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Abstract

This paper provides a unifying approach for valuing contingent claims on a portfolio of credits, such as collateralized debt obligations (CDOs). We introduce the defaultable (T, x)-bonds, which pay one if the aggregated loss process in the underlying pool of the CDO has not exceeded x at maturity T, and zero else. Necessary and sufficient conditions on the stochastic term structure movements for the absence of arbitrage are given. Background market risk as well as feedback contagion effects of the loss process are taken into account. Moreover, we show that any exogenous specification of the volatility and contagion parameters actually yields a unique consistent loss process and thus an arbitrage-free family of (T, x)-bond prices. For the sake of analytical and computational efficiency we then develop a tractable class of doubly stochastic affine term structure models.

Key words: affine term structure, collateralized debt obligations, loss process, single tranche CDO, term structure of forward spreads

1 Introduction

This paper provides a unifying approach for valuing contingent claims on a portfolio of credits, such as *collateralized debt obligations (CDOs)*. CDOs are securities backed by a pool of reference entities such as bonds, loans or credit default swaps. The reference entities form the *asset side* of a CDO-structure. Traded products are notes on the CDO *tranches*. They have different seniorities and build the *liability side* of the CDO.

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CDO markets have witnessed an extraordinary growth in the last decade. The most liquidly traded CDOs are those based on so-called indices. In 2004 the CDX in North-America and the Itraxx in Europe have been created and are nowadays very liquid. Both indices consist of the most liquidly traded and quoted credit default swaps in the given market, for example the corporate investment grade iTraxx in Europe consists of the 125 most liquid investment grades corporate credit default swaps. In addition to contingent claims based on the indices, there are also many options on the market spread of those indices and *single tranche CDOs (STCDOs)* (a STCDO is a credit default swap on a tranche, see Section 4 below for a definition); typically calls and puts. However, the corresponding products are less liquid and only quoted by a few market makers. For more background and references we refer to the respective chapters in [21].

Concerning the valuation of the basic STCDOs, the one-factor Gaussian copula approach [20] has emerged as the industry standard. It is basically a static description of the default times on the asset side of the CDO. However, it is well acknowledged that this approach has a number of deficiencies. First of all, there are only two parameters: the average default probability and a correlation parameter, which is a stylized version of Mertons asset correlation. This model is not able to capture all market quotes on the liability side with these two parameters. Therefore each tranche can only be priced with a different correlation, the so-called implied correlation. Moreover, the latest credit crises has illustrated that [20] appears to be insufficient to capture the dependence structure between the single names.

Recently, there have emerged several new attempts on CDO valuation based on the aggregate loss function ("top-down"), as opposed to the above mentioned ("bottom-up") single tranche default models. Giesecke and Goldberg [15] decompose the portfolio loss process into single name loss processes. Bennani [2] models, for a fixed maturity T and loss variable L_T, the T-Forward Loss process $L(t,T) = \mathbb{E}[L_T \mid \mathcal{F}_t]$ and the T-Forward Outstanding Notional $ON(t,T) = 1 - L(t,T)$. From there the excess loss $\mathbb{E}[(L(T,T) - K)^+]$ for strike K and maturity T can be computed numerically by Monte Carlo methods. However, this approach focuses on one maturity date T only, and neither market interest rate and nor spread risk is considered. Schönbucher [22] introduces the forward loss distributions and finds a Markov chain with the same marginal distribution as the loss process. Ehlers and Schönbucher [11] extend [22] by considering non-constant interest rates for pricing. They introduce conditional forward interest-rates $f_n(t,T)$ and forward protection rates (spreads) $F_n(t,T)$ given the realization of the loss process $L(t) = n$. An HJM-type specification of the loss-contingent forward interest and loss rates f_n and F_n is then proposed and no-arbitrage conditions are given. Ehlers and Schönbucher [12] analyze the interplay of the background (forward interest and protection rates, say) and loss process conditional on an increasing sequence of filtrations. However, the technical analysis in [22, 11, 12] relies on the assumption that the loss process lives on a finite grid, and their extension to multi-step increments (loss given default risk) becomes notationally demanding. The paper of Sidenius et al. (SPA) [24] is closest to our framework. However, SPA assume zero risk-free rates. Moreover, some crucial problems, e.g. regarding the construction of a consistent loss process, have remained open in [24] and will be completed from a global point of view in our paper. Some corresponding efficient calibration algorithms have recently been developed in Cont et al. [7, 8]. Cont and Minca [7] consider a finite set of maturities for traded CDO tranches which is in the spirit to the so-called term structure market models.

The aim of our paper is to provide a unifying approach for valuing contingent claims on CDOs, which encompasses the above mentioned and puts them on a common mathematical basis. We therefore introduce the defaultable (T, x)-bonds, which pay one if the aggregated CDO loss process has not exceeded x at maturity T, and zero else. It turns out that essentially all contingent claims on the CDO-pool, such as STCDOs, can be written—and thus hedged and priced—as linear combinations of (T, x)-bonds. We then model the corresponding (T, x)-forward rates, which equal the sum of risk-free forward rates plus forward spreads, as semimartingales driven by some Brownian motion, reflecting market information, and the jump measure associated to the loss process. This setup is universal, and allows for feedback, or contagion effects, from the loss process on the forward curve. As a first result, we provide necessary and sufficient conditions for the absence of arbitrage in terms of a drift condition and a relation between the short end of the spread curve and the prevailing loss intensity. Most important from a modelling point of view, we then provide mathematical evidence that arbitrage-free (T, x)-bond models uniquely exist under general assumptions. This is very much in the spirit of the Heath–Jarrow–Morton [16] approach to the modelling of the term structure of risk free interest rates. Risk-neutral pricing has to be done numerically in general. We will sketch a generic Monte-Carlo algorithm below. By omitting the aforementioned contagion effects we find an efficient CDO derivatives pricing formula. This extends the doubly stochastic framework for single name models to our multivariate setup. For the sake of analytical and computational efficiency we then develop a tractable class of doubly stochastic affine term structure models, which lead to closed form STCDO formulas.

The significance of our approach is its focus on the (T, x)-bonds and their exogenous stochastic specification. Albeit (T, x)-bonds are not directly traded, this perspective facilitates the mathematical analysis since the absence of arbitrage is expressed by two simple and clear formal conditions on the drift and short end of the spread curve. Correspondingly, on an integrated level, (T, x)-bonds are factorized into their default and market (forward spread) risk components. This representation corresponds to a stylized fact of financial markets: spread risk is what primarily drives CDO values; the objective default risk is secondary. Our focus on the (T, x)-bonds should also facilitate the empirical estimation for dynamic CDO term structure modelling, as it is the case for Heath–Jarrow–Morton [16] type forward rate models. The forward curve can be estimated from market data using bootstrapping and interpolation techniques. In some ongoing project, we use standardized swaps and STCDOs on standardized credit indexes such as iTraxx and CDX where liquid quotes are available for standard maturities, like 3, 5, 7 and 10 years for iTraxx, to calibrate our model. The next instruments one can use are call and put options on swaps and STCDOs. Our framework can also be used for an effective CDO risk management. Indeed, the forward spread volatilities can be exogenously specified, and the sensitivity analysis of the CDO portfolio with respect to various maturity and rating buckets is straightforward. Hedging via traded STCDOs is possible in principle. The development of market models with a focus on quoted values within this framework is ongoing research.

The structure of the paper is as follows. In Section 2, we formally introduce the (T, x)-bonds. In Section 3, we provide necessary and sufficient conditions for the absence of arbitrage. In Section 4, we derive STCDO and swaption price formulas. In Section 5, we give sufficient conditions on the stochastic basis such that arbitrage-free (T, x)-bond models uniquely exist under general assumptions. This is then further improved in the doubly stochastic framework in Section 6. In Section 7, we provide an affine specification for the doubly stochastic framework. We conclude in Section 8.

2 (T, x)-Bonds

As stochastic basis, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$. We assume that $\mathbb{Q}$ is a risk-neutral pricing measure. An equivalent measure change will be discussed below in Remark 3.4.

Consider a pool of credits (the CDO-pool) with an overall nominal normalized to 1, and let $\mathcal{I} = [0, 1]$ denote the set of loss fractions, i.e. $x \in \mathcal{I}$ represents the state where $100x\%$ of the overall nominal has defaulted.

We denote by L the $\mathcal{I}$-valued increasing aggregate CDO-*loss process*. That is, L_t represents the ratio of CDO-losses occurred by time t.

The basic instrument that we consider is a (T, x)*-bond* which pays $1_{\{L_T \leq x\}}$ at maturity T, for $x \in \mathcal{I}$. Its price at time $t \leq T$ is denoted by $P(t, T, x)$. Obviously, $P(t, T, x)$ is increasing in x and decreasing in T. Since $L_t \leq 1$ for all t, the risk free T*-bond* price $P(t, T)$ at time $t \leq T$ equals

$$P(t, T) = P(t, T, 1). \tag{1}$$

(T, x)-bonds are the fundamental components for the hedging and pricing of CDO-derivatives. Indeed, any European type contingent claim on the loss process with (regular enough) payoff function $F(L_T)$ at maturity T can be decomposed into a linear combination of (T, x)-bonds

$$F(L_T) = F(1) - \int_{\mathcal{I}} F'(x) 1_{\{L_T \leq x\}} \, dx.$$

Hence the static portfolio

$$F(1)P(t, T) - \int_{\mathcal{I}} F'(x) P(t, T, x) \, dx$$

replicates, and thus prices the claim at any time $t \leq T$, model independently. For example, the basic components of the payment leg of the STCDO in Section 4 below are put options with payoff $(K - L_T)^+ = \int_{(0,K]} 1_{\{L_T \leq x\}}\, dx$.

Remark 2.1. *Note that this setup contains the finite case $\mathcal{I} = \{\frac{i}{n} \mid i = 0, \ldots, n\}$ in particular. Indeed, if L can only assume fractions $\frac{i}{n}$, $i = 0, \ldots, n$, then $1_{\{L_T \leq x\}} = 1_{\{L_T \leq \frac{i}{n}\}}$, and hence $P(t,T,x) = P(t,T,\frac{i}{n})$, for all $x \in [\frac{i}{n}, \frac{i+1}{n})$.*

3 Arbitrage-free Term Structure Movements

Our aim is to describe the (T,x)-bond price term structure movements explicitly in the form

$$P(t,T,x) = 1_{\{L_t \leq x\}} e^{-\int_t^T f(t,u,x)\, du} \tag{2}$$

where $f(t,T,x)$ denotes the (T,x)-*forward rate* prevailing at date t. That is, $f(t,T,x)$ is the rate that one can contract for at time t, given that $L_t \leq x$, on a defaultable forward investment of one euro that begins at date T and is returned an instant dT later conditional on $L_{T+dT} \leq x$. The following forward rate agreement replicates the corresponding cash flows:

- at t: sell one (T,x)-bond and buy $\frac{P(t,T,x)}{P(t,T+dT,x)}$ $(T+dT,x)$-bonds.

This zero net investment at t yields the following future cash flows:

- at T: pay (“invest”) $1_{\{L_T \leq x\}}$ euro.
- at $T+dT$: receive $\frac{P(t,T,x)}{P(t,T+dT,x)} 1_{\{L_{T+dT} \leq x\}}$ euros.

The corresponding continuously compounded forward rate $f(t,T,x)$ is given, in first order in dT, by

$$e^{f(t,T,x)\, dT} \approx \frac{P(t,T,x)}{P(t,T+dT,x)}.$$

In the limit $dT \to 0$, we obtain (2).

Note that, in line with (1),

$$f(t,T) = f(t,T,1) \quad \text{and} \quad \phi(t,T,x) = f(t,T,x) - f(t,T) \tag{3}$$

are the corresponding risk free T-*forward rate* and (T,x)-*forward spread*, respectively.

The (T,x)-bond specification (2) combines default risk, $1_{\{L_t \leq x\}}$, and market risk, $f(t,T,x)$, in one instrument. This is a slight, but crucial difference to the SPA [24] framework where the market and default risks are specified by a two layer-process. In what follows we analyze the consistency of the loss process L and market term structure $f(t,T,x)$ movements in order that the (T,x)-bond market be free of arbitrage.

As for the loss process, we make the general assumption

(A1) $L_t = \sum_{s \le t} \Delta L_s$ is an $\mathcal{I}$-valued increasing marked point process[1] which admits an absolutely continuous compensator $\nu(t, dx)\, dt$.

This setup implies totally inaccessible default times of the (T, x)-bonds and a fundamental relation between their intensity processes and the compensator $\nu(t, dx)$:

Lemma 3.1. *Assume that* **(A1)** *holds. Then, for any $x \in \mathcal{I}$, the indicator process $1_{\{L_t \le x\}}$ is càdlàg with intensity process*

$$\lambda(t, x) = \nu(t, (x - L_t, 1] \cap \mathcal{I}). \tag{4}$$

That is,

$$M_t^x = 1_{\{L_t \le x\}} + \int_0^t 1_{\{L_s \le x\}} \lambda(s, x)\, ds \tag{5}$$

is a martingale. Moreover, $\lambda(t, x)$ is progressive, decreasing and càdlàg in $x \in \mathcal{I}$ with $\lambda(t, 1) = 0$.

Conversely, $\lambda(t, x)$ uniquely determines $\nu(t, dx)$ via

$$\nu(t, (0, x]) = \lambda(t, L_t) - \lambda(t, L_t + x), \quad x \in \mathcal{I}, \tag{6}$$

where we denote $\lambda(t, x) = 0$ for $x \ge 1$.

Note that SPA [24] postulate that $\lambda(t, x)$ exist with the above properties. With Lemma 3.1 we now put their assumption on a sound mathematical basis.

Proof. Right-continuity of $1_{\{L_t \le x\}}$ follows from the structure **(A1)** of L. By the very definition of $\nu(t, dx)$,

$$F(L_t) - \int_0^t \int_{\mathcal{I}} (F(L_s + y) - F(L_s)) \nu(s, dy)\, ds \tag{7}$$

is a martingale, for any bounded measurable function F. In particular, for $F(L_t) = 1_{\{L_t \le x\}}$ we have

$$F(L_s + y) - F(L_s) = -1_{\{L_s + y > x\}} 1_{\{L_s \le x\}}. \tag{8}$$

This proves (5). The other properties of $\lambda(t, x)$ hold by inspection. □

We next assume that any (T, x)-forward rate process follows a semimartingale of the form

$$f(t, T, x) = f(0, T, x) + \int_0^t a(s, T, x) ds + \int_0^t b(s, T, x)^\top \cdot dW_s + \int_0^t \int_{\mathcal{I}} c(s, T, x; y)\, \mu(ds, dy) \tag{9}$$

[1] Also called multivariate point process. For a definition see e.g. [17] or [3].

where W is some d-dimensional Brownian motion (*market noise*) and

$$\mu(\omega; dt, dx) = \sum_{s>0} 1_{\{\Delta L_s(\omega)\neq 0\}} \delta_{(s,\Delta L_s(\omega))}(dt, dx)$$

denotes the integer-valued random measure associated to the jumps of L, where we write δ_a for the Dirac measure at point a.

This specification is universal[2], and it allows for two kinds of feedback, or contagion, of the loss process on the forward rates:

(C1) direct, via simultaneous jumps driven by L: $\Delta f(t,T,x) = c(t,T,x;\Delta L_t)$.

(C2) indirect, via letting the model parameters a, b, and c be explicit functions of the prevailing loss path L ("regime switching").

Indeed, there seems to be empirical evidence for such contagion effects in U.S. industrial firm data covering the last three decades, see [9, 1]. We will provide, in Section 5 below, mathematical evidence that CDO term structure models (9) with properties **(C1)** and **(C2)** exist under very general assumptions.

To assert that the subsequent analysis and formal manipulations be meaningful, we make the following technical assumptions, where $\mathcal{O}$ and $\mathcal{P}$ denote the optional and predictable σ-algebra on $\Omega \times \mathbb{R}_+$, respectively:

(A2) the initial forward curve $f(0,T,x)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable, and locally integrable:

$$\int_0^T |f(0,u,x)|\, du < \infty \quad \text{for all } (T,x),$$

(A3) the *drift* parameter $a(t,T,x)$ is $\mathbb{R}$-valued $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable, and locally integrable:

$$\int_0^T \int_0^T |a(t,u,x)|\, dt\, du < \infty \quad \text{for all } (T,x),$$

(A4) the *volatility* parameter $b(t,T,x)$ is $\mathbb{R}^d$-valued $\mathcal{O}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathcal{I})$-measurable, and locally bounded:

$$\sup_{t\le u\le T} \|b(t,u,x)\| < \infty \quad \text{for all } (T,x),$$

(A5) the *contagion* parameter $c(t,T,x;y)$ is $\mathbb{R}$-valued $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathcal{I})\otimes\mathcal{B}(\mathcal{I})$-measurable, and locally bounded:

$$\sup_{t\le u\le T,\, y\in\mathcal{I}} |c(t,u,x;y)| < \infty \quad \text{for all } (T,x).$$

[2]This framework can obviously be further generalized by adding Lévy and/or Poisson random measure driven market noise to W.

Conditions **(A2)**–**(A5)** assert that the risk free *short rate* $r_t = f(t,t)$ has a progressive version and satisfies $\int_0^T |r_t|\,dt < \infty$ for all T, see e.g. [14]. Hence the *savings account* $e^{\int_0^t r_s ds}$ is well defined.

It is well known that there exists no admissible[3] arbitrage strategy in the (T,x)-bond market if the discounted price processes

$$e^{-\int_0^t r_s ds} P(t,T,x) \text{ are local martingales for all } (T,x). \tag{10}$$

We now give necessary and sufficient conditions for (10) to hold.

Theorem 3.2. *Assume* **(A1)**–**(A5)** *hold. Then the no-arbitrage condition* (10) *is equivalent to*

$$\begin{aligned}\int_t^T a(t,u,x)\,du &= \frac{1}{2}\left\|\int_t^T b(t,u,x)\,du\right\|^2 \\ &\quad + \int_{\mathcal{I}} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1\right) 1_{\{L_t + y \le x\}}\,\nu(t,dy), \qquad (11)\\ r_t + \lambda(t,x) &= f(t,t,x) \qquad (12)\end{aligned}$$

on $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$*-a.s. for all* (T,x).

Note that (12) is part of the assumptions in the SPA [24] framework, while here we demonstrate that this is in fact a necessary consequence of the no-arbitrage condition (10).

Proof. We denote

$$p(t,T,x) = e^{-\int_t^T f(t,u,x)\,du} \tag{13}$$

so that $P(t,T,x) = 1_{\{L_t \le x\}} p(t,T,x)$. Using a stochastic Fubini argument proposed by Heath et al. [16], see also [14], we transform

$$\int_t^T \int_0^t \cdots ds\,du = \int_0^t \int_t^T \cdots du\,ds = \int_0^t \int_s^T \cdots du\,ds - \int_0^t \int_0^u \cdots ds\,du,$$

and similarly for $dW_s\,du$ and $\mu(ds,dy)\,du$. We thus derive by Itô's formula

$$\begin{aligned}\frac{dp(t,T,x)}{p(t-,T,x)} = \Bigg\{ f(t,t,x) &- \int_t^T a(t,u,x)\,du + \frac{1}{2}\left\|\int_t^T b(t,u,x)\,du\right\|^2 \\ &+ \int_{\mathcal{I}} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1\right)\nu(t,dy)\Bigg\} dt \\ &- \int_t^T b(t,u,x)^\top\,du \cdot dW_t \\ &+ \int_{\mathcal{I}} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1\right)\left(\mu(dt,dy) - \nu(t,dy)\,dt\right). \qquad (14)\end{aligned}$$

[3] A self-financing trading strategy is admissible if its discounted value process is a supermartingale, e.g. bounded from below or a martingale.

Denote $Z(t,T,x) = e^{-\int_0^t r_s ds} P(t,T,x)$. Integrating by parts and using (5) yields

$$\frac{dZ(t,T,x)}{Z(t-,T,x)} = -r_t dt + dM_t^x - \lambda(t,x)dt + \frac{dp(t,T,x)}{p(t-,T,x)} + \frac{d\left[M_t^x, p(t,T,x)\right]}{p(t-,T,x)}. \quad (15)$$

Combining (14) and (15) shows that (10) holds if and only if

$$\begin{aligned} - r_t - \lambda(t,x) + f(t,t,x) - \int_t^T a(t,u,x)\,du + \frac{1}{2}\left\|\int_t^T b(t,u,x)\,du\right\|^2 \\ + \int_{\mathcal{I}} \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1\right) 1_{\{L_t + y \le x\}}\, \nu(t,dy) = 0 \quad (16) \end{aligned}$$

on $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$-a.s. for all (T,x).

Setting $T = t$, we obtain that (16) is equivalent to (11)–(12). $\square$

Static arbitrage strategies, such as holding long a (T,x)-bond and short a (T,y)-bond if $P(0,T,x) > P(0,T,y)$ for some $x < y$, may not be admissible and hence not excluded by condition (10). Hence (10) does not necessarily assert monotonicity of $P(t,T,x)$ in T and x. Note, however, that (12) implies monotonicity of $f(t,t,x)$ in x in any case. The following corollary is obvious:

Corollary 3.3. *If the discounted prices processes in* (10) *are true martingales then $P(t,T,x)$ is decreasing in T and increasing in x. This holds in particular if* (11)–(12) *are satisfied and forward rates are positive: $f(t,T,x) \ge 0$.*

Remark 3.4. *We present our approach under the assumption that $\mathbb{Q}$ is a risk-neutral measure, i.e. the no-arbitrage condition* (10) *is supposed to hold under $\mathbb{Q}$. It is of course possible to consider the above dynamic equations with respect to some objective probability measure $\mathbb{P} \sim \mathbb{Q}$. The measure change from $\mathbb{P}$ to $\mathbb{Q}$ will have the following impact:*

$$\begin{aligned} a(t,T,x) &\rightsquigarrow a(t,T,x) + b(t,T,x)^\top \cdot \Phi(t) \\ \nu(t,dx) &\rightsquigarrow \Psi(t,x)\nu(t,dx) \end{aligned}$$

for some appropriate stochastic processes $\Phi(t)$ and $\Psi(t,x)$ with values in $\mathbb{R}^d$ and $(0,\infty)$, respectively. We do not intend to provide further general results on this, as it is rather standard and regularity conditions have to be checked from case to case. For a general reference see Theorem III.3.24 in [19], for Markovian models see also [6].

4 Single Tranche CDOs (STCDOs)

The standard instrument for investing in a CDO-pool is a *single tranche CDO (STCDO)*, also called *tranche credit default swap*. In this intermediary section, we formally define and value this key instrument.

A STCDO is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$,
- a *tranche* with lower and upper detachment points $x_1 < x_2$ in $\mathcal{I}$,
- a fixed swap rate κ.

We write

$$H(x) := (x_2 - x)^+ - (x_1 - x)^+ = \int_{(x_1,x_2]} 1_{\{x \le y\}} dy.$$

An investor in this STCDO

- receives $\kappa H(L_{T_i})$ at T_i, $i = 1, \dots, n$ (payment leg),
- pays $-dH(L_t) = H(L_{t-}) - H(L_t)$ at any time $t \in (T_0, T_n]$ where $\Delta L_t \neq 0$ (default leg).

As in (13), we denote the $(\mathcal{G}_t)$-adapted part of the (T,x)-bond price by $p(t,T,x)$, so that $P(t,T,x) = 1_{\{L_t \le x\}} p(t,T,x)$.

Lemma 4.1. *The value of the STCDO at time $t \le T_0$ is*

$$\Gamma(t,\kappa) = \int_{(x_1,x_2]} 1_{\{L_t \le y\}} \left(\kappa \sum_{i=1}^{n} p(t,T_i,y) - p(t,T_0,y) + p(t,T_n,y) + \gamma(t,y) \right) dy \tag{17}$$

where $\gamma(t,y) = \int_{T_0}^{T_n} \mathbb{E}\left[r_u e^{-\int_t^u r_s\, ds} 1_{\{L_u \le y\}} \mid \mathcal{F}_t \right] du$.

Moreover, if the risk free rates $f(s,u)$ and the loss process L_s, for $t \le s \le u$, are $\mathcal{F}_t$-conditionally independent then $\gamma(t,y)$ in (17) can be replaced by

$$\gamma(t,y) = \int_{T_0}^{T_n} f(t,u) p(t,u,y)\, du.$$

Proof. The value of the payment leg at time $t \le T_0$ is

$$\mathbb{E}\left[\sum_{i=1}^{n} e^{-\int_t^{T_i} r_s ds} \kappa H(L_{T_i}) \mid \mathcal{F}_t \right] = \kappa \sum_{i=1}^{n} \int_{(x_1,x_2]} P(t,T_i,y) dy.$$

Next we use integration by parts to calculate

$$\begin{aligned} &\int_{T_0}^{T_n} e^{-\int_t^u r_s ds} dH(L_u) \\ &\quad = e^{-\int_t^{T_n} r_s ds} H(L_{T_n}) - e^{-\int_t^{T_0} r_s ds} H(L_{T_0}) + \int_{T_0}^{T_n} r_u e^{-\int_t^u r_s ds} H(L_u) du. \end{aligned}$$

The (negative) value of the default leg at time $t \le T_0$ for the investor is then given as $\mathcal{F}_t$-conditional expectation. Summing up the two legs, we obtain (17).

The second part of the lemma follows since

$$\mathbb{E}\left[r_u e^{-\int_t^u r_s\,ds} \mid \mathcal{F}_t\right] = f(t,u)P(t,u).$$

$\square$

The *forward STCDO swap rate* κ_t^* prevailing at $t \le T_0$ is the rate which gives $\Gamma(t,\kappa_t^*) = 0$. In view of (17) hence

$$\kappa_t^* = \frac{\int_{(x_1,x_2]} 1_{\{L_t \le y\}} \left(p(t,T_0,y) - p(t,T_n,y) - \gamma(t,y)\right) dy}{\sum_{i=1}^n \int_{(x_1,x_2]} 1_{\{L_t \le y\}} p(t,T_i,y)\,dy}.$$

A STCDO *swaption* with strike rate K gives the holder the right to enter the above STCDO with swap rate K at swaption maturity T_0. Its value at T_0 is thus $\Gamma(T_0,K)^+$. Note that, since $\Gamma(T_0,\kappa_{T_0}^*) = 0$, this swaption payoff can also be written as

$$\left(\sum_{i=1}^n \int_{(x_1,x_2]} 1_{\{L_t \le y\}} p(T_0,T_i,y)\,dy\right)\left(K - \kappa_{T_0}^*\right)^+. \tag{18}$$

As it is the case for single name models, e.g. [10, 23, 5], there is no closed form solution for swaption prices available in general. See however Remark 7.3 below.

5 A Martingale Problem

Theorem 3.2 states that, under the no-arbitrage condition (10), the drift parameter $a(t,T,x)$ is determined by the volatility and contagion parameters $b(t,T,x)$ and $c(t,T,x)$, respectively. However, there is still an implicit relation between the loss process L and the short end of the forward curve $f(t,t,x)$ in (12) which cannot be expressed directly in terms of the volatility and contagion parameters.

This circumstance has been addressed in the previous works [2, 22, 11, 12, 24] by ad-hoc methods, such as the construction of conditional Markov loss processes given the market information. This special case will be further discussed in Section 6 below.

In this section, we provide mathematical evidence that arbitrage-free CDO term structure models (9) with properties **(C1)** and **(C2)**, in fact, uniquely exist under general assumptions. Our framework contains and unifies the approaches in [2, 22, 11, 12, 24] as particular cases.

Without loss of generality we henceforth assume that the stochastic basis satisfies:

(A6) $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}$, $\mathbb{Q}(d\omega) = \mathbb{Q}_1(d\omega_1)\mathbb{Q}_2(\omega_1, d\omega_2)$, where $\omega = (\omega_1,\omega_2) \in \Omega$, and $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$, where

(i) $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ is some filtered probability space carrying the market information, in particular the Brownian motion $W(\omega) = W(\omega_1)$,

(ii) $(\Omega_2, \mathcal{H})$ is the canonical space of paths for $\mathcal{I}$-valued increasing marked point processes endowed with the minimal filtration $(\mathcal{H}_t)$: the generic $\omega_2 \in \Omega_2$ is a càdlàg, increasing, piecewise constant function from $\mathbb{R}_+$ to $\mathcal{I}$. Henceforth, we let the loss process

$$L_t(\omega) = \omega_2(t)$$

be the coordinate process. The filtration $(\mathcal{H}_t)$ is thus $\mathcal{H}_t = \sigma(L_s \mid s \leq t)$, and $\mathcal{H} = \mathcal{H}_\infty$,

(iii) $\mathbb{Q}_2$ is a probability kernel from $(\Omega_1, \mathcal{G})$ to $\mathcal{H}$ to be determined below.

This setup implies that the volatility and contagion parameters

$$b(\omega; t, T, x) = b(\omega_1, \omega_2; t, T, x), \quad c(\omega; t, T, x) = c(\omega_1, \omega_2; t, T, x)$$

in **(A4)**–**(A5)** actually are functions of the loss path ω_2. Hence the indirect contagion property **(C2)** is satisfied. The evolution equation (9) can thus be solved on the stochastic basis $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ along any genuine loss path $\omega_2 \in \Omega_2$. Indeed, the integral with respect to μ in (9) is path-wise in ω_2. However, in view of (6), condition (12) is equivalent to

$$\nu(\omega; t, dx) = -f(\omega; t, t, \omega_2(t) + dx), \quad (\text{set } f(t, t, x) \equiv r_t \text{ for } x \geq 1). \tag{19}$$

Hence, unless the contagion parameter c is zero,

$$a(t, T, x) = a(t, T, x, f(t, \cdot))$$

in (11) becomes via (19) an explicit linear functional of the (short end of the) prevailing spread curve. In fact, there may result an implicit non-linear smooth dependence on the entire prevailing spread curve $f(t, \cdot)$ via b and c in (11), respectively. But since this dependence on $f(t, \cdot)$ is smooth, for any given loss path $\omega_2 \in \Omega_2$, equation (9) will generically be uniquely solvable.

It thus remains to find a probability kernel $\mathbb{Q}_2$ such that ν in (19) becomes the compensator of L. This is a martingale problem for marked point processes, which has completely been solved by Jacod [17]. It turns out that $\mathbb{Q}_2$ exists and is unique.

Theorem 5.1. *Assume* **(A6)** *holds. Let $f(0, T, x)$, $b(t, T, x)$ and $c(t, T, x)$ satisfy* **(A2)**, **(A4)** *and* **(A5)**, *respectively. Define $\nu(t, dx)$ by* (19) *and $a(t, T, x)$ by* (11) *for all (t, T, x).*

Suppose, for any loss path $\omega_2 \in \Omega_2$, there exists a solution $f(t, T, x)$ of (9) *such that $f(t, t, x)$ is progressive, decreasing and càdlàg in $x \in \mathcal{I}$. Then*

(i) **(A3)** *is satisfied.*

(ii) *there exists a unique probability kernel $\mathbb{Q}_2$ from $(\Omega_1, \mathcal{G})$ to $\mathcal{H}$, such that the loss process $L_t(\omega) = \omega_2(t)$ satisfies* **(A1)** *and the no-arbitrage condition* (10) *holds.*

(iii) $\nu(t,dx)\,dt$ *is the compensator of* L *with respect to* $(\mathcal{G}\otimes\mathcal{H}_t)$*. Moreover,*

$$\begin{aligned}&\mathbb{Q}\left[\tau_{n+1}-\tau_n>t \mid \mathcal{G}\otimes\mathcal{H}_{\tau_n}\right]\\&\quad=\mathbb{Q}_2\left[\tau_{n+1}-\tau_n>t \mid \mathcal{H}_{\tau_n}\right]=\mathrm{e}^{-\int_{\tau_n}^{\tau_n+t}\nu(\omega_1,\omega_2(\tau_n);s,\mathcal{I})\,ds},\quad t\geq 0 \qquad (20)\end{aligned}$$

and

$$\mathbb{Q}\left[\Delta L_{\tau_n}\in A \mid \mathcal{G}\otimes\mathcal{H}_{\tau_n-}\right]=\mathbb{Q}_2\left[\Delta L_{\tau_n}\in A \mid \mathcal{H}_{\tau_n-}\right]=\frac{\nu(\tau_n,A)}{\nu(\tau_n,\mathcal{I})} \qquad (21)$$

for all $A\in\mathcal{B}(\mathcal{I})$ *on* $\{\tau_n<\infty\}$*, where* $0<\tau_1<\tau_2<\cdots$ *denote the successive jump times of* L*.*

(iv) $\mathbb{Q}_2(\cdot,A)$ *is* $\mathcal{G}_t$*-measurable for all* $A\in\mathcal{H}_t$ *and* t*. Consequently, every* $(\mathcal{G}_t)$*-martingale is a* $(\mathcal{F}_t)$*-martingale.*

Remark 5.2. *Property* (iv) is known as "(H)-hypothesis", see [4, 12]. Property (iii) has been explored in [12] as "successive (H)-property" for finite $\mathcal{I}$. The formulas (20)–(21) will be most useful for numerical implementations as sketched in Section 5.1 below.

Proof. That **(A3)** holds follows from the local boundedness assumptions **(A4)**–**(A5)**, which in particular imply that $\int_0^T |f(t,t,0)|\,dt<\infty$ for all T, see [14]. Whence (i).

By Lemma 3.1 and (12), if the loss process L satisfies **(A1)** and the no-arbitrage condition (10) holds, its compensator is necessarily of the form (19). Theorem 3.6 in [17] now implies that there exists a unique probability kernel $\mathbb{Q}_2$ from Ω_1 to $\mathcal{H}$, such that ν is the compensator of L. Indeed, Jacod's [17] notation (in quotation marks) corresponds to ours as follows: "$\Omega=\Omega'\times\Omega''$"$\leftrightarrow$ $\Omega=\Omega_1\times\Omega_2$, "$\mathcal{F}_0$"$\leftrightarrow\mathcal{G}\otimes\{\emptyset,\Omega_2\}$, "$\mathcal{F}_t$"$\leftrightarrow\mathcal{G}\otimes\mathcal{H}_t$, "$P_0$"$\leftrightarrow\mathbb{Q}_1$ (its trivial extension to $\mathcal{G}\otimes\{\emptyset,\Omega_2\}$, respectively). In particular, what Jacod [17] refers to as "past" ("$\mathcal{F}_0$") corresponds to our market information $\mathcal{G}$. In [17, eqn (10)], Jacod then recursively defines a unique probability measure "P" on "$(\Omega,\mathcal{F}_\infty)$", starting with a transition probability from "$(\mathcal{F}_0,P_0)$". This yields our probability kernel $\mathbb{Q}_2$. By construction, in [17], ν becomes the predictable projection of μ on "$(\Omega,\mathcal{F},(\mathcal{F}_t),P)$", which is $(\mathcal{G}\otimes\mathcal{H}_t)$. This proves (ii) and the first part of (iii). In view of [3, Section VIII.1], $\nu(t,\mathcal{I})$ is the $(\mathcal{G}\otimes\mathcal{H}_t)$-intensity of the point process $\{\tau_n\}$. Hence, on $\{\tau_n<\infty\}$,

$$\begin{aligned}\Phi(t)&=\mathbb{Q}[\tau_{n+1}-\tau_n>t \mid \mathcal{G}\otimes\mathcal{H}_{\tau_n}]\\&=1+\mathbb{E}\left[\int_{\tau_n}^{\tau_n+t}\nu(s,\mathcal{I})1_{\{\tau_{n+1}-\tau_n>s\}}\,ds \mid \mathcal{G}\otimes\mathcal{H}_{\tau_n}\right]\\&=1+\int_{\tau_n}^{\tau_n+t}\nu(\omega_1,\omega_2(\tau_n);s,\mathcal{I})\Phi(s)\,ds,\end{aligned}$$

which implies (20). Formula (21) follows from [3, Theorem 6, Chapter VIII].

Now fix T and replace "$\mathcal{F}_0$" above by $\mathcal{G}_T \otimes \{\emptyset, \Omega_2\}$ and "$\mathcal{F}_t$" by $\mathcal{G}_T \otimes \mathcal{H}_t$. Then $\nu(t, dx)\, dt$, $t \le T$, is "$(\mathcal{F}_t)$"-predictable and Jacod's [17] argument above yields a unique probability kernel $\mathbb{Q}_2^{(T)}$ from $(\Omega_1, \mathcal{G}_T)$ to $\mathcal{H}_T$ such that ν becomes the predictable projection of μ, restricted on the time interval $[0, T]$. But then, by uniqueness again, we conclude that $\mathbb{Q}_2(\cdot, A) = \mathbb{Q}_2^{(T)}(\cdot, A)$ is $\mathcal{G}_T$-measurable for all $A \in \mathcal{H}_T$. From this, (iv) follows. $\square$

Remark 5.3. *The SPA [24] paper leaves open the question whether a "conditional Markov" loss process consistent with the loss distributions for all $x \in \mathcal{I}$ can be constructed in their way. Theorem 5.1 now resolves and clarifies this in full generality, as it implies that the law of the loss process L is* uniquely *determined by $f(0, T, x)$, $b(t, T, x)$ and $c(t, T, x)$.*

Remark 5.4. *Equation* (9) *falls essentially into the class of "non-classical" stochastic differential equations of Jacod and Protter [18] where the characteristics of certain driving semimartingales depend on the solution-process. However, their framework is univariate and does not explicitly address marked point process drivers and nor* **(C2)**.

5.1 A Monte-Carlo Algorithm

The pricing of loss path-dependent CDO derivatives, such as the default leg of a STCDO or swaptions in Section 4 above, becomes a computational issue. In general, one has to resolve to Monte-Carlo methods such as the following. Fix a finite time horizon τ, and denote by $f(0, T, x)$ a given initial forward curve, for $T \le \tau$ and $x \in \mathcal{I}$. We now sketch an algorithm to simulate N trajectories for the joint process $(f(t, T, x), L_t)$, $t \le T \le \tau$.

(i) Simulate N independent samples of ω_1, i.e. standard Brownian paths $\omega_1^{(1)}(t), \dots, \omega_1^{(N)}(t)$, $t \in [0, \tau]$.

(ii) Initialize: set $T^{(i,0)} := 0$, $f^{(i,0)}(0, T, x) := f(0, T, x)$, $j := 1$, and the initial placeholder loss process $\omega_2^{(i,j)}(t) := 0$, $t \in [0, \tau]$, for $i = 1, \dots, N$.

(iii) Solve (9) along $(\omega_1^{(i)}, \omega_2^{(i,j)})$, e.g. via Euler scheme. This gives $f^{(i,j)}(t, T, x)$ for all $t \le T \le \tau$ and $x \in \mathcal{I}$. In fact, you can set $f^{(i,j)}(t, T, x) := f^{(i,j-1)}(t, T, x)$ for $t < T^{(i,j-1)}$.

(iv) Simulate an independent standard exponential random variable $\epsilon^{(j)}$, set $\lambda^{(i,j)}(t, x) := f^{(i,j)}(t, t, x) - f^{(i,j)}(t, t, 1)$ ($:= 0$ for $x \ge 1$), and determine the jth jump time via

$$T^{(i,j)} = \inf \left\{ t \ge T^{(i,j-1)} \mid \int_{T^{(i,j-1)}}^{t} \lambda^{(i,j)}(s, \omega_2^{(i,j)}(T^{(i,j-1)}))\, ds \ge \epsilon^{(j)} \right\}.$$

This is justified by (20).

(v) If $T^{(i,j)} > \tau$, set $\omega_2^{(i)}(t) := \omega_2^{(i,j)}(t)$ for all $t \in [0, \tau]$. The ith path $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)})$ is thus fully simulated and can be omitted in the following. Continue with those i where $T^{(i,j)} \leq \tau$. The algorithm terminates if no such i is left.

(vi) Simulate the jth loss jump size $\Delta L_{T^{(i,j)}}$ as independent $\mathcal{I}$-valued random variable with distribution function

$$\frac{\lambda^{(i,j)}(T^{(i,j)}, \omega_2^{(i,j)}(T^{(i,j)})) - \lambda^{(i,j)}(T^{(i,j)}, \omega_2^{(i,j)}(T^{(i,j)}) + x)}{\lambda^{(i,j)}(T^{(i,j)}, \omega_2^{(i,j)}(T^{(i,j)}))}, \quad x \in \mathcal{I}.$$

This is justified by (21). Update the loss path

$$\omega_2^{(i,j+1)}(t) = \begin{cases} \omega_2^{(i,j)}(t), & t < T^{(i,j)}, \\ \omega_2^{(i,j)}(t) + \Delta L_{T^{(i,j)}}, & t \geq T^{(i,j)}. \end{cases}$$

(vii) Set $j := j + 1$ and repeat from (iii).

Note that, in every iteration, we have to assume that $f^{(i,j)}(t, t, x)$ is decreasing and càdlàg in $x \in \mathcal{I}$.

6 Doubly Stochastic Framework

As a special case of the above general framework we now omit both contagion effects **(C1)** and **(C2)** and assume that

(A7) the volatility parameter $b(\omega; t, T, x) = b(\omega_1; t, T, x)$ is $(\mathcal{G}_t)$-adapted, and $c(t, T, x; y) \equiv 0$.

Note that $a(t, T, x)$ in (11) simplifies considerably. In particular, there is no explicit dependence on $f(t, t, \cdot)$ via (19) anymore. Moreover, the forward curve $f(\omega; t, T, x) = f(\omega_1; t, T, x)$ becomes a function of ω_1 only.

Theorem 5.1 can now be improved and extended by a very useful formula for CDO derivatives pricing.

Theorem 6.1. *Suppose* **(A7)** *and the assumptions in Theorem 5.1 hold.*

Then the loss process L becomes $\mathcal{G}$-conditional Markov under $\mathbb{Q}_2$. Moreover, for any positive $\mathcal{G}$-measurable random variable X and all $x \in \mathcal{I}$,

$$\mathbb{E}[X 1_{\{L_T \leq x\}} \mid \mathcal{F}_t] = 1_{\{L_t \leq x\}} \mathbb{E}\left[X e^{-\int_t^T \lambda(s,x) ds} \mid \mathcal{G}_t \right]. \tag{22}$$

Proof. The $\mathcal{G}$-conditional Markov property of L follows since, for given ω_1, the compensator $\nu(\omega; t, dx) = -f(\omega_1; t, t, \omega_2(t) + dx)$ in (19) is now a function of the current loss level $\omega_2(t)$ only.

Now let $X \geq 0$ be $\mathcal{G}$-measurable. Since $\lambda(t,x)$ is $\mathcal{G}$-measurable, we obtain

$$\begin{aligned}
\Phi(T) &:= \mathbb{E}\left[X 1_{\{L_T \leq x\}} \mid \mathcal{G} \otimes \mathcal{H}_t\right] \\
&= \mathbb{E}\left[X M_T^x - \int_0^T X 1_{\{L_s \leq x\}} \lambda(s,x) ds \mid \mathcal{G} \otimes \mathcal{H}_t\right] \\
&= X M_t^x - \int_0^T \lambda(s,x) E\left[X 1_{\{L_s \leq x\}} \mid \mathcal{G} \otimes \mathcal{H}_t\right] ds \\
&= X 1_{\{L_t \leq x\}} - \int_t^T \lambda(s,x) \Phi(s) ds.
\end{aligned}$$

We infer that

$$\Phi(T) = X 1_{\{L_t \leq x\}} e^{-\int_t^T \lambda(s,x) ds}.$$

Conditioning on $\mathcal{F}_t$ yields (22). □

Formula (22) states that the market information $\mathcal{G}$ is enough to price any, possibly loss path-dependent, CDO derivative. Indeed, a simple iteration of (22) yields

$$\mathbb{Q}\left[\bigcap_{i=1}^n \{L_{t_i} \leq x_i\} \cap A \mid \mathcal{F}_{t_0}\right] = 1_{\{L_{t_0} \leq \min_i x_i\}} \mathbb{E}\left[\prod_{i=1}^n \mathrm{e}^{-\int_{t_{i-1}}^{t_i} \lambda(s,x_i)\, ds} 1_A \mid \mathcal{G}_{t_0}\right]$$

for all $t_0 < \cdots < t_n$ and $A \in \mathcal{G}$. Hence the $\mathcal{F}_{t_0}$-conditional law of the loss process is given by a $\mathcal{G}_{t_0}$-conditional expectation of a functional of the $\mathcal{G}$-measurable compensator $\lambda(t,x)$. This property generalizes the concept of a doubly stochastic Poisson process to marked point processessee e.g. [3, Section II.1] or [21, Chapter 9].

Applying (22) and (12) to the STCDO formula (17), we obtain the following corollary.

Corollary 6.2. *If the doubly stochastic assumptions of Theorem 6.1 are in force then $\gamma(t,y)$ in (17) can be replaced by*

$$\gamma(t,y) = \int_{T_0}^{T_n} \mathbb{E}\left[r_u e^{-\int_t^u f(s,s,y) ds} \mid \mathcal{G}_t\right] du.$$

7 Doubly Stochastic Affine Term Structure

In this section we consider an analytically tractable class of Markov factor models for the term structure movements (9) in the doubly stochastic framework. We assume that **(A6)** holds. Let $\mathcal{Z} \subset \mathbb{R}^d$ be some closed state space with non-empty interior and Z some $\mathcal{Z}$-valued diffusion process satisfying

$$\begin{aligned}
dZ_t &= \mu(Z_t) dt + \sigma(Z_t) \cdot dW_t, \\
Z_0 &= z
\end{aligned} \tag{23}$$

where μ and σ are continuous functions from $\mathbb{R}_+ \times \mathcal{Z}$ into $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively.

In what follows we consider *affine term structure* models of the form

$$f(t,T,x) = A'(t,T,x) + B'(t,T,x)^\top \cdot Z_t$$

that is, in terms of (9),

$$\begin{aligned} a(t,T,x) &= \partial_t A'(t,T,x) + \partial_t B'(t,T,x)^\top \cdot Z_t + B'(t,T,x)^\top \cdot \mu(Z_t) \\ b(t,T,x) &= B'(t,T,x)^\top \cdot \sigma(Z_t) \end{aligned} \tag{24}$$

for some functions $A'(t,T,x)$ and $B'(t,T,x)$ with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote

$$A(t,T,x) = \int_t^T A'(t,u,x)du, \quad B(t,T,x) = \int_t^T B'(t,u,x)du.$$

The following theorem gives a characterization of those affine term structure models which satisfy the no-arbitrage condition (10).

Theorem 7.1. *Assume that, for all $z \in \mathcal{Z}$, there exists a $\mathcal{Z}$-valued continuous solution $Z = Z^z$ of* (23) *such that the coefficients given in* (24) *satisfy* (11) *for all $t \le T$ and x a.s. If the $d + \frac{d(d+1)}{2}$ functions in (T,x),*

$$B_i(0,T,x), \quad B_k(0,T,x)B_l(0,T,x), \quad k \le l, \tag{25}$$

are linearly independent, then Z is necessarily affine. *That is, drift and diffusion matrix are affine functions of $z = (z_1,\ldots,z_d) \in \mathcal{Z}$:*

$$\mu(z) = \mu_0 + \sum_{i=1}^d z_i\mu_i, \quad \frac{1}{2}\sigma\cdot\sigma^\top(z) = \nu_0 + \sum_{i=1}^d z_i\nu_i \tag{26}$$

for some vectors $\mu_i \in \mathbb{R}^d$ and matrices $\nu_i \in \mathbb{R}^{d\times d}$. Moreover, A and B solve the following system of Riccati equations, *for $t \le T$,*

$$\begin{aligned} -\partial_t A(t,T,x) &= A'(t,t,x) + \mu_0^\top \cdot B(t,T,x) - B(t,T,x)^\top \cdot \nu_0 \cdot B(t,T,x) \\ A(T,T,x) &= 0 \\ -\partial_t B_i(t,T,x) &= B_i'(t,t,x) + \mu_i^\top \cdot B(t,T,x) - B(t,T,x)^\top \cdot \nu_i \cdot B(t,T,x) \\ B(T,T,x) &= 0 \end{aligned} \tag{27}$$

for all (T,x).

Proof. Note that

$$\int_t^T \partial_t A'(t,u,x)du = \partial_t A(t,T,x) + A'(t,t,x),$$

and analogously for B'. Hence (11) yields

$$\partial_t A(t,T,x) + A'(t,t,x) + (\partial_t B(t,T,x) + B'(t,t,x))^\top \cdot Z_t + B(t,T,x)^\top \cdot \mu(Z_t) \\ = \frac{1}{2} B(t,T,x)^\top \cdot \sigma \cdot \sigma^\top(Z_t) \cdot B(t,T,x). \quad (28)$$

Letting $t \downarrow 0$, by continuity, we obtain the respective equality for Z_t replaced by z, for all T, x and z. We infer that

$$B(0,T,x)^\top \cdot \mu(z) + B(0,T,x)^\top \cdot \frac{\sigma \cdot \sigma^\top(z)}{2} \cdot B(0,T,x)$$

is an affine function in z, for all T and x. By assumption (25), we conclude that μ and $\sigma \cdot \sigma^\top/2$ must be affine functions of the form (26).

Plugging (26) back in (28) and separating first order terms in z_i, we obtain (27). □

The next theorem is the converse to Theorem 7.1 and gives sufficient conditions for the existence of an arbitrage-free affine term structure model.

Theorem 7.2. *Assume μ and $\sigma\sigma^\top$ are affine of the form* (26)*. Let $A'(t,t,x)$ and $B'(t,t,x)$ be some bounded $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{I})$-measurable functions with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that $A'(t,t,x) + B'(t,t,x)^\top \cdot z$ is decreasing and càdlàg in $x \in \mathcal{I}$ for all t and $z \in \mathcal{Z}$.*

Let A and B be given as solutions of the Riccati equations (27)*, and let Z be a continuous $\mathcal{Z}$-valued solution of* (23)*, for some $z \in \mathcal{Z}$. Then the conclusions of Theorem 6.1 apply and*

$$P(t,T,x) = 1_{\{L_t \leq x\}} e^{-A(t,T,x) - B(t,T,x)^\top \cdot Z_t}$$

defines an arbitrage-free, doubly stochastic (T,x)-bond market.

Proof. It follows as in the proof of Theorem 7.1 that (11) is equivalent to (28), which again is implied by (27). Moreover,

$$f(t,t,x) = A'(t,t,x) + B'(t,t,x)^\top \cdot Z_t \quad (29)$$

satisfies the required properties in Theorem 6.1. Hence the conclusions of Theorem 6.1 apply. □

Remark 7.3. *Using the affine toolbox, as developed in e.g. [10, 23, 5], and the fact that $f(t,t,x)$ in* (29) *is an affine function of the affine process Z, derivative prices such as in Lemma 4.1 and* (18) *can now efficiently be computed.*

7.1 Example

As simple example, we consider: $d = 1$, $\mathcal{Z} = \mathbb{R}_+$, $\mu_0 \geq 0$, $\mu_1 \in \mathbb{R}$, $\nu_1 = \sigma^2/2$, for some $\sigma > 0$. That is, Z is a Feller square root process:

$$dZ_t = (\mu_0 + \mu_1 Z_t)dt + \sigma\sqrt{Z_t}dW_t, \quad Z_0 = z \in \mathbb{R}_+.$$

Moreover, we let $A'(t,t,x) = \alpha(t,x)$ and $B'(t,t,x) = \beta(x)$, for some $\mathbb{R}_+$-valued bounded measurable functions $\alpha(t,x)$ and $\beta(x)$ which are decreasing and càdlàg in $x \in \mathcal{I}$ with $\alpha(t,1) \equiv r \geq 0$ and $\beta(1) = 0$. That is, we have a constant risk free short rate

$$r_t \equiv r, \quad \text{and} \quad \lambda(t,x) = \alpha(t,x) - r + \beta(x) Z_t.$$

The Riccati equations (27) become

$$\begin{aligned} A(t,T,x) &= \int_t^T (\alpha(s,x) + \mu_0 B(s,T,x))\, ds \\ -\partial_t B(t,T,x) &= \beta(x) + \mu_1 B(t,T,x) - \frac{\sigma^2}{2} B(t,T,x)^2, \quad B(T,T,x) = 0. \end{aligned}$$

The equation for B has the solution

$$B(t,T,x) \equiv B(T-t,x) = \frac{2\beta(x)\left(e^{\rho(x)(T-t)} - 1\right)}{\rho(x)\left(e^{\rho(x)(T-t)} + 1\right) - \mu_1\left(e^{\rho(x)(T-t)} - 1\right)}$$

where $\rho(x) = \sqrt{\mu_1^2 + 2\sigma^2 \beta(x)}$. Note that

$$\partial_T A(t,T,x) = \alpha(T,x) + \mu_0 B(T-t,x).$$

Hence, we obtain

$$\begin{aligned} f(t,T,x) &= \alpha(T,x) + \mu_0 B(T-t,x) + \partial_T B(T-t,x) Z_t \\ f(t,T) &\equiv r. \end{aligned}$$

Since the independence assumption in the second part of Lemma 4.1 is clearly met, we conclude that $\gamma(t,y)$ in (17) can be replaced by

$$\gamma(t,y) = r \int_{T_0}^{T_n} p(t,u,y)\, du,$$

where

$$p(t,T,x) = \mathrm{e}^{-A(t,T,x) - B(T-t,x) Z_t}.$$

Hence STCDO values, and thus swap rates and swaptions, are efficiently computable via (17). We conclude with the remarkable fact that this simple model is capable of capturing any given initial forward curve $f(0,T,x)$ by an appropriate choice of the function $\alpha(T,x)$.

8 Conclusion

We have provided a universal framework for arbitrage-free CDO term structure movements, where forward rates are driven by Brownian market noise and contagion feedback from the loss process. This extends the Heath–Jarrow–Morton

[16] methodology to the defaultable case. Our first main result was a necessary and sufficient condition on the drift and default intensity for the absence of arbitrage. As second main result, we have shown that any exogenous specification of volatility and contagion parameters uniquely determines the loss process such that the no-arbitrage conditions are met. Moreover, we have provided formulas and an algorithm for CDO derivative pricing, both for the general and the doubly stochastic case, which was then further specified for affine term structures.

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