta=-1} \left(\frac{1}{\theta} \frac{\partial}{\partial t} \mathbb{E} \left(e^{\theta \Lambda(t)} \right) \right) = \frac{1}{k!} \frac{\partial^k}{\partial \theta^k} \Big|_{\theta=-1} \frac{\partial}{\partial t} \frac{1}{\theta} S(\theta, 0, t). \quad \blacksquare \end{aligned}$$

APPENDIX B. EXAMPLE - SOME COMPUTATIONS

In our concrete illustration, the system of ODEs in (11)-(12) reduces to

$$\begin{aligned} \frac{\partial A}{\partial t}(\theta, t, T) + \kappa b B(\theta, t, T) &= 0 \\ \frac{\partial B}{\partial t}(\theta, t, T) + \kappa B + \frac{1}{2} \sigma^2 B^2(\theta, t, T) &= -\theta \end{aligned}$$

with the boundary conditions $A(\theta, T, T) = 0, B(\theta, T, T) = 0$ for all θ . The second ODE is a scalar Riccati equation with well-known solution.

$$B(\theta, t, T) = \frac{2\theta [e^{\psi(T-t)} - 1]}{(\psi - \kappa) [e^{\psi(T-t)} - 1] + 2\psi}$$

with $\psi = \sqrt{\kappa^2 - 2\sigma^2\theta}$. Given B , A can then be obtained by simple integration.⁸

$$A(\theta, t, T) = \kappa b \int_t^T B(\theta, u, T) du = \frac{2\kappa b}{\sigma^2} \ln \left(\frac{(\psi - \kappa) [e^{\psi(T-t)} - 1] + 2\psi}{2\psi e^{(\psi - \kappa)\frac{(T-t)}{2}}} \right)$$

To compute D we make use of the Laplace transform of the $\chi^2(2)$ distribution⁹. For $u < 0.5$ this is $\varphi(u) = \mathbb{E}(e^{u\chi^2_2}) = (1 - 2u)^{-1}$. Then we need to compute

$$D(\theta, t, T) = l \left[\int_t^T \varphi \left(\frac{\theta}{c} (1 - e^{-c(T-u)}) \right) du - (T - t) \right].$$

The integral equals

$$\int_t^T \frac{1}{1 - 2\frac{\theta}{c}(1 - e^{-c(T-u)})} du = \int_t^T \frac{1}{K_1 + K_2 \exp(cu)} du, \quad (18)$$

where $K_1 = 1 - 2\frac{\theta}{c}$ and $K_2 = 2\frac{\theta}{c} \exp(-cT)$. Thus,

$$(18) = \frac{\ln(K_2 + K_1 e^{-ct}) - \ln(K_2 + K_1 e^{-cT})}{K_1 c} = \frac{c(T-t) - \ln\left(1 + 2\frac{\theta}{c}(e^{-c(T-t)} - 1)\right)}{c - 2\theta}$$

and we obtain that

$$D(\theta, t, T) = l \left(\frac{c(T-t) - \ln\left(1 + 2\frac{\theta}{c}(e^{-c(T-t)} - 1)\right)}{c - 2\theta} - (T-t) \right).$$

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⁸A detailed treatment of solutions for scalar Riccati equations and how to compute some operators on those solutions (including the integral operator) can be found in Gaspar (2006).

⁹Recall that for $u \geq 0$ the Laplace transform of random variable which has χ^2 distribution with ν degrees of freedom, equals $\mathbb{E}(e^{-u\chi^2_\nu}) = (1 + 2u)^{-\nu/2}$.

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