Credit Risk - A Survey

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Abstract. This paper presents a review of the developments in the area of credit risk. Starting in 1974, Merton developed a pricing method for a bond facing default risk, which was mainly settled in the framework of Black and Scholes (1973). Certain attempts have been made to relax the assumptions, giving rise to a class of models called structural models. A second class, called hazard rate models, was first addressed in Pye (1974) and more recently reached attention with the works of, e.g., Lando (1994). There are extensions in different directions, e.g., models which incorporate ratings, models for a portfolio of bonds or market models. The so called commercial models are readily implemented models which are widely accepted in practice. Finally we describe certain credit derivatives.

1. Structural Models

The first class of models tries to measure the credit risk of a corporate bond by relating the firm value of the issuing company to its liabilities. If the firm value at maturity T is below a certain level, the company is not able to pay back the full amount of money, so that a default event occurs.

1.1. Merton (1974). In his landmark paper Merton (1974) applied the framework of Black and Scholes (1973) to the pricing of a corporate bond. A corporate bond promises the repayment F at maturity T. Since the issuing company might not be able to pay the full amount of money back, the payoff is subject to default risk.

Let V_t denote the firm's value at time t. If, at time T, the firm's value V_T is below F, the company is not able to make the promised repayment so that a default event occurs. In Merton's model it is assumed that there are no bankruptcy costs and that the bond holder receives the remaining V_T, thus facing a financial loss.

If we consider the payoff of the corporate bond in this model, we see that it is equal to F in the case of no default ($V_T \geq F$) and V_T otherwise, i.e.,

$$1_{\{V_T > F\}} F + 1_{\{V_T \leq F\}} V_T = F - (F - V_T)^+.$$

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If we split the single liability into smaller bonds with face value 1, then we can replicate the payoff of this bond by a portfolio of a riskless bond $B(t,T)$ with face value 1 (long) and $1/F$ puts with strike F (short).

Consequently the price of the corporate bond at time t, which we denote by $\bar{B}(t,T)$, equals the price of the replicating portfolio:

$$\begin{aligned}
\bar{B}(t,T) &= B(t,T) - 1/F \cdot P(F, V_t, t, T, \sigma_V) \\
&= e^{-r(T-t)} - \frac{1}{F}\Big(Fe^{-r(T-t)}\Phi(-d_2) - V_t\Phi(-d_1)\Big) \\
&= e^{-r(T-t)}\Phi(d_2) + \frac{V_t}{F}\Phi(-d_1), \qquad (1.1)
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

Furthermore, $P(F, V_t, t, T, \sigma_V)$ denotes the price of a European put on the underlying V with strike F, evaluated at time t, when maturity is T and the volatility of the underlying is σ_V. This price is calculated using the Black and Scholes option pricing formula. The constants d_1 and d_2 are

$$\begin{aligned}
d_1 &= \frac{\ln \frac{V_t}{Fe^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}.
\end{aligned}$$

If the current firm value V_t is far above F the put is worth almost nothing and the price of the corporate bond equals the price of the riskless bond. If, otherwise, V_t approaches F the put becomes more valuable and the price of the corporate bond reduces significantly. This is the premium the buyer receives as a compensation for the credit risk included in the contract. Price reduction implies a higher yield for the bond. The excess yield over the risk-free rate is directly connected to the creditworthiness of the bond and is called the *credit spread*. In this model the credit spread at time t equals

$$\begin{aligned}
s(t,T) &= -\frac{1}{T-t}\ln\left[\bar{B}(t,T)e^{r(T-t)}\right] \\
&= -\frac{1}{T-t}\ln\left(\Phi(d_2) + \frac{V_t}{F\cdot e^{-r(T-t)}}\Phi(-d_1)\right),
\end{aligned}$$

see Figure 1.

The question of *hedging* the corporate bond is easily solved in this context, as hedging formulas for the put are readily available. To replicate the bond the hedger has to trade the risk-free bond and the firm's share simultaneously[1]. This reveals the fact that in Merton's model the corporate bond is a derivative on the risk-free bond and the firm's share.

[1]The hedge consists primarily of hedging $\frac{1}{F}$ put and is a straightforward consequence of the Black-Scholes Delta-Hedging.

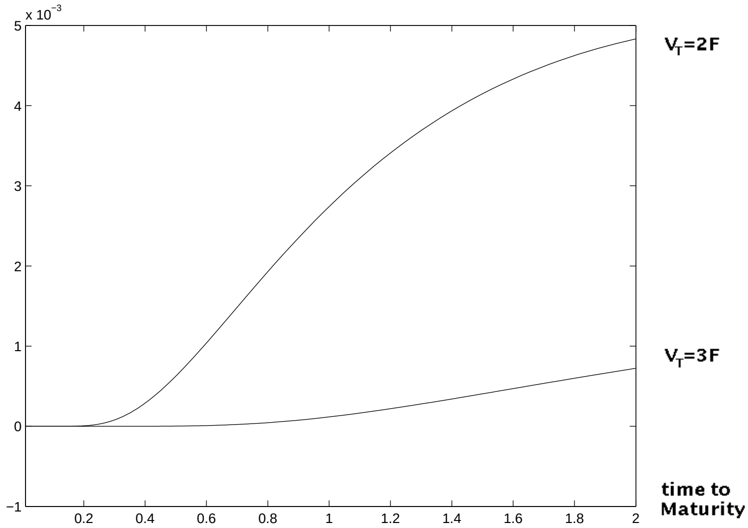

FIGURE 1. This plot shows the credit spread versus time to maturity in the range from zero to two years. The upper line is the price of a bond issued by a company whose firm value equals twice the liabilities while for the second the liabilities are three times as high. Note that if maturity is below 0.3 years the credit spreads approach zero.

We face the following problems within this model:

- The credit spreads for short maturity are close to zero if the firm value is far above F. This is in contrast to observations in the credit markets, where these *short maturity spreads* are not negligible because even close to maturity the bond holder is uncertain whether the full amount of money will be paid back or not; cf. Wei and Guo (1991) and Jones, Mason, and Rosenfeld (1984).
  The reason for this are the assumptions of the model, in particular continuity and log-normality of the firm value process. On the other hand, the intrinsic modeling of the default event may also be questionable. In reality there can be many reasons for a default which are not covered by this model.
- The model is not designed for different bonds with different maturities. Also it can happen that not all bonds default at the same time (*seniority*).
- In practice not all liabilities of a firm have to be paid back at the same time. One distinguishes between short-term and long-term liabilities. To determine the critical level where the company might default Vasiček (1984) introduced the *default point* as a mixture of the level of outstandings. This concept is discussed in Section 7.1.
- The interest rates are assumed to be constant. This assumption is relaxed, for example, by Kim, Ramaswamy, and Sundaresan (1993), as discussed in Section 1.4.
- As there are only few parameters which determine the price of the bond, this model cannot be calibrated to all traded bonds on the market, which reveals arbitrage possibilities.

Geske and Johnson (1984) extended the Merton model to coupon-bearing bonds while Shimko, Tejima, and van Deventer (1993) considered stochastic interest rates using the interest rate model proposed in Vasiček (1977). The second extension is essentially equivalent to pricing a European put option with Vasiček interest rates, where closed-form solutions are available. Of course, any other interest rate model can be used in this framework, like Cox, Ingersoll, and Ross (1985) or Heath, Jarrow, and Morton (1992).

1.2. Longstaff and Schwartz (1995). As already mentioned defaults in the Merton model are restricted to happen only at maturity, if at all. In practice defaults may happen at any time. Also, when a company offers more than one bond with different maturities or seniorities, inconsistencies in the Merton model show up which can be solved by the following approach.

Black and Cox (1976) first used *first passage time models* in the context of credit risk. This means that a default happens at the first time, when the firm value falls below a pre-specified level. They used a time dependent boundary, $F(t) = ke^{-\gamma(T-t)}$, which resulted in a random default time τ. Unfortunately, this framework proves to be unsatisfactory.

Longstaff and Schwartz (1995) extended the Merton, respectively Black and Cox, framework with respect to the following issues:

- Default may happen at the first time, denoted by τ, when the firm value V_t drops below a certain level F.
- Interest rates are stochastic and assumed to follow the Vasiček model.

As a consequence, the firm value at default equals F. In the Merton model the value of the defaulted bond was assumed to be V_T/F which equals 1 in this context. The recovery value of the bond is therefore assumed to be a pre-specified constant $(1-w)$. This is the fraction of the principal the bond holder receives at maturity. Since further defaults are excluded in this model, the bond value at default equals $\bar{B}(\tau,T) = (1-w)B(\tau,T)$, where $B(t,T)$ is the value of a risk-free bond maturing at T. This assumption is often referred to as *recovery of treasury value.*

In the following, we present the model of Longstaff and Schwartz (1995) in greater detail. The firm value is assumed to follow the stochastic differential equation

$$\frac{dV(t)}{V(t)} = \mu(t)\,dt + \sigma\,dW_V(t),$$

and the spot rate is modeled according to the model of Vasiček (1977):

$$dr(t) = \nu(\theta - r(t))\,dt + \eta\,dW_r(t). \tag{1.2}$$

Moreover,

$$\mathbb{E}(W_V(s)\,W_r(t)) = \rho \cdot (s \wedge t) \qquad \text{for all } t \text{ and } s.$$

The last equation reveals a possible correlation between the two Brownian Motions W_V and W_r.

The Vasiček model exhibits a mean-reversion behavior at level θ and easily allows for an explicit representation of r_t. It is a classical model used in interest rate theory and often taken as a starting point for more sophisticated models. A drawback of this model is the fact that it may exhibit negative interest rates with positive probability. See, for example, Brigo and Mercurio (2001) and the discussions therein.

For the price of the defaultable bond they obtain

$$\begin{aligned}\bar{B}_{\mathrm{LS}}(t,T) &= B(t,T)\cdot \mathbb{E}^{Q^T}\left[1_{\{\tau>T\}}+(1-w)1_{\{\tau\le T\}}\Big|\,\mathcal{F}_t\right]\\ &= B(t,T)\cdot\left[w\,Q^T(\tau>T|\mathcal{F}_t)+(1-w)\right].\end{aligned} \tag{1.3}$$

Note that $Q^T(\tau > T|\mathcal{F}_t)$ is the conditional probability (under the T-forward measure[2]) that the default does not happen before T.

To the best of our knowledge, a closed-form solution for this probability is not available[3]. Nevertheless there are certain quasi-explicit results provided by Longstaff and Schwartz (1995). See also Lehrbass (1997) for an implementation of the model.

In the empirical investigation of Wei and Guo (1991), the Longstaff and Schwartz model reveals a performance worse than the Merton model. According to these authors this is mainly due to the exogenous character of the recovery rate.

1.3. Jump Models - Zhou (1997). Another approach to solve the problem of *short maturity spreads* is to extend the firm value process to allow for jumps. Mason and Bhattacharya (1981) extended the Black and Cox (1976) model to a pure jump process for the firm value. The size of the jumps has a binomial distribution. In this model there is some considerable probability for the default to happen even just before maturity.

Alternatively, Zhou (1997) extended the Merton model by assuming the firm value to follow a jump-diffusion process. The immediate consequence is that defaults are not predictable. The model is formulated directly under an equivalent martingale measure Q, and the firm value is assumed to follow

$$dV_t/V_{t-} = (r_t-\lambda\nu)dt+\sigma dW_V(t)+(\Pi_t-1)dN_t. \tag{1.4}$$

N_t is a Poisson process with constant intensity λ. The jumps are $\Pi_t = U_{N_t}$, where $U_1, U_2, \dots$ are i.i.d. and assumed to be independent of N, r_t and W_V. Denote $\nu := \mathbb{E}(U_i) - 1$. Note that the integral of $(\Pi_t - 1)\, dN_t$ is shorthand for

$$Y_s := \int_0^s (\Pi_t-1)\,dN_t = \sum_{i=1}^{N_s}(U_i-1),$$

so that Y_t is a marked point process. It can be proved[4] that $Y_t - \lambda\nu t$ is a martingale so that consequently the discounted firm value is a martingale under the measure Q.

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[2]The T-forward measure is the risk neutral measure which has the risk-free bond with maturity T as numeraire. For details see Björk (1997).

[3]See discussions in Bielecki and Rutkowski (2002) and **?**).

[4]See, for example, Brémaud (1981).

The interest rate is assumed to be stochastic and follow the Vasiček model; see (1.2). The recovery rate is determined by a deterministic function w, so that the bond holder receives

$$\big(1 - w(V_\tau/F)\big)$$

at default. The function w represents the loss of the bond's value due to the reorganization of the firm. For $w = 1$ we have the zero recovery case.

Zhou considers two models. The first, more general model, assumes that default happens at the first time when the firm value falls below a certain threshold. See the previous chapter for more examples of this class of models. Since in this case no closed-form solutions are available, the author proposes an implementation via Monte-Carlo techniques.

In the second, more restrictive model, the author obtains closed form solutions. For this a constant interest rate and log-normality of the U_i's is assumed and default happens only at maturity T, when $V_T < F$. Furthermore w is assumed to be linear, i.e., $w(x) = 1 - \tilde{w}\,x$. For $\tilde{w} = 1$ we obtain the recovery structure of the Merton model.

Equation (1.4) takes the form of a Doleans-Dade exponential and can be explicitly solved under these assumptions, cf. Protter (2004, p. 77):

$$V_t = V_0 \exp\Big[\sigma_V W_V(t) + (r - \frac{1}{2}\sigma_V^2 - \lambda\nu)t\Big] \prod_{i=1}^{N_t} U_i.$$

Denote by σ_U^2 the variance of $\ln U_1$. We then have the following

PROPOSITION 1.1 (Zhou). *Denote $\tilde{\nu} := 1 + \nu$. Then the price of a defaultable bond in the above model equals*

$$\begin{aligned} &\bar{B}_{ZH}(0,T) = \\ &\quad \frac{\tilde{w}}{F} V_0 e^{-\lambda T \tilde{\nu}} \sum_{j=0}^{\infty} \frac{(\lambda\nu T)^j}{j!} \Phi\Big(\frac{\ln\frac{F}{V_0} - (r + \frac{1}{2}\sigma_V^2 - \lambda\nu)T - j(\ln\tilde{\nu} + \frac{1}{2}\sigma_U^2)}{\sqrt{\sigma_V^2 T + j\sigma_U^2}} \Big) \\ &\quad + e^{-(r+\lambda)T} \sum_{J=0}^{\infty} \frac{(\lambda T)^j}{j!} \Phi\Big(- \frac{\ln\frac{F}{V_0} - (r - \frac{1}{2}\sigma_V^2 - \lambda\nu)T - j(\ln\tilde{\nu} - \frac{1}{2}\sigma_U^2)}{\sqrt{\sigma_V^2 T + j\sigma_U^2}} \Big). \end{aligned}$$

PROOF. The payoff of the bond equals

$$\begin{aligned} \bar{B}_{ZH}(t,T) &= 1_{\{\tau > T\}} + 1_{\{\tau \le T\}}\big(1 - w(V_T/F)\big) \\ &= 1_{\{\tau > T\}} + 1_{\{\tau \le T\}} \tilde{w}\frac{V_T}{F} = 1 + 1_{\{\tau \le T\}}\big(\tilde{w}\frac{V_T}{F} - 1\big). \end{aligned}$$

To compute the present value of the bond we consider the expectation of the discounted payoff

$$\begin{aligned} \bar{B}_{ZH}(t,T) &= \mathbb{E}^Q\Big[e^{-r(T-t)} \cdot \Big(1 + 1_{\{\tau \le T\}}\big(\tilde{w}\frac{V_T}{F} - 1\big)\Big)\Big| \mathcal{F}_t\Big] \\ &= e^{-r(T-t)}\Big[1 + \mathbb{E}^Q\Big(1_{\{V_T < F\}}\big(\tilde{w}\frac{V_T}{F} - 1\big)\Big)\Big| \mathcal{F}_t\Big] \\ &= e^{-r(T-t)}\Big[1 + \frac{\tilde{w}}{F}\mathbb{E}^Q\Big(1_{\{V_T < F\}} V_T\Big| \mathcal{F}_t\Big) - \mathbb{E}^Q\Big(1_{\{V_T < F\}}\Big| \mathcal{F}_t\Big)\Big]. \end{aligned}$$

Note that conditionally on $\{N_T = j\}$ we obtain a log-normal distribution for V_T:

$$\begin{aligned}
\mathbb{P}(V_T < F|N_T = j) &= \mathbb{P}\Big(V_0 e^{(r-\frac{1}{2}\sigma_V^2-\lambda\nu)T}\exp[\sigma_V W_V(T)]\prod_{i=1}^{N_T} U_i < F\big|N_T = j\Big) \\
&= \mathbb{P}\Big(\ln V_0 + (r - \frac{1}{2}\sigma_V^2 - \lambda\nu)T + \sigma_V W_V(T) + \sum_{i=1}^{j}\ln U_i < \ln F\Big) \\
&=: \mathbb{P}(\xi_j < \ln F),
\end{aligned}$$

where $\sigma_V W(T) + \sum_{i=1}^{j}\ln U_i$ as a sum of independent normally distributed random variables is again normally distributed. Recall σ_U^2, the variance of $\ln U_1$. As $\mathbb{E}(\ln U) = \ln(1+\nu) - \frac{1}{2}\sigma_U^2$, we get

$$\begin{aligned}
\xi_j &\sim \mathcal{N}\Big(\ln V_0 + (r - \frac{1}{2}\sigma_V^2 - \lambda\nu)T + j(\ln\nu - \frac{1}{2}\sigma_U^2), \sigma_V^2 T + j\sigma_U^2\Big) \\
&=: \mathcal{N}(\tilde{\mu}(j), \tilde{\sigma}^2(j)).
\end{aligned}$$

It is an easy exercise to verify that for $\xi \sim \mathcal{N}(\mu, \sigma_V^2)$

$$\mathbb{E}\big(e^{\xi}1_{\{e^{\xi}<F\}}\big) = e^{\mu+\frac{1}{2}\sigma_V^2}\Phi\Big(\frac{\ln F - \mu}{\sigma_V} - \sigma_V\Big).$$

Conclude that

$$\begin{aligned}
\mathbb{E}^Q\Big[1_{\{V_T<F\}}V_T\Big] &= \sum_{j=0}^{\infty} Q(N_T = j)\mathbb{E}^Q(1_{\{V_T<F\}}V_T|N_T = j) \\
&= \sum_{j=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^j}{j!}\exp(\frac{1}{2}\tilde{\sigma}^2(j) + \tilde{\mu}(j))\Phi\Big(\frac{\ln F - \tilde{\mu}(j)}{\tilde{\sigma}(j)} - \tilde{\sigma}(j)\Big) \\
&= e^{-\lambda T}V_0 e^{(r-\lambda\nu)T}\sum_{j=0}^{\infty}\frac{(\lambda\nu T)^j}{j!} \\
&\quad\cdot\Phi\Big(\frac{\ln\frac{F}{V_0} - (r + \frac{1}{2}\sigma_V^2 - \lambda\nu)T - j(\ln(1+\nu) + \frac{1}{2}\sigma_U^2)}{\sqrt{\sigma_V^2 T + j\sigma_U^2}}\Big).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
&\bar{B}_{ZH}(0,T) = \\
&\quad e^{-rT} + \frac{\tilde{w}}{F}V_0 e^{-\lambda T(1+\nu)} \\
&\quad\cdot\sum_{j=0}^{\infty}\frac{(\lambda\nu T)^j}{j!}\Phi\Big(\frac{\ln\frac{F}{V_0} - (r + \frac{1}{2}\sigma_V^2 - \lambda\nu)T - j(\ln(1+\nu) + \frac{1}{2}\sigma_U^2)}{\sqrt{\sigma_V^2 T + j\sigma_U^2}}\Big) \\
&\quad -e^{-(r+\lambda)T}\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}\Phi\Big(\frac{\ln\frac{F}{V_0} - (r - \frac{1}{2}\sigma_V^2 - \lambda\nu)T - j(\ln(1+\nu) - \frac{1}{2}\sigma_U^2)}{\sqrt{\sigma_V^2 T + j\sigma_U^2}}\Big).
\end{aligned}$$

Noting that

$$e^{-rT} = e^{-(r+\lambda)T}\sum(\lambda T)^j/(j!),$$

the proof is complete.

$\square$

In the case where no jumps are present, i.e., $\lambda = 0$, the sum reduces to the summand with $j = 0$ so that the bond price formula of Merton (1.1) is obtained as a special case.

This model features some properties which are also found in empirical investigations on credit risk:

- The term structure of the credit spreads can be "upward-sloping", flat, humped or "downward-sloping".
- The "short maturity spreads" can be significantly higher than in the Merton model.
- As the firm value at default is random, especially not equal to F as in the Longstaff and Schwartz (1995) model, the recovery is more realistic.
- The recovery rate is correlated with the firm value also just before default.

1.4. Further Structural Models. Kim, Ramaswamy, and Sundaresan (1993) extended the first passage time models to also incorporate stochastic interest rates following the model of Cox, Ingersoll, and Ross (1985). In their model there is an additional possibility for a default to happen at maturity. The payoff they considered equals $\min(F, V)$. Possibly the company is not able to meet its liabilities at maturity but did not face a default up to this time.

Nielsen, Saà-Requejo, and Santa-Clara (1993) extended these models to incorporate a stochastic default boundary. For the interest rate they used the model of Hull and White (1990) but were only able to obtain explicit formulas in the special case of the Vasiček model, cf. formula (1.2).

In the work of Ammann (1999) *vulnerable claims* are considered. These are possibly stochastic payoffs which face a counterparty risk. Counterparty risk plays a role if the buyer of a claim considers the default probability of the seller as significant. He therefore will ask for a risk premium which compensates for the possible loss in case of a default. The default is assumed to happen if $V_T < F$, similar to Merton's model. In that case the buyer of the claim X receives the fraction $\frac{V_T}{F} \cdot X$. Explicit prices are derived for the Heath, Jarrow, and Morton (1992) forward rate structure and Merton-like firm dynamics.

This section on structural models heavily relies on the assumption that the firm's value is observable or even tradeable. From a practical point of view this seems not justifiable as the firm's value is not tradeable and even difficult to observe. This difficulty is discussed by Buffett (2002) and also solved in the KMV-model; see Section 7.1.

2. Hazard Rate Models

In comparison to structural models, intensity based models or hazard rate models use a totally different approach for modeling the default. In the structural approach default occurs when the firm value falls below a certain boundary. The hazard rate approach takes the default time as an exogenous random variable and tries to model or fit its probability to default. The main tool for this is a Poisson process with possibly random intensity λ_t, and jumps denoting the default events. As in the first passage time models recovery is not intrinsic to this model and is often assumed to be a somehow determined constant.

The reason for this new approach lies in the very different causes for default. Precise determination as done in structural models seems to be very difficult. Furthermore, in structural models the calibration to market prices often causes difficulties, while intensity based models allow for a better fit to available market data.

In some approaches basic ideas of these model classes are combined, for example by Madan and Unal (1998) and Ammann (1999) where the default intensity explicitly depends on the firm value. These models are called *hybrid models* and will be discussed in Section 5. As the firm value approaches a certain boundary, intensity increases sharply and default becomes very likely. So basic features of the structural models are mimicked.

A more involved hybrid model is presented by Duffie and Lando (2001) where a firm value model with incomplete accounting data is considered.

Basically we may distinguish three types of hazard rate models. In the first approach the default process is assumed to be independent of most economic factors, sometimes it is even modeled independently from the underlying.

The *rating based approach* incorporates the firm's rating as this constitutes readily available information on the company's creditworthiness. In principle one tries to model the company's way through different rating classes up to a possible fall to the lowest rating class which determines the default.

A third and very recent class is in the line of the famous market models of Jamshidian (1997) and Brace, Gatarek, and Musiela (1995), see Chapter 6.

2.1. Mathematical Preliminaries. In this section we consider the modeling of the default process in greater detail. The approach is mainly based on Lando (1994) and also discussed in many articles and books like Jeanblanc (2002) and Bielecki and Rutkowski (2002). We first present a brief introduction to *Cox processes.*

As already mentioned different stopping times denoting the default events need to be modeled. The Poisson process is taken as a starting point. Constant intensity seems too restrictive so one uses Cox processes, which can be considered as Poisson processes with random intensities[5]. A special case which suits well for our purposes is the following:

Consider a stochastic process λ_t which is adapted to some filtration $\mathcal{G}_t$. For a Poisson process N_t with intensity 1 independent of $\sigma(\lambda_s : 0 \le s \le T^*)$ set

$$\tilde{N}_t := N\Big(\int_0^t \lambda_u \, du\Big), \qquad t \le T^*.$$

[5] For a full treatment of Cox processes see Brémaud (1981) and Grandell (1997).

$\tilde{N}_t$ is a Cox process. Observe that for positive λ_t the process $\int_0^t \lambda_u \, du$ is strictly increasing and so $\tilde{N}$ can be viewed as a Poisson process under a random change of time. This reveals a very powerful concept for the problems considered in credit risk.

If just one default time τ is considered, this will be equal to the first jump τ_1 of $\tilde{N}_t$. If more default events are considered, for example, transition to other rating classes, further jumps τ_i are taken into account. The bigger λ is, the sooner the next jump may be expected to occur. We obtain, for any $t < T^*$,

$$\begin{aligned} \mathbb{P}(\tau > t) &= \mathbb{E}\big[\mathbb{P}\left(\tau > t | (\lambda_s)_{0 \le s \le t}\right)\big] \\ &= \mathbb{E}\Big[\exp\big(- \int_0^t \lambda_u \, du\big)\Big]. \end{aligned}$$

Conclude that conditionally on $\sigma(\lambda_s : 0 \le s \le T^*)$ the jumps are exponentially distributed with parameter $\int_0^t \lambda_u \, du$.

It may be recalled that a fundamental assumption to obtain this is the independence of λ and N.

2.2. Jarrow and Turnbull (1995-2000). In the work of Jarrow and Turnbull (1995) a binomial model is considered. In extension of the classical Cox, Ross, and Rubinstein (1979) approach the authors also modeled the non-default and the default state. So for every time period four possible states may be attained: {up,down} $\times$ {non-default,default}. They discovered an analogy to the foreign-exchange markets. As the intensity of the model is assumed to be constant we do not discuss it in greater detail.

In Jarrow and Turnbull (2000) a Vasiček model for the spot rate is used and the hazard rate is explicitly modeled. Correlation of the hazard rate and spot rates are allowed. Denote by Z_t and W_t Brownian motions under the risk neutral measure Q, with constant correlation ρ. Z_t can be some economic factor, like an index or the logarithm of the firm value.

Assume the following dynamics

$$\begin{aligned} dr_t &= \kappa(\theta - r_t)\, dt + \sigma dW_t, \\ \lambda_t &= a_0(t) + a_1(t) r_t + a_2(t) Z_t. \end{aligned}$$

Note that λ may take on negative values with positive probability.

Recovery must be modeled exogenously and the authors use the already mentioned *recovery of treasury value*[6]. This means if default happens prior to maturity of the bond, the bond holder receives a fraction $(1 - w)$ of the principal at maturity. For the value of the bond we calculate the expectation of the discounted payoff under the risk-neutral measure Q. For ease of notation we consider $t = 0$. By equation (1.3),

$$\bar{B}(0,T) = (1-w)B(0,T) + w\mathbb{E}^Q\Big[\exp\Big(- \int_0^T r_s \, ds\Big) 1_{\{\tau > T\}}\Big].$$

[6] See the Longstaff and Schwartz model, Section 1.2.

In the model of Jarrow and Turnbull we obtain

$$\begin{aligned}
\bar{B}(0,T) &= (1-w)B(0,T) + w\mathbb{E}^Q\Big[\exp(-\int_0^T r_u\,du)Q(\tau \le T|\lambda_s : 0 \le s \le T)\Big] \\
&= (1-w)B(0,T) + w\mathbb{E}^Q\Big[\exp[-\int_0^T (r_u+\lambda_u)\,du]\Big] \\
&= (1-w)B(0,T) + w\exp(-\mu_T + \frac{1}{2}v_T).
\end{aligned}$$

In the last equation μ_T and v_T denote expectation and variance of $\int_0^T (r_u+\lambda_u)\,du$. Under the stated assumptions this integral is normally distributed and μ and v can be easily calculated.

The flexibility of the model leads to a good fit to market data, which is not obtained by most structural models. Also the model incorporates economic factors (Z_t).

2.3. Duffie and Singleton (1999). The paper by Duffie and Singleton (1999) combines two very successful model classes in interest rate modeling to access Credit Risk: exponential affine models and the Heath, Jarrow, and Morton (1992) methodology.

For the exponential affine model the authors model a vector of hidden factors which underlie the term structure of interest rates. This vector is assumed to follow a multidimensional Cox-Ingersoll-Ross model:

$$d\mathbf{y}(t) = \mathbf{K}(\mathbf{\Theta} - \mathbf{y}(t))dt + \mathbf{\Sigma}\ \mathrm{diag}(\mathbf{y}(t))^{1/2}d\mathbf{W}(t).$$

Consequently the components of $\mathbf{y}$ are nonnegative random numbers. Spot and hazard rate are assumed to be linear in $\mathbf{y}(t)$:

$$\begin{aligned}
r(t) &= \delta_0 + \delta'\mathbf{y}(t), \\
\lambda(t)(1-\theta(t)) &= \gamma_0 + \gamma'\mathbf{y}(t).
\end{aligned}$$

A main feature of the exponential affine models is that the solution of the above SDE can be explicitly expressed in an exponential affine form. Hence we obtain deterministic functions $a(), b()$ such that

$$\mathbb{E}\left[\exp\left(i\boldsymbol{\xi}'\int_0^t \mathbf{y}(u)\,du\right)\right] = \exp[a(t,\boldsymbol{\xi}) + b(t,\boldsymbol{\xi}'\mathbf{y}(0))].$$

Thus the price of the defaultable bond can be calculated in closed form as the value of the characteristic function at a proper point.

The second approach uses the well known Heath-Jarrow-Morton model of forward rates. Denote by $\bar{f}(t,T)$ the forward rates determined by the term structure of the defaultable bond prior to default[7] and by $\mathbf{W}(t,T)$ a d-dimensional standard Brownian motion. Assume the dynamics of the forward rate to be

$$\bar{f}(t,T) = \bar{f}(0,T) + \int_0^t \mu(u,T)\,du + \int_0^t \boldsymbol{\sigma}(u,T)\,d\mathbf{W}(u).$$

[7]The forward rate is by definition $\bar{f}(t,T) = -\frac{\partial}{\partial T}\ln\bar{B}(t,T)$.

Similar to Heath, Jarrow, and Morton (1992) the authors specify the dynamics under the objective measure and consider an equivalent measure Q. For arbitrage-freeness it is sufficient - see the work of Harrison and Pliska (1981) - that all discounted price processes are martingales. Naturally this heavily relies on the recovery assumption.

Duffie and Singleton (1999) introduced the *recovery of market value* which means that immediately at default the bond loses a fraction of its value. This setup is particularly well suited for working with SDEs. The loss rate w_t is assumed to be an adapted process. Hence

$$\bar{B}(\tau, T) = (1 - w_t)\bar{B}(\tau-, T).$$

Under these assumptions the authors derived the following *drift condition* for μ and σ:

$$\mu(t, T) = \sigma(t, T)\Big(\int_t^T \sigma(u, T)\, du\Big)'.$$

On the other hand, using the above mentioned *recovery of treasury value* (cf. 1.2) and denoting the riskless forward rate by $f(t, T)$, the authors obtained

$$\mu(t, T) = \sigma(t, T)\Big(\int_t^T \sigma(u, T)\, du\Big)' + \theta(t)\lambda(t)\frac{v(t, T)}{p(t, T)}(\bar{f}(t, T) - f(t, T)).$$

3. Credit Ratings Based Methods

Simple hazard rate models are often criticized because they do not incorporate available economic fundamental information like firm value or credit ratings. This section reveals some models which incorporate these data. This is also a basic feature of commercial models; see Section 7.

Credit ratings constitute a published ranking of the creditor's ability to meet his obligations. Such ratings are provided by independent agencies, for example Standard & Poor's or Moody's and mostly financed by the gauged companies. The firms are rated even if they are not willing to pay, but for a fee they get detailed insight in the results of the examinations and might retain fundamental insights in their internal divisions to identify weaknesses.

Each rating company uses a different system of letters to classify the creditworthiness of the rated agencies. Standard & Poor's, for example, describes the highest rated debt (triple-A=AAA) with the words "Capacity to pay interest and repay principal is extremely strong". An obligation with the lowest rating, 'D', is in state of default or is not believed to make payments in time or even during a grace period. The lower the rating, the greater is the risk that interest or principal payments will not be made.

3.1. Jarrow, Lando and Turnbull (1997). The model proposed by Jarrow, Lando, and Turnbull (1997) circumvents some disadvantages of the hitherto introduced models. Especially the use of credit ratings is an attractive feature. The movements between the single rating classes is modeled by a time homogenous Markov chain, the entry into the lowest rating class yielding a default. For example, if a bond is rated AAA, it is a member of the highest rating class (= class

1). If there exist $K-1$ rating classes, denote by K the class of default. Default is assumed to be an absorbing state, restructuring after default is not considered in this model. The generator of the Markov chain is defined as

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1K} \\ \lambda_{21} & -\lambda_2 & \lambda_{23} & \cdots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ \lambda_{K-1,1} & \lambda_{K-1,2} & \cdots & -\lambda_{K-1} & \lambda_{K-1,K} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

The transition rates for the first rating class are in the first row. So $\lambda_1 = \sum_{j\neq 1} \lambda_{1j}$ is the rate for leaving this class, while λ_{12} is the rate for downgrading to class 2 and so on. The rate for a default directly from class one is λ_{1K}.

We denote

$$q_{ij}(0,t) := \mathbb{P}(\text{Rating is in class } i \text{ at } 0 \text{ and in class } j \text{ at } t),$$

and by $\mathbf{Q}(t)$ the matrix of the transition probabilities $q_{ij}(0,t)$. The transition probabilities can be computed from the intensity matrix via[8]

$$\mathbf{Q}(t) = \exp(t\mathbf{\Lambda}) := \mathrm{id}_n + t\mathbf{\Lambda} + \frac{1}{2!}(t\mathbf{\Lambda})^2 + \frac{1}{3!}(t\mathbf{\Lambda})^3 + \ldots,$$

where id_n is the $n \times n$ identity-Matrix.

Under the recovery of treasury assumption[9] we obtain for the price of a zero coupon bond under default risk

$$\begin{aligned} \bar{B}(t,T) &= 1_{\{\tau>t\}}\mathbb{E}_t\Big[e^{-\int_t^\tau r_s\,ds}\cdot \delta B(\tau,T)1_{\{\tau\leq T\}} + e^{-\int_t^T r_s\,ds}\cdot 1_{\{\tau>T\}}\Big] \\ &= 1_{\{\tau>t\}}\mathbb{E}_t\Big[\delta 1_{\{\tau\leq T\}}e^{-\int_t^T r_s\,ds} + 1_{\{\tau>T\}}e^{-\int_t^T r_s\,ds}\Big] \\ &= 1_{\{\tau>t\}}\Big[\delta B(t,T) + \mathbb{E}_t\Big((1-\delta)e^{-\int_t^T r_s\,ds}1_{\{\tau>T\}}\Big)\Big] \\ &= 1_{\{\tau>t\}}B(t,T)\Big[\delta + (1-\delta)Q_t^T(\tau>T)\Big]. \qquad (3.1) \end{aligned}$$

Q^T is the T-forward measure[10]. It is therefore crucial to have a model which determines the transition probabilities under this measure. While rating agencies estimate the transition probabilities using historical observations, i.e., under the objective measure P, Jarrow, Lando, and Turnbull (1997) propose a method which uses the defaultable bond prices and calculates transition probabilities under the the risk-neutral measure Q.

Consider the bond with rating "i" and set $Q_t^{T,i}(\tau > T)$ the probability that the bond will not default until T given it is rated "i" at t. As it makes no sense to talk about bond prices after default, we further on just consider the bond price on

[8]See, for example, Israel, Rosenthal, and Wei (2001).

[9]The bond holder receives δ equivalent and riskless bonds in case of default. See Section 1.2.

[10]The T-forward measure is the risk neutral measure which has the risk-free bond with maturity T as numeraire. For details see Björk (1997).

$\{\tau > t\}$ and get

$$\bar{B}_i(t,T) = B(t,T)\left(\delta + (1-\delta)Q_t^{T,i}(\tau > T|\tau > t)\right). \tag{3.2}$$

Jarrow, Lando, and Turnbull (1997) split the intensity matrices into an empirical part (under P) and a risk adjustment like a market price of risk: They assume that the intensities under Q^T have the form $\mathbf{U\Lambda}$ and $\mathbf{U}$ denotes a diagonal matrix where the entries are the risk adjusting factors μ_i. For the transition probabilities this yields that $q_{ij}(t,T)$ is the ij'th entry of the matrix $\exp(\mathbf{U\Lambda})$. Time homogeneity of μ would entail exact calibration being impossible.

For the discrete time approximation, $[0,T]$ is divided into steps of length 1. Starting with (3.2) one obtains

$$\begin{aligned} Q_0^{T,i}(\tau > T) &= \frac{B(0,T)(1-\delta) - \bar{B}_i(0,T) + \delta B(0,T)}{B(0,T)(1-\delta)} \\ &= \frac{B(0,T) - \bar{B}_i(0,T)}{B(0,T)(1-\delta)}. \end{aligned} \tag{3.3}$$

Denote the empirical probabilities from the rating agency by $p_{ij}(t,T)$. This leads to $Q_0^{T,i}(\tau \le 1) = \mu_i(0)p_{iK}(0,1)$, and we obtain

$$\mu_i(0) = \frac{Q_0^{T,i}(\tau > 1)}{p_{iK}(0,1)} = \frac{B(0,1) - \bar{B}_i(0,1)}{p_{iK}(0,1)\cdot B(0,1)(1-\delta)}.$$

By this one obtains $(\mu_1,\dots,\mu_{K-1})'$ and consequently $q_{ij}(0,1)$. For the step from t to $t+1$ use

$$Q_0^{T,i}(\tau \le t+1) = Q_0^{T,i}(\tau \le t+1|\tau > t)\cdot Q_0^{T,i}(\tau > t)$$

to get

$$\begin{aligned} Q_0^{T,i}(\tau \le t+1) &= \mu_i(t)P^i(\tau \le t+1|\tau > t)\cdot \sum_{j=1}^{K-1} q_{ij}(0,t) \\ &= \mu_i(t)p_{iK}(t,t+1)\cdot \sum_{j=1}^{K-1} q_{ij}(0,t). \end{aligned}$$

This leads to

$$\begin{aligned} \mu_i(t) &= \frac{Q_0^{T,i}(\tau \le t+1)}{\sum_{j=1}^{K-1} q_{ij}(0,t)\cdot p_{iK}(t,t+1)} \\ &\overset{(3.3)}{=} \frac{B(0,t+1) - \bar{B}_i(0,t+1)}{B(0,t+1)(1-\delta)\left(\sum_{j=1}^{K-1} q_{ij}(0,t)\right)p_{iK}(t,t+1)}, \end{aligned}$$

and, via $q_{ij}(0,t+1) = \mu_i(t)p_{ij}(0,t+1)$, the required probabilities are obtained.

This model extends Jarrow and Turnbull (1995) using time dependent intensities but still working with constant recovery rates. Das and Tufano (1996) propose a model which also allows for correlation between interest rates and default intensities.

It seems problematic that all bonds with the same rating automatically have the same default probability. In reality this is definitely not the case. Naturally different credit spreads occur for bonds with the same rating.

A further restrictive assumption is the time independence of the intensities. The yield of a bond in this model may only change if the rating changes. Usually the market price precedes the ratings with informations on a possible rating change which is an important insight of the KMV model; see Section 7.1.

3.2. Lando (1998). The work of Lando (1998) uses a conditional Markov chain[11] to describe the rating transitions of the bond under consideration. All available market information like interest rates, asset values or other company specific information is modeled as a stochastic process $(X_t)_{t\geq 0}$. This is analogous to the case without ratings, where Lando used $\lambda_t = \lambda(X_t)$.

Assume that a risk-neutral martingale measure Q is already chosen. Then the arbitrage-free price of a contingent claim is the conditional expectation under this measure Q. The author lays out the framework for rating transitions where all probabilities are already under the risk-neutral measure and calibrates them to available market prices. As no historical information is used the probability distribution under the objective measure is not needed. If one wants to consider risk-measures like Value-at-Risk, note that the objective measure is still required.

We denote the generator of the conditional Markov chain C_t by

$$\mathbf{\Lambda}(s) = \begin{pmatrix} -\lambda_1(s) & \lambda_{12}(s) & \lambda_{13}(s) & \cdots & \lambda_{1K}(s) \\ \lambda_{21}(s) & -\lambda_2(s) & \lambda_{23}(s) & \cdots & \lambda_{2K}(s) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \lambda_{K-1,1}(s) & \lambda_{K-1,2}(s) & \cdots & -\lambda_{K-1}(s) & \lambda_{K-1,K}(s) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

We assume $\lambda_{ij}(t)$ to be adapted processes and nonnegative for $i \neq j$. Furthermore, for all s

$$\lambda_i(s) = \sum_{j\neq i}^{K} \lambda_{ij}(s), \quad i = 1, \dots, K-1.$$

It is important for the intensities to depend on both time and interest rates. Especially for low rated companies the default rates vary considerably over time[12]. It was observed by Duffee (1999), e.g., that default rates significantly depend on the term structure of interest rates. It is certainly bad news for companies with high debt when interest rates increase whereas for other companies it might be good news.

Consider a series of independent exponential(1)-distributed random variables E_{11}, $\dots, E_{1K}, E_{21}, \dots, E_{2K}, \dots$ which are also independent of $\sigma(\mathbf{\Lambda}(s) : s \geq 0)$ and denote the rating class of the company at the beginning of the observation by η_0.

[11] See also Section 11.3 in Bielecki and Rutkowski (2002).

[12] Cf. Chapter 15 in Caouette, Altmann, and Narayanan (1998).

Define

$$\tau_{\eta_0,i} := \inf\{t : \int_0^t \lambda_{\eta_0,i}(X_s)\, ds \geq E_{1i}\}, \quad i = 1, \dots, K$$

and

$$\tau_0 := \min_{i \neq \eta_0} \tau_{\eta_0,i}, \qquad \eta_1 := \arg\min_{i \neq \eta_0} \tau_{\eta_0,i}.$$

The $\tau_{\eta_0,i}$ model the possible transitions to other rating classes starting from rating η_0. The first transition to happen determines the transition that really takes place. The reached rating class is denoted by η_1 while τ_0 denotes the time at which this occurs. Analogously, the next change in rating starting in η_1 is defined, and similarly for η_i and τ_i.

Default is assumed to be an absorbing state of the Markov chain and we denote the overall-time to default by τ. This is the first time when $\eta_i = K$.

The transition probabilities $P(s,t)$ for the time interval (s,t) satisfy Kolmogorov's backward differential equation[13]

$$\frac{\partial P_X(s,t)}{\partial s} = -\mathbf{\Lambda}(s)\, P_X(s,t).$$

Consider the price of a defaultable zero recovery bond at time t, $\bar{B}^i(t,T)$, which has maturity T and is rated in class i at time t. Then we obtain the following Theorem.

THEOREM 3.1. *Under the above assumptions the price of the defaultable bond equals*

$$\bar{B}^i(t,T) = \mathbb{E}\Big(\exp\big(-\int_t^T r_s\, ds\big)(1 - P_X(t,T)_{i,K}) \Big| \mathcal{F}_t \Big).$$

Here $P_X(t,T)_{i,K}$ is the (i,K)-th element of the matrix of transition probabilities for the time interval (t,T), $P_X(t,T)$.

PROOF. As already mentioned the Markov chain is modeled under Q so that the arbitrage-free price of the bond is the following conditional expectation:

$$\bar{B}^i(t,T) = \mathbb{E}\Big(\exp\big(-\int_t^T r_s\, ds\big) 1_{\{\tau > T\}} \Big| \mathcal{F}_t \Big).$$

Using conditional expectations and the independence of E_{1K} and $(\mathbf{\Lambda}(s))$ one concludes

$$\begin{aligned} \bar{B}^i(t,T) &= 1_{\{C_t = i\}} \mathbb{E}\Big(\exp\big(-\int_t^T r_s\, ds\big) \mathbb{P}\big(\tau > T \big| \sigma(\mathbf{\Lambda}_s : 0 \leq s \leq T) \vee \mathcal{F}_t\big) \Big| \mathcal{F}_t \Big) \\ &= \mathbb{E}\Big(\exp\big(-\int_t^T r_s\, ds\big)(1 - P_X(t,T)_{i,K}) \Big| \mathcal{F}_t \Big). \end{aligned}$$

□

For the calibration to observed credit spreads explicit formulas are needed and therefore further assumptions will be necessary. Lando chooses an Eigenvalue-representation of the generator.

[13] For non-commutative $\mathbf{\Lambda}$ the solution is in general not of the form $P_X(s,t) = \exp \int_s^t \mathbf{\Lambda}(u)\, du$. See Gill and Johannsen (1990) for solutions using product integrals.

Denote with $\mathbf{A}(s)$ the matrix with entries $\lambda_1(s), \ldots, \lambda_{K-1}(s), 0$ on the diagonal and zero otherwise. Assume that $\mathbf{\Lambda}(s)$ admits the representation

$$\mathbf{\Lambda}(s) = \mathbf{B}\,\mathbf{A}(s)\,\mathbf{B}^{-1},$$

where $\mathbf{B}$ is the $K \times K$-matrix of the Eigenvectors of $\mathbf{\Lambda}(s)$.

We conclude $P_X(s,t) = \mathbf{B}\,\mathbf{C}(s,t)\,\mathbf{B}^{-1}$ with

$$\mathbf{C}(s,t) = \begin{pmatrix} \exp\int_s^t \lambda_1(u)du & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \exp\int_s^t \lambda_{K-1}(u)du & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is easy to see that $P_X(s,t)$ satisfies the Kolmogorov-backward differential equation. For uniqueness, see Gill and Johannsen (1990).

Under these additional assumptions the price of the defaultable bond in Theorem 3.1 simplifies considerably.

Proposition 3.2. *Denoting by b_{ij} the entries of $\mathbf{B}$, the price of the defaultable bond equals*

$$\bar{B}^i(t,T) = \sum_{j=1}^{K-1} -\frac{b_{ij}}{b_{jK}} I\!E\Big[\exp\Big(\int_t^T (\lambda_j(u) - r_u)\,du\Big)\Big|\,\mathcal{F}_t\Big].$$

Proof. In this setup the conditional probability for a default when the bond is in rating class i equals

$$I\!P_X(t,T)_{i,K} \quad = \quad 1_{\{\tau>t\}} \sum_{j=1}^{K} b_{ij}\,\exp(\int_t^T \lambda_j(u)du) b_{jK}^{-1}.$$

With $b_{iK} b_{KK}^{-1} = 1$ we obtain

$$1 - I\!P_X(t,T)_{i,K} = \sum_{j=1}^{K-1} -\frac{b_{ij}}{b_{jK}}\,\exp(\int_t^T \lambda_j(u)du)$$

and the conclusion follows as in 3.1. □

Using the readily available tools for hazard rate models it is now easy to consider options which explicitly depend on the credit rating or credit derivatives with a credit trigger.

3.2.1. *Calibration.* Assuming a Vasiček model[14] for the interest rate we are in the position to use the model laid out above for calibration to observed credit spreads. There are no economic factors considered other than the interest rate and, as a consequence, λ_t must be adapted to $\mathcal{G}_t = \sigma(r_s : 0 \le s \le t)$.

[14] see equation (1.2).

Furthermore, we assume

$$\lambda_j(s) = \gamma_j + \kappa_j r_s, \quad j = 1, \dots, K-1,$$

with constants γ_j, κ_j.

The dynamics of the generator matrix is $\mathbf{\Lambda}(s) = \mathbf{B}\,\mathbf{A}(s)\,\mathbf{B}^{-1}$ and $\mathbf{B}$ has to be estimated from historical data while γ_j, κ_j are calibrated.

The credit spread is the difference of the offered yield to the spot rate. By Theorem 3.1 the bond price satisfies

$$\bar{B}^i(t,T) = -\sum_{j=1}^{K-1} -\frac{b_{ij}}{b_{jK}} \mathbb{E}\Big[\exp\Big(\int_t^T \gamma_j - (1-\kappa_j) r_u\, du\Big)\Big|\, \mathcal{F}_t\Big].$$

Therefore, we obtain for the bond's yield

$$\begin{aligned} -\frac{\partial}{\partial T}\Big|_{T=t} \log \bar{B}^i(t,T) &= -\frac{\partial}{\partial T}\Big|_{T=t} \sum_{j=1}^{K-1} \beta_{ij} \mathbb{E}\Big[\exp\Big(\int_t^T \gamma_j - (1-\kappa_j) r_u\, du\Big)\Big|\, \mathcal{F}_t\Big] \\ &= -\sum_{j=1}^{K-1} \beta_{ij} \lim_{T\to t} \mathbb{E}\Big[(\gamma_j + (\kappa_j - 1) r_T) \exp\Big(\int_t^T \gamma_j + \kappa_j r_s - r_s ds\Big)\Big|\, \mathcal{F}_t\Big] \\ &= -\sum_{j=1}^{K-1} \beta_{ij} (\gamma_j + (\kappa_j - 1) r_t). \end{aligned}$$

Hence the credit spread equals

$$s^i(t) = -\sum_{j=1}^{K-1} \beta_{ij} (\gamma_j + \kappa_j r_t).$$

For calibration a second relation is needed. Lando uses the sensitivity of the credit spreads w.r.t. the spot rate:

$$\frac{\partial}{\partial r_t} s^i(t) = -\sum_{j=1}^{K-1} \beta_{ij} \kappa_j.$$

Denote by $\hat{s}_0, d\hat{s}_0$ the observed credit spreads and their estimated sensitivities. One finally has to solve the following equation to calibrate the model:

$$\begin{aligned} -\beta(\gamma + \kappa r_0) &= \hat{s}_0 \\ -\beta\kappa &= d\hat{s}_0. \end{aligned}$$

It turns out to be problematic that observed credit spreads are not always monotone with respect to the ratings. The author argues that in practice this would occur rather seldom.

4. Basket Models

Usually there is a whole portfolio under consideration instead of just one single asset. Therefore the so far presented models were extended to models which may handle the behavior of a larger number of individual assets with default risk, a so-called *portfolio* or *basket*.

There are several approaches in the literature and they can be grouped into models which use a conditional independence concept and others which are based on copulas.

From the first class we present the methods of Kijima and Muromachi (2000), which provide a pricing formula for a credit derivative on baskets with a first- or second-to-default feature. An example is the first-to-default put, which covers the loss of the first defaulted asset in the considered portfolio, see also Section 8.6. From the second class we discuss an implementation based on the normal copula in Section 4.2.

Besides that, Jarrow and Yu (2001) model a kind of direct interaction between default intensities of different companies. In their model the default of a *primary* company has some impact on the hazard rate of a *secondary* company, whose income significantly depends on the primary company.

4.1. Kijima and Muromachi (2000). Consider a portfolio of n defaultable bonds and denote by τ_i the default time of the i-th bond. Let $(\mathcal{G}_t)_{t\geq 0}$ represent the general market information and assume that for any $t_1, \dots, t_n \leq T$

$$Q(\tau_1 > t_1, \dots, \tau_n > t_n | \mathcal{G}_T) = Q(\tau_1 > t_1 | \mathcal{G}_T) \cdot \dots \cdot Q(\tau_n > t_n | \mathcal{G}_T), \tag{4.1}$$

where Q is assumed to be the unique risk neutral measure. Using the representation via Cox processes, this yields

$$(4.1) = \exp(-\sum_{i=1}^{n} \int_0^{t_i} \lambda_i(s)\, ds).$$

In the recovery of treasury model, the loss of bond i upon default equals the pre-specified constant $w_i := (1 - \delta_i)$. So the first-to-default put is the option which pays w_i if the ith asset is the first one to default before T and zero if there is no default. Denote the event that the first defaulted bond is number i by

$$D_i := \{\tau_i \leq T, \tau_j > \tau_i, \forall j \neq i\}.$$

Then, using the risk neutral valuation principle, the price of the bond can be computed as the expectation w.r.t. the risk-neutral measure Q and equals

$$\begin{aligned} \bar{S}_F &= \mathbb{E}\Big[\exp(-\int_0^T r_u\, du) \sum_{i=1}^{n} w_i 1_{A_i}\Big] \\ &= \sum_{i=1}^{n} w_i \mathbb{E}\big[\exp(-\int_0^T r_u\, du) Q(A_i | \mathcal{G}_T)\big]. \end{aligned}$$

We obtain this probability using the factorization

$$\begin{aligned} &\mathbb{P}(\tau_i \leq T, \tau_k > \tau_i,\, \forall k \neq i | \mathcal{G}_T \vee \{\tau_i = x\}) \\ &\qquad = 1_{\{x \leq T\}} \mathbb{P}(\tau_k > x,\, \forall k \neq i | \mathcal{G}_T \vee \{\tau_i = x\}) \\ &\qquad = 1_{\{x \leq T\}} \exp(-\sum_{k \neq i} \int_0^x \lambda_k(s)\, ds). \end{aligned}$$

We therefore obtain[15]

$$
\begin{aligned}
&\mathbb{P}(\tau_i \leq T, \tau_k > \tau_i,\, \forall k \neq i | \mathcal{G}_T) \\
&\qquad = \mathbb{E}\Big[1_{\{\tau_i \leq T\}} \exp(-\sum_{k \neq i} \int_0^{\tau_i} \lambda_k(s)\, ds) | \mathcal{G}_T\Big] \\
&\qquad = \mathbb{E}\Big[\int_0^T \lambda_i(u) \exp(-\int_0^u \lambda_i(s)\, ds) \exp(-\sum_{k \neq i} \int_0^u \lambda_k(s)\, ds)\, du\Big] \\
&\qquad = \int_0^T \mathbb{E}\big[\lambda_i(u) \exp(-\int_0^u \sum_{k=1}^n \lambda_k(s)\, ds)\big]\, du.
\end{aligned}
$$

We conclude for the price of the first-to-default put:

$$
\bar{S}_F = \sum_{i=1}^n \delta_i \int_0^T \mathbb{E}\Big[\lambda_i(u) \exp(-\int_0^T r_s\, ds - \sum_{k=1}^n \int_0^u \lambda_k(s)\, ds)\Big]\, du.
$$

This formula simplifies considerably if $w_i \equiv w$, as in that case

$$
\begin{aligned}
\bar{S}_F &= w\mathbb{E}\Big[\int_0^T \sum_{i=1}^n \lambda_i(u) \exp(-\int_0^u \sum_{k=1}^n \lambda_k(s)\, ds)\, du \exp(-\int_0^T r_s\, ds)\Big] \\
&= w\mathbb{E}\Big[\big(-\exp(-\sum_{i=1}^n \int_0^T \lambda_i(u)\, du)\big)\big|_0^T \cdot \exp(-\int_0^T r_s\, ds)\Big] \\
&= (1-\delta)B(0,T)\big[1 - \mathbb{E}^T\big(\exp(-\int_0^T \sum_{i=1}^n \lambda_i(u)\, du)\big)\big].
\end{aligned}
$$

Using similar methods, we determine the swap-price, if w_i is paid immediately at default to the swap-holder. Set

$$
\bar{S}_F^* = \mathbb{E}\Big[\exp(-\int_0^\tau r_u\, du) \cdot \sum_{i=1}^n w_i 1_{A_i}\Big].
$$

Certainly, $\int_0^\tau r_u\, du$ is not $\mathcal{G}_T$-measurable, so that a slight modification of the previously used method is necessary. We obtain for the factorization

$$
\begin{aligned}
&\mathbb{E}\big[\exp(-\int_0^x r_u\, du) 1_{\{x \leq T\}} 1_{\{\tau_k > x, \forall k \neq i\}} \big| \mathcal{G}_T \vee \{\tau_i = x\}\big] \\
&\qquad = 1_{\{x \leq T\}} \exp(-\int_0^x r_u + \sum_{k \neq i} \lambda_k(u)\, du)
\end{aligned}
$$

and conclude

$$
\bar{S}_F^* = \sum_{i=1}^n w_i \int_0^T \mathbb{E}\big[\lambda_i(u) \exp(-\int_0^u r_s + \sum_{k=1}^n \lambda_k(s)\, ds)\big]\, du.
$$

[15] See Bielecki and Rutkowski (2002, Proposition 5.1.1.).

Similarly, the authors provide the following price of a (first and) second-to-default swap, which protects the holder against the first two defaults in the portfolio:

$$\begin{aligned}
\bar{S}_S &= \sum_{i=1}^{n} \delta_i \mathbb{E}\big[\exp(-\int_0^T \lambda_i(u)\,du)\big] - B(0,T)\sum_{i=1}^{n}\delta_i \\
&\quad + \sum_{i\neq j}(\delta_i+\delta_j)\int_0^T \mathbb{E}\Big[\lambda_k(u)\exp(-\int_0^T r_s\,ds - \sum_{j=1}^{n}\int_0^u \lambda_j(u)\,du)\Big] \\
&\quad -(n-2)\sum_{i=1}^{n}\delta_i\int_0^T \mathbb{E}\big[\lambda_i(u)\exp(-\int_0^T r_s\,ds - \int_0^u \sum_{j=1}^{n}\lambda_j(s)\,ds)\big]
\end{aligned}$$

4.1.1. *Extended Vasiček implementation.* Kijima and Muromachi (2000) discuss a special case of the above implementation. The main idea is to perform a calibration similar to the one of Hull and White (1990) for credit risk models. Assume for the dynamics of the hazard rates

$$d\lambda_i(t) = \big(\phi_i(t) - a_i\lambda_i(t)\big)\,dt + \sigma_i\,dw_i(t), \qquad i = 1,\dots,n, \tag{4.2}$$

where w_i are standard Brownian motions with correlation ρ_{ij}, which is sometimes stated as $dw_i dw_j = \rho_{ij}\,dt$. Furthermore, assume for the short rate r_t

$$dr_t = \big(\phi_0(t) - a_0 r_t\big)\,dt + \sigma_0\,dw_0(t).$$

Note that equations of the type (4.2) admit explicit solutions, see Schmidt (1997). From this, we get

$$\lambda_i(t) = \lambda_i(0)e^{-a_i t} + \int_0^t \phi_i(s)e^{-a_i(t-s)}\,ds + \sigma_i\int_0^t e^{-a_i(t-s)}\,dw_i(s).$$

Using the recovery of treasure assumption the bond price equals

$$\bar{B}_i(0,t) = \delta_i B(0,t) + (1-\delta_i)\mathbb{E}\big[\exp(-\int_0^t (r_u + \lambda_i(u))\,du)\big].$$

Note that $\int(r_u + \lambda_i(u))\,du$ is normally distributed and therefore the expectation equals the Laplace transform of a normal random variable with mean

$$\begin{aligned}
\mathbb{E}\big[-\int_0^t (r_u+\lambda_i(u))\,du\big] &= -\int_0^t \big(r_0e^{-a_0u} + \int_0^u \phi_0(s)e^{-a_0(u-s)}\,ds\big) \\
&\quad -\int_0^t \big(\lambda_i(0)e^{-a_iu} + \int_0^u \phi_i(s)e^{-a_i(u-s)}\,ds\big)\,du
\end{aligned}$$

and variance

$$\begin{aligned}
&\mathrm{Var}\big[\int_0^t (r_u+\lambda_i(u))\,du\big] \\
&\qquad = \mathrm{Var}\Big[\int_0^t \sigma_0\int_0^u e^{-a_0(u-s)}dz_0(s)\,du + \int_0^t \sigma_i\int_0^u e^{-a_i(u-s)}dw_i(s)\,du\Big].
\end{aligned}$$

To compute the variances it is sufficient to calculate the variances of all summands and the covariances. Setting $\rho_{ii} = 1$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t\int_0^t \sigma_i\sigma_j \int_0^{u_1}\int_0^{u_2} \exp(-a_i(u_1-s_1)-a_j(u_2-s_2))\,dw_j(s_2)\,dw_i(s_1)\,du_2\,du_1\Big]\\
&= \sigma_i\sigma_j\mathbb{E}\Big[\int_0^t\int_0^t\int_0^{s_1}\int_0^{s_2} \exp(-a_i(u_1-s_1)-a_j(u_2-s_2))\,du_2\,du_1\,dw_j(s_2)\,dw_i(s_1)\Big]\\
&= \sigma_i\sigma_j\mathbb{E}\Big[\int_0^t\int_0^t e^{a_is_1+a_js_2}\frac{1}{a_ia_j}(1-e^{-a_is_1})(1-e^{-a_js_2})\,dw_j(s_2)\,dw_i(s_1)\Big]\\
&= \sigma_i\sigma_j\rho_{ij}\int_0^t e^{a_is+a_js}\frac{1}{a_ia_j}(1-e^{-a_is})(1-e^{-a_js})\,ds\\
&= \frac{\sigma_i\sigma_j\rho_{ij}}{a_ia_j}\Big[t+\frac{1}{a_i}(e^{-a_it}-1)+\frac{1}{a_j}(e^{-a_jt}-1)+\frac{1}{a_i+a_j}(1-e^{-(a_i+a_j)t})\Big]\\
&=: c_{ij}(t)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Var}\Big[\int_0^t \sigma_i\int_0^u e^{-a_i(u-s)}dw_i(s)\,du\Big]&\\
&= \frac{\sigma_i^2}{a_i^2}\Big[t+\frac{2}{a_i}(e^{-a_it}-1)+\frac{1}{2a_i}(1-e^{-2a_it})\Big] =: v_2(t).
\end{aligned}
$$

Recall that we want to calibrate the model to the bond prices, which means calculating $\phi_i(s)$. $\phi_0(s)$ is computed as in the risk neutral case, see Hull and White (1990). Consider

$$
\frac{1}{B(0,t)}\mathbb{E}\big[\exp(-\int_0^t(r_u+\lambda_i(u))\,du)\big] = \frac{1}{1-\delta_i}\Big[\frac{B_i(0,t)}{B(0,t)}-\delta_i\Big] =: \gamma_i(t),
$$

which can be obtained from available prices, since δ_i is assumed to be known. Note that $\gamma_i(t)$ does not involve $\phi_0(s)$ as

$$
\begin{aligned}
\gamma_i(t) &= \exp\Big[-\int_0^t\big(\lambda_i(0)e^{-a_iu}+\int_0^u\phi_i(s)e^{-a_i(u-s)}\,ds\big)\,du\\
&\qquad +\frac{1}{2}\big(c_{0i}(t)+v_2(t)\big)\Big].
\end{aligned}
$$

As we want to solve this expression for ϕ_i, we consider the following derivatives:

$$
\begin{aligned}
-\frac{\partial}{\partial t}\ln\gamma_i(t) &= \lambda_i(0)e^{-a_it}+\int_0^t\phi_i(s)e^{-a_i(t-s)}\,ds-\frac{1}{2}\Big[c_{0i}(t)+v_2(t)\Big]'\\
&=: g_i(t)
\end{aligned}
$$

With

$$
\frac{\partial}{\partial t}g_i(t) = -a_i\lambda_i(0)e^{-a_it}+\phi_i(t)-a_ie^{-a_it}\int_0^t\phi_i(s)e^{a_is}\,ds-\frac{1}{2}\Big[c_{0i}(t)+v_2(t)\Big]''
$$

we conclude

$$
\phi_i(t) = \frac{\partial}{\partial t}g_i(t)+a_ig_i(t)+a_i\frac{1}{2}\Big[c_{0i}(t)+v_2(t)\Big]'+\frac{1}{2}\Big[c_{0i}(t)+v_2(t)\Big]''.
$$

Hence

$$
\begin{aligned}
a_i c_{0i}(t)' + c_{0i}(t)'' &= \sigma_0\sigma_i\rho_{0i}\Big[\frac{1}{a_0} - \frac{1}{a_0}e^{-a_0 t} - \frac{1}{a_0}e^{-a_i t} + \frac{1}{a_0}e^{-(a_0+a_i)t}\Big] \\
&\quad +\sigma_0\sigma_i\rho_{0i}\Big[\frac{1}{a_i}e^{-a_0 t} + \frac{1}{a_0}e^{-a_i t} - \frac{a_0+a_i}{a_0 a_i}e^{-(a_0+a_i)t}\Big] \\
&= \sigma_0\sigma_i\rho_{0i}\Big[\frac{1-e^{-a_0 t}}{a_0} + e^{-a_0 t}\frac{1-e^{-a_i t}}{a_i}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
a_i v_2(t)' + v_2(t)'' &= \sigma_i^2\Big[\frac{1}{a_i} - \frac{2}{a_i}e^{-a_i t} - \frac{1}{a_i}e^{-2a_i t} + \frac{2}{a_i}e^{-a_i t} + \frac{2}{a_i}e^{-2a_i t}\Big] \\
&= \frac{\sigma_i^2}{a_i}\Big[1 + e^{-2a_i t}\Big]
\end{aligned}
$$

which finally leads to

$$
\begin{aligned}
\phi_i(t) &= \frac{\partial}{\partial t}g_i(t) + a_i g_i(t) + \frac{\sigma_i^2}{2a_i}(1-e^{-2a_i t}) \\
&\quad +\frac{1}{2}\sigma_0\sigma_i\rho_{0i}\Big[\frac{1-e^{-a_0 t}}{a_0} + e^{-a_0 t}\frac{1-e^{-a_i t}}{a_i}\Big].
\end{aligned}
$$

Using similar methods Kijima and Muromachi (2000) obtain an explicit formula for the first-to-default swap. In Kijima (2000) these methods are extended to pricing a credit swap on a basket, which might incorporate a first-to-default feature.

4.2. Copula Models. The concept of copulas is well known in statistics and probability theory, and has been applied to finance quite recently. Modeling dependent defaults using copulas can be found, for example, in Li (2000) or Frey and McNeil (2001). We give an outline of Schmidt and Ward (2002), who apply a special copula, the normal copula, to the pricing of basket derivatives.

Fix $t = 0$. The goal of the model is to present a calibration method. Consider the default times $\tau_1, \dots, \tau_n$ and assume for the beginning that $t = 0$. The link between the marginals $Q_i(t) := Q(\tau_i \le t)$ and the joint distribution is the so-called copula $C(t_1, \dots, t_n)$. Assuming continuous marginals, $U_i := Q_i(\tau_i)$ is uniformly distributed. The joint distribution of the transformed random times is the copula

$$C(u_1, \dots, u_n) := Q(U_1 \le u_1, \dots, U_n \le u_n)$$

and defines the joint distribution of the τ_i's via

$$Q(\tau_1 \le t_1, \dots, \tau_n \le t_n) = C\big(Q_1(t_1), \dots, Q_n(t_n)\big).$$

For more detailed information on copulas see Nelsen (1999).

The choice of the copula certainly depends on the application. Schmidt and Ward (2002) choose the normal copula because in a Merton framework with correlated firm value processes such a dependence is obtained, and secondary the normal copula is determined by correlation coefficients which can be estimated from data.

Assume that $(Y_1, \dots, Y_n)$ follows an $n-$dimensional normal distribution with correlation matrix $\mathbf{\Sigma} = (\rho_{ij})$, where $\rho_{ii} = 1$ for all i. Denoting their joint distribution function by $\Phi_n(y_1, \dots, y_n, \mathbf{\Sigma})$ yields the *normal copula*

$$C(u_1, \dots, u_n) = \Phi_n\big(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)\big).$$

For modeling purposes it is useful to note that setting

$$\tau_i := Q_i^{-1}(\Phi(Y_i)),$$

results in $\{\tau_1, \dots, \tau_n\}$ having a normal copula with correlation matrix $\mathbf{\Sigma}$.

The above methods enable us to calculate the joint distribution of n default times, and the required correlations can be estimated using historical data. Thus, a value at risk can be determined.

For the pricing of a derivative with first-to-default feature, note that

$$Q(\tau^{1\text{st}} \le T) = 1 - Q(\tau_1 > T, \dots, \tau_n > T) \tag{4.3}$$

which can be calculated from the copula and the marginals. A more involved, but also explicit formula can be obtained for a kth-to-default option.

For example, consider a *first-to-default swap*, which is also discussed in Section 8.6. This is a derivative which offers default protection against the first defaulted asset in a specified portfolio. Under the assumption, that all credits have the same recovery rate $\delta_i \equiv \delta$, the swap pays $(1-\delta)$ at $\tau^{1\text{st}}$ if $\tau^{1\text{st}} \le T$. In exchange to this, the swap holder pays the premium S at times $T_1, \dots, T_m$, but at most until $\tau^{1\text{st}}$. As explained in Section 8.3, calculating expectations of the discounted cash flows yields the first-to-default swap premium. Thus, using Equation (8.1), we obtain

$$S_{1\text{st}} \quad = \quad \frac{(1-\delta)\mathbb{E}\big[\exp(-\int_t^{\tau^{1\text{st}}} r_u\,du) 1_{\{\tau^{1\text{st}} \le T\}}\big]}{\sum_{i=1}^m \mathbb{E}\big[\exp(-\int_0^{T_i} r_u\,du) 1_{\{\tau^{1\text{st}} > T_i\}}\big]}.$$

To calculate the expectations, the distribution of $\tau^{1\text{st}}$ under any forward measure is needed. Assuming, for simplicity, independence of the default intensity and the risk-free interest rate, one obtains

$$\mathbb{E}\big[\exp(-\int_0^{T_i} r_u\,du) 1_{\{\tau^{1\text{st}} > T_i\}}\big] = B(0, T_i) Q(\tau^{1\text{st}} > T_i).$$

The bond prices are readily available and the probability can be calculated via (4.3), once the copula is determined.

For the second expectation, use

$$\begin{aligned}
&\mathbb{E}\big[\exp(-\int_t^{\tau^{1\text{st}}} r_u\,du) 1_{\{\tau^{1\text{st}} \le T\}}\big] \\
&\qquad = \int_0^T B(0,s) \mathbb{E}\big[\exp(-\int_t^s \lambda_u^{1\text{st}}\,du) \lambda^{1\text{st}}(s)\big].
\end{aligned}$$

Note that this expectation can be obtained via

$$\begin{aligned}\frac{\partial}{\partial s} Q(\tau^{1\text{st}} > s) &= \frac{\partial}{\partial s} \mathbb{E}\big[\exp(-\int_t^s \lambda_u^{1\text{st}}\, du)\big] \\ &= \mathbb{E}\big[\exp(-\int_t^s \lambda_u^{1\text{st}}\, du)\lambda^{1\text{st}}(s)\big].\end{aligned}$$

Further on, Schmidt and Ward (2002) derive interesting results on spread widening, once a default occurred. For example, if one of two strongly related companies defaults, it might be likely that the remaining one gets into difficulties, and therefore credit spreads increase. It seems interesting that traders have a good intuition on this amount of spread widening, which also could be used as an input parameter to the model, which determines the copulas.

5. Hybrid models

Hybrid models incorporate both preceding models, for example the firm value is modeled, and a hazard rate framework is derived within this model.
The approach of Madan and Unal (1998) mimics the behavior of the Merton model in a hazard rate framework. They assume the following structure for the default intensity:

$$\lambda(t) = \frac{c}{\left(\ln \frac{V(t)}{F \cdot B(t)}\right)^2}.$$

Here $V(t)$ denotes the firm value which as in Merton's model is assumed to follow a geometric Brownian motion. $B(t)$ is the discounting factor $\exp(-\int_0^t r_u\, du)$ and F is the amount of outstanding liabilities. If the firm value approaches F the default intensity increases sharply and it is very likely that the bond defaults. As defaults can happen at any time this model is much more flexible than the Merton model. Unlike in Longstaff and Schwartz's model, the default can even happen when the firm value is far above F, though with low probability.

The authors also consider parameter estimation in their model. A closed form solution for the bond price is not available and for calculating the prices of derivatives numerical methods need to be used.

Further hybrid models of this type can be found in Ammann (1999) or Bielecki and Rutkowski (2002).

The approach of Duffie and Lando (2001) accounts for the fact that bond holders only obtain imperfect information on the firm value. Thus, starting in a structural framework, this leads to a hazard rate model.

6. Market Models with Credit Risk

Schönbucher (2000) discusses the framework for a defaultable *market model*. The difference between the market models and the continuous time models is that market models rely only on a finite number of bonds, whereas continuous time models assume a continuity of bonds traded in the market. As a matter of fact, many important variables are not available in these models as, for example, the short rate or continuously derived forward rates, which form the basis for the setting in Heath, Jarrow, and Morton (1992). Introductions to market models without default risk can be found for example in Brace, Gatarek, and Musiela (1995), Rebonato (1996) or Brigo and Mercurio (2001).

Assume we are given a collection of settlement dates $T_1 < \cdots < T_K$, the tenor structure, which denotes the maturities of all traded bonds.

Denote by $B_k(t) := B(t, T_k)$ the riskless bonds traded in the market. The discrete forward rate for the interval $[T_k, T_{k+1}]$ is defined as

$$F(t, T_k, T_{k+1}) =: F_k(t) = \frac{1}{T_{k+1} - T_k}\Big(\frac{B_k(t)}{B_{k+1}(t)} - 1\Big).$$

The defaultable zero coupon bond is denoted by $\bar{B}(t, T_k)$. As a starting point for modeling, it is assumed that this is a zero recovery bond, i.e., at default the value of the bond falls to zero. Put $\bar{B}_k(t) = \bar{B}(t, T_k) = 1_{\{\tau > t\}}\bar{B}(t, T_k)$. The *default risk factor* is denoted by

$$D_k(t) := \frac{\bar{B}_k(t)}{B_k(t)}.$$

If there exists an equivalent martingale measure Q we have

$$\begin{aligned} D_k(t) &= \frac{1}{B_k(t)} \mathbb{E}^Q\Big[\exp(-\int_t^{T_k} r_u\,du) 1_{\{\tau > T_k\}}\Big|\,\mathcal{F}_t\Big] \\ &= \frac{B_k(t)}{B_k(t)} \mathbb{E}^{T_k}\big[1_{\{\tau > T_k\}}\big|\,\mathcal{F}_t\big] \\ &= Q^{T_k}\big(\tau > T_k\big|\mathcal{F}_t\big) \end{aligned}$$

where Q^{T_k} denotes the T_k-forward measure[16] and $\mathbb{E}^{T_k}$ the expectation w.r.t. this measure. So $D_k(t)$ denotes the probability that, under the forward measure, the bond survives time T_k.

Define

$$H(t, T_k, T_{k+1}) := H_k(t) = \frac{1}{T_{k+1} - T_k}\Big(\frac{D_k(t)}{D_{k+1}(t)} - 1\Big).$$

To simplify the notation we write B_1 for $B_1(t)$ (similarly for F, D, H) and $T_{j+1} - T_j = \delta_j$.

---

[16]The T_k-forward measure is the risk neutral measure which has the risk-free bond with maturity T_k as numeraire. For details see Björk (1997).

This leads to the following decomposition

$$\begin{aligned}
\bar{B}_k &= \bar{B}_1 \prod_{j=1}^{k-1} \frac{\bar{B}_{j+1}}{\bar{B}_j} \\
&= \bar{B}_1 \prod_{j=1}^{k-1} \frac{\bar{B}_{j+1}}{B_{j+1}} \frac{B_j}{\bar{B}_j} \frac{B_{j+1}}{B_j} \\
&= D_1 \prod_{j=1}^{k-1} \frac{D_{j+1}}{D_j} \, B_1 \prod_{j=1}^{k-1} \frac{B_{j+1}}{B_j} \\
&= D_1 B_1 \prod_{j=1}^{k-1} \big(1 + \delta_j H_j\big)^{-1} \cdot \big(1 + \delta_j F_j\big)^{-1}.
\end{aligned}$$

The discrete forward rates of the defaultable bond are split into a risk-free part and a risky part which is represented by the "discrete-tenor hazard rate" H.

Defining the *credit spread*

$$S_k(t) = S(t, T_k, T_{k+1}) := \bar{F}_k(t) - F_k(t),$$

we immediately obtain

$$\begin{aligned}
S_k(t) &= \frac{1}{\delta_k}\Big(\frac{\bar{B}_k}{\bar{B}_{k+1}} - 1\Big) - \frac{1}{\delta_k}\Big(\frac{B_k}{B_{k+1} - 1}\Big) \\
&= \frac{B_k}{B_{k+1}} \frac{1}{\delta_k}\Big(\frac{\bar{B}_k \, B_{k+1}}{\bar{B}_{k+1} \, B_k} - 1\Big) \\
&= (1 + \delta_k F_k) \, H_k.
\end{aligned}$$

The main motivation for market models was to reproduce Black-like formulas for prices of caps and swaptions. This was particularly possible in the so-called LIBOR-market models. The basic assumption in these models is that the discrete forward rate has a log-normal distribution. There are also other models, see, for example, Andersen and Andreasen (2000).

Schönbucher (2000) concentrates on LIBOR-like models and assumes

$$\begin{aligned}
\frac{dF_k(t)}{F_k(t)} &= \mu_k^F(t)\, dt + \boldsymbol{\sigma}_k^F \cdot d\mathbf{W}(t) \\
\frac{dS_k(t)}{S_k(t)} &= \mu_k^S(t)\, dt + \boldsymbol{\sigma}_k^S \cdot d\mathbf{W}(t).
\end{aligned}$$

Here $\mathbf{W}$ denotes a N-dimensional standard Brownian motion, whereas $\boldsymbol{\sigma}_k$ are constant vectors and μ_k are adapted processes.

Alternatively, also the dynamics of H could be specified and the dynamics of S derived.

Since $H_k = S_k/(1+\delta_k F_k)$, we obtain

$$\begin{aligned} dH_k(t) &= \frac{1}{(1+\delta_k F_k)^2}\Big[(1+\delta_k F_k)S_k(\mu_k^S(t)\,dt + \boldsymbol{\sigma}_k^S \cdot d\mathbf{W}_t) \\ &\qquad\qquad - S_k\delta_k F_k(\mu_k^F(t)\,dt + \boldsymbol{\sigma}_k^F \cdot d\mathbf{W}_t) - S_k\delta_k F_k \boldsymbol{\sigma}_k^S \cdot \boldsymbol{\sigma}_k^F\,dt\Big] \\ &\quad -\frac{S_k}{(1+\delta_k F_k)^3}\delta_k^2 F_k^2 \sigma_k^F \cdot \sigma_k^F\,dt \\ &= \ldots\,dt + \frac{S_k}{1+\delta_k F_k}\Big[\boldsymbol{\sigma}_k^S - \frac{\delta_k F_k}{1+\delta_k F_k}\boldsymbol{\sigma}_k^F\Big]\cdot d\mathbf{W}_t \\ &=: H_k(t)\Big[\mu_k^H(t)\,dt + \boldsymbol{\sigma}_k^H(t)\cdot d\mathbf{W}_t\Big]. \end{aligned}$$

Note that $\boldsymbol{\sigma}_k^H$ is not a constant, but an adapted process with

$$\boldsymbol{\sigma}_k^H(t) = \boldsymbol{\sigma}_k^S - \frac{\delta_k F_k(t)}{1+\delta_k F_k(t)}\boldsymbol{\sigma}_k^F.$$

Using Itô's formula we obtain for the dynamics of the defaultable forward rates

$$\begin{aligned} d\bar{F}_k(t) &= dS_k(t) + dF_k(t) + d < S_k, F_k >_t \\ &= \Big[S_k\mu_k^S + F_k\mu_k^F + S_kF_k\boldsymbol{\sigma}_k^S\cdot\boldsymbol{\sigma}_k^F\Big]\,dt + \big(S_k\boldsymbol{\sigma}_k^S + F_k\boldsymbol{\sigma}_k^F\big)\cdot d\mathbf{W}_t \\ &=: \bar{F}_k(t)\Big[\mu_k^{\bar{F}}(t)\,dt + \boldsymbol{\sigma}_k^{\bar{F}}(t)\cdot d\mathbf{W}_t\Big]. \end{aligned}$$

The main reason for the popularity of the market models lies in the agreement between the model and well-established market formulas for basic derivative products. Therefore the model is usually calibrated to actual market data and afterwards used, for example, to price more complicated derivatives. For this reason the dynamics are directly modeled under the risk-neutral measure, or even more conveniently, under the T_k-forward measures. In search of something analogous for market models with credit risk, the T_k-survival measure turns up naturally. It is the measure under which the defaultable bond $\bar{B}_k(t)$ becomes a numeraire.

The T_k-*survival measure* $\bar{Q}_k$ is defined by the density

$$\bar{L}_k := \frac{\beta(t)1_{\{\tau > T_k\}}}{\bar{B}_k(0)} = \frac{d\bar{Q}_k}{dQ}.$$

Note that the density has Q-expectation 1 but becomes zero at default. In view of this, $\bar{Q}_k$ is not equivalent to Q but only absolutely continuous w.r.t. Q.

At this point different changes of measures can be obtained. Changes from the survival to the forward measure and the analogy of the spot LIBOR measure in a credit risk context are also discussed in Schönbucher (2000).

Finally, consider an $\mathcal{F}_T$-measurable claim X_T, which is paid only when $\tau > T$. Assuming zero recovery, then this claim can be valued by the following result, see Bielecki and Rutkowski (2002):

$$S_t = \bar{B}(t,T)\bar{E}_k\big(X_T\big|\mathcal{F}_t\big).$$

Here $\bar{E}_k$ denotes the expectation with respect to $\bar{Q}_k$.

7. Commercial Models

The models presented in this section, the so-called commercial models, are quite different from the models presented up to now. These models were developed by several companies and are widely accepted in practice. They all offer an implemented software, but the complete procedure of this implementation is published only for some models.

7.1. The KMV Model (1995) - CreditMonitor. The procedure of KMV is based on Merton's approach (see Section 1.1) and combines it with historical information via a statistical procedure.

KMV do not publish the exact procedure implemented in their software but the following illustrative example may be considered to be very close to their approach.

In Merton's model the firm value of the company was assumed to be observable. In reality this is unfortunately not the case. Usually shares of a company are traded but the real firm value is even difficult to estimate for internals. Using the traded shares as an estimate of the unknown firm value dates back to Modigliani and Miller, see Caouette, Altmann, and Narayanan (1998, p. 142 p.p.) for more information. The share is viewed as a call option on the firm value, where the exercise price is the level of the company's debt.

With the dynamic chosen as in Merton's model and denoting by D the debt level at time T, the value of the shares E corresponds to the Black-Scholes formula

$$E = V\,\Phi(d_1) - De^{-r(T-t)}\Phi(d_2),$$

where the constants d_1, d_2 are

$$\begin{aligned} d_1 &= \frac{\ln \frac{V}{De^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Inverting this relation results in the firm value. Also an estimate for the volatility of the share results in an estimate of the firm's value.

KMV found that in general firms do not default when their asset value reaches the book value of their total liabilities. This is due to the long-term nature of some of their liabilities which provides some breathing space. The *default point* therefore lies somewhere in between the total liabilities and the short-term (or current) liabilities. For this reason set

$$\text{default point} := \text{short-term debt} + 50\%\ \text{long-term debt}.$$

In the next step they calculate the *distance-to-default*

$$DD = \frac{\text{firm value} - \text{default point}}{\text{firm value} \times \text{vola of firm value}}.$$

Finally KMV obtains the default probability from data on historical default and bankruptcy frequencies including over 250,000 company-years of data and over 4,700 incidents of bankruptcy[17].

7.2. Moody's. Besides Merton's approach, which is often stated as contingent claims analysis (CCA), there are statistical approaches, pioneered by Altman (1968), which predict default events using market information and accounting variables via econometric methods. Moody's public firm risk model bridges between these models and is therefore named a 'hybrid' model. The procedure, as described in Sobehart and Klein (2000), uses a variant of Merton's CCA as well as rating information (if available), certain reported accounting information and some macroeconomic variables to represent the state of the economy and of specific industries through logistic regression. On this basis they provide a one-year estimated default probability (EDP).

7.3. CreditMetrics. CreditMetrics was originally developed by J.P. Morgan and belongs to RiskMetrics Group since 1998. The procedure is totally published to clarify the model and the used data are provided in the Internet.

The target of CreditMetrics is a full valuation of a whole portfolio. This includes different assets and derivatives like loans, bonds, commitments to lend, financial letters-of-credit, receivables and market driven instruments like swaps, forwards and options.

The determination of the actual price of the portfolio proceeds in three steps. First the probability of a default is determined, second the probability of changes in rating (which directly results in a different price) and third the determination of the changes in value which are evoked by either a default or a change in rating.

For the three steps certain inputs are needed. They can be obtained by historical estimation or are observable in the market[18]:

- Transition matrices - transition probabilities for changes in rating,
- Recovery rates in default - ordered by seniority, countries and sectors,
- Risk-free yield curve,
- Credit spreads - for all maturities and ratings.

The transition matrices are also provided by Moody's and Standard & Poor's and therefore have to be listed separately (Moody's rates in eight and Standard & Poor's in 18 classes). In our example we consider the Table 1.
Observe that there are some unusual figures in this table. For example, the probability that a company rated CCC is rated AAA after one year equals 0.22 %. This seems to be unusually high in comparison to the other entries. As there are few CCC ratings this seems to be a consequence of an exceptional event. Also critical is that the probability to default for a company rated AAA or AA equals zero. For sure there is a small but positive probability that such an event may happen. At this point smoothing algorithms are recommended to obtain a transition-matrix

[17] See Crosbie and Bohn (2001) for further information.
[18] See www.riskmetrics.com/products/data/datasets/creditmetrics.

Table 1

Rating (now)	Rating in 1 year - Prob. in %							
	AAA	AA	A	BBB	BB	B	CCC	D
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.7	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.3	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.2
CCC	0.22	0	0.22	1.3	2.38	11.24	64.86	19.79

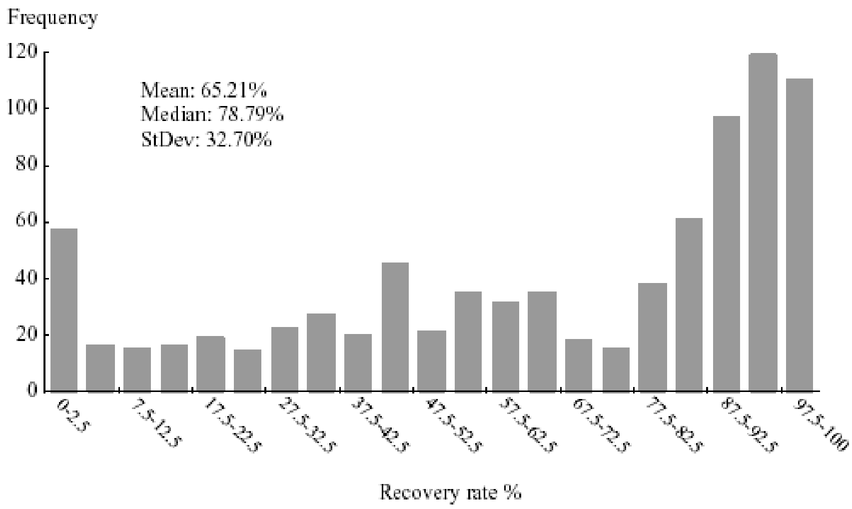

Figure 2. Recovery Rates

which is well suited for further calculations; see Gupton, Finger, and Bhatia (1997, p. 66-67).

For the second set of data, recovery rates are estimated on a historical basis. Usually this information is provided by rating agencies. There are some studies on recovery rates, and we discuss an example of Asarnow and Edwards (1995). CreditMetrics though uses just mean and standard deviation. The use of a beta distribution is discussed but not implemented.

The seniority of the bond certainly has a significant influence on the recovery rate. Table 2 illustrates this.

Table 2

Seniority	mean (%)	SD (%)
Senior Secured	53.80	26.86
Senior Unsecured	51.13	25.45
Senior Subordinated	38.52	23.81
Subordinated	32.74	20.18
Junior Subordinated	17.09	10.90

CreditMetrics also uses the actual term structure of interest rates and observable credit spreads. As the target is the valuation of bonds in a year's horizon not only default information should be used but also price changes due to rating changes. One needs to answer the question "What will be the value of a bond rated XXX in a year?". This is done by calculating stripped forward rates with respect to the rating. Stripping is the procedure to calculate zero coupon prices from a set of bonds offering coupons.

Assume for now that the current credit spreads do not change. The risk-free term structure provides forward rates and the current credit spreads are added to obtain the future (defaultable) forward-rates.

We show the full procedure in the context of an example. We face the problem to price a BBB-rated senior unsecured bond with maturity 5Y and annual coupons of 6%. Face value is 100 USD.

As described above one strips the bond prices to obtain the defaultable forward zero coupon curve. We want to explain this procedure in greater detail using the figures in Table 3.

Table 3

Category	1Y	2Y	3Y	4Y (in %)
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

Assume the bond has rating A at the end of the year. The forward value then becomes

$$FV = 6 + \frac{6}{1+3,72\%} + \frac{6}{(1+4.32\%)^2} + \frac{6}{(1+4.93\%)^3} + \frac{106}{(1+5.32\%)^4} = 108.64.$$

The other forward values are

Rating	AAA	AA	A	BBB	BB	B	CCC
Forward Value($)	109.35	109.17	108.64	107.53	102.01	98.09	83.63

The results may be found in Table 4.

Table 4

State in 1Y	Prob. (%)	Forward Value	$(FV - \bar{FV})^2$
AAA	0.02	109.35	5.21
AA	0.33	109.17	4.42
A	5.95	108.64	2.48
BBB	86.93	107.53	0.21
BB	5.3	102.01	25.63
B	1.17	98.09	80.70
CCC	0.12	83.63	549.60
Default	0.18	51.13	3129.21
	mean/ SD:	**107.07**	**8.94**

The value at default is assumed to be the mean of historical recovery values for senior unsecured debt. In the above calculation we followed the CreditMetrics Technical Document. For the standard deviation they do not include the estimated standard deviation of the recovery rates. If this is incorporated (SD for senior unsecured debt = 25.45%, see the table on the previous page) one obtains a standard deviation of 10.11 which is considerably higher.

8. Credit Derivatives

In this section we introduce several types of derivatives that relate to credit risk. Unless explicitly mentioned, we assume that the protection seller has no default risk. In reality, strong correlations between protection seller and underlying prove to be quite dangerous. The protection seller might default shortly after the underlying and the protection becomes worthless.

Additionally to the derivatives presented in this section, there exist so-called *vulnerable options*. These are derivatives whose writer may default, thus facing a counterparty risk. They are considered, for example, in Ammann (1999) or Bielecki and Rutkowski (2002). We do not consider derivatives on large baskets like *collateralized debt obligations* or others. See Blum, Overbeck, and Wagner (2003) for more information.

8.1. Credit Default Swaps and Options. A *credit default swap* or a *credit default option* is an exchange of a fee for a contingent payment if a credit default event occurs. The fee is usually called *default swap premium*. The difference between swap and option is determined by the way the fee is paid. If the fee is paid up-front, the agreement is called option, while if the fee is paid over time, it is called swap[19].

The "default event" is not a precise notion. Quite contrary, the event, which triggers the payment, is negotiable. It could be a certain level of spread widening, occurrence of publicly available information of failure to pay or an event, that the partners can agree upon. See Das (1998) for examples of credit derivatives and the underlying contracts. Not surprisingly, terms of documentation risk or legal risk arise in the context of credit risk.

If the payoff is some predetermined constant, the derivative is called *digital*, for example default digital put or default digital swap.

There are also options on a basket which have specific features. For example, a first-to-default swap is based on a basket of underlyings, where the protection seller agrees to cover the exposure of the first entity triggering a default event. The first-to-default structure is similar to a collateralized bond or loan obligation. Usually there are bonds or loans with similar credit ratings in the basket, because otherwise the weakest credit would dominate the derivative's behavior.

Like in the interest rate case, there are options with early exercise possibility, called American, credit derivatives with knock-in/out features, options directly on the credit spreads or leveraged credit default structures, see Tavakoli (1998). Also reduced loss credit default options are mentioned therein, which yields a way to reduce the cost of default protection. In this contract the protection buyer still takes a fixed percentage of the loss on a default event, while the further loss is covered by the protection seller.

8.2. Digital Options. In the case of a digital swap or option the payment, which is exchanged if the default event occurs within the lifetime of the option, is fixed. Assume, for simplicity, that the payoff equals 1. There are two possibilities for the time, when the payoff is exchanged, either at maturity T of the option or directly at default τ:

[19]See, for example, Tavakoli (1998, p.61 p.p.).

1. If the payoff takes place at maturity, the price of the option (usually called put) at time t, if there was no default before t, equals[20]

$$\begin{aligned} 1_{\{\tau>t\}} P_d(t,T) &= 1_{\{\tau>t\}} \mathbb{E}_t\Big[\exp(-\int_t^T r_u\,du) 1_{\{\tau\le T\}}\Big] \\ &= 1_{\{\tau>t\}} B(t,T)\, Q_t^T\big[\tau \le T\big]. \end{aligned}$$

   This default digital put is closely related to a zero recovery bond, as

$$P_d(t,T) + B^0(t,T) = B(t,T), \qquad \forall t.$$

2. If the payoff is done at default, we obtain[21]

$$\begin{aligned} 1_{\{\tau>t\}} P_d(t,T) &= 1_{\{\tau>t\}} \mathbb{E}_t\Big[\exp(-\int_t^\tau r_u\,du) 1_{\{\tau\le T\}}\Big] \\ &= 1_{\{\tau>t\}} \mathbb{E}_t\Big[\int_t^T \exp(-\int_t^s (r_u+\lambda_u)\,du)\lambda_s\,ds\Big] \\ &= 1_{\{\tau>t\}} \int_t^T B(t,s)\,\mathbb{E}_t^s\big[\exp(-\int_t^s \lambda_u\,du)\lambda_s\big]\,ds. \end{aligned}$$

Remark 8.1. The payoff of the digital default put is similar to the payoff of the zero recovery bond. In fact, if we denote the defaultable bond with zero recovery and maturity T by $B^0(\cdot,T)$, we obtain

$$\begin{aligned} 1_{\{\tau>t\}} P_d(t,T) &= 1_{\{\tau>t\}} \mathbb{E}_t\big[\exp(-\int_t^T r_u\,du)(1-1_{\{\tau>T\}})\big] \\ &= 1_{\{\tau>t\}} [B(t,T) - B^0(t,T)]. \end{aligned}$$

So, once the price of the zero recovery bond is known, the price of the default put can be easily calculated. Economically spoken, as a defaultable put and a zero recovery bond with same maturities guarantee the payoff 1, their price must be equal to the price of a risk-free bond, which is $B(t,T)$.

8.3. Default Option and Default Swap. To clarify the payments taking place for a *default option* or a *default swap*, consider figures 3 and 4. In the case of the default option, the protection buyer pays a fee up-front, which equals the price of the option. For the default swap the premium S is paid at time points $t_1,\dots,t_n$ until either maturity of the contract or default.

There are several structural options for the default payment[22]:
Difference to par: If a default event occurs, the protection seller has either to pay the par value (which we always assume to be 1) in exchange for the defaulted bond, or pay the par value minus the post-default price of the underlying bond. The payoff is equivalent to

$$1 - \bar{B}(\tau,T), \qquad \text{if } \tau \le T.$$

[20]For convenience we write $\mathbb{E}_t(\cdot)$ for $\mathbb{E}^Q(\cdot|\mathcal{F}_t)$ and $\mathbb{E}_t^T(\cdot)$ for $E^{Q^T}(\cdot|\mathcal{F}_t)$, when Q^T is the T-forward measure.

[21]See Bielecki and Rutkowski (2002, Proposition 5.1.1.).

[22]See, for example, Das (1998, p. 63).

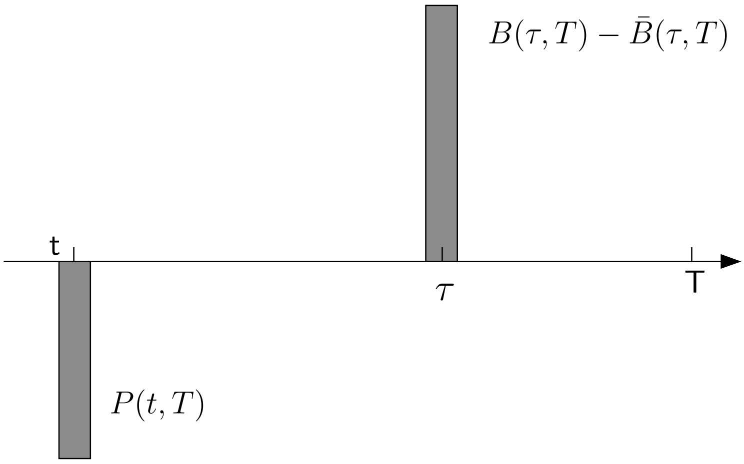

FIGURE 3. Cash flows for a default put. Default occurs at τ before the option expires. The payoff is agreed to be the "difference to an equivalent default-free bond", which is denoted by $B(\tau,T) - \bar{B}(\tau,T)$. The price of the default put is denoted by $P(t,T)$ and is paid initially at t.

Difference to an equivalent bond: The payoff in the case that a default event occurs is the value of an equivalent, default-free bond minus the market value of the defaulted bond. In this case the payoff equals

$$B(\tau,T) - \bar{B}(\tau,T), \qquad \text{if } \tau \le T.$$

In the case of a coupon bond, there is usually a protection of the principal, and possibly of the accrued interest.

The first step in pricing the defaultable swap is the pricing of the defaultable option with the same payoff. The price of the option, denoted by $P(t,T)$, yields the discounted value of the payoff at time t. The premium S is paid at times $t_1,\dots,t_n$ until a default event occurs. Denoting the price of a zero recovery bond by $B^0(t,T)$, this yields

$$P(t,T) = \sum_{i=1}^{n} S \cdot B^0(t,T_i).$$

Consequently, the swap premium can be obtained, once the price of the defaultable option and the zero recovery bond prices are known, as

$$S(t) = \frac{P(t,T)}{\sum_{i=1}^{n} B^0(t,T_i)}. \tag{8.1}$$

For example, if we assume recovery of treasury for the defaultable bond, we have

$$P(t,T) = \mathbb{E}_t\big[\exp(-\int_t^\tau r_u\,du)\,(1-\delta)1_{\{t<\tau\le T\}}\big],$$

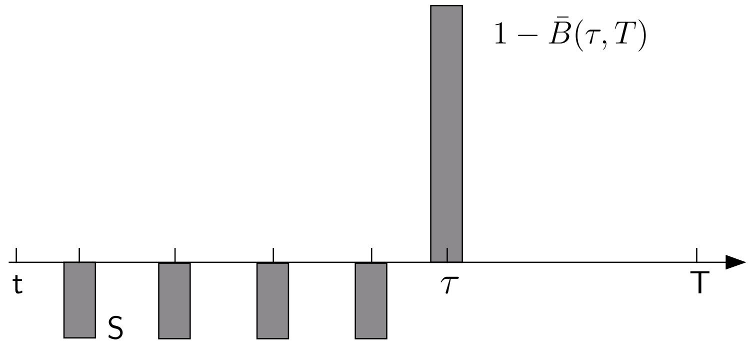

FIGURE 4. Cash flows for a credit default swap. Default occurs at τ before the option expires. The payoff is agreed to be the "difference to par", $1-\bar{B}(\tau,T)$. The default swap spread, S, is paid regularly at times $t_1,\ldots,t_4$ until default.

which can be expressed using the default digital put as

$$P(t,T)=\mathbb{E}_t(1-\delta)P_d(t,T).$$

As already mentioned, this gets slightly more difficult if the underlying is a coupon bond, see Schmid (2002) for details.

8.4. Default Swaptions. A *credit default swaption* offers the right, but not the obligation, to buy or sell a credit default swap at a future time point T for a pre-specified swap premium K. The contract is knocked out if a default of the reference entity occurs before T. We refer to a *credit default swap call* (CDS call) if the assigned right is to buy a credit default swap and otherwise to a credit default put (CDS put). Credit default swaptions are not yet standard instruments which are liquidly traded, but, for example, Hull and White (2002) report that a market for such contracts is developing.

Denoting the tenor structure of the underlying swap by $\mathcal{T}=\{T_1,\ldots,T_n\}$ and the price of the CDS call at time t by $C_S(t,T,\mathcal{T})$, we obtain for the payoff of the CDS call at maturity $T\le T_1$

$$C_S(T,T,\mathcal{T})=\left[\bar{S}(T)-K\right]^+\sum_{i=1}^n B^0(T,T_i)1_{\{\tau>T\}}.$$

$\bar{S}(T)$ is the swap rate at time T. For simplicity we set the day-count fraction to one[23].

If the swap offers the replacement of the difference to an equivalent default-free bond in the case of a default, the swap rate equals

$$\bar{S}(T)=\frac{B(T,T_n)-\bar{B}(T,T_n)}{\sum_{i=1}^n B^0(T,T_i)}.$$

[23]For a discussion on the different day-count fractions, see James and Webber (2000, p. 51 p.p.). With arbitrary day-count fraction Δ_i we would have to consider $\sum_{i=1}^n \Delta_i B^0(T,T_i)$.

We conclude for the price of the CDS call

$$C_S(0,T,\mathcal{T}) = \\ \mathbb{E}\Big[\exp(-\int_0^T r_u\,du)\Big(B(T,T_n) - \bar{B}(T,T_n) - K\sum_{i=1}^n B^0(T,T_i)\Big)^+ 1_{\{\tau>T\}}\Big].$$

Otherwise, if difference to par is considered, the swap price depends on the recovery. In a recovery of treasury model, the swap rate, as shown in the previous section, equals

$$\bar{S}(T) = \frac{(1-\delta)\,P_d(T,T_n)}{\sum_{i=1}^n B^0(T,T_i)}.$$

This yields that the price of the CDS call can be computed via

$$C_S(0,T,\mathcal{T}) \;=\; \mathbb{E}\Big[\exp(-\int_0^T r_u\,du)\Big((1-\delta)\,P_d(T,T_n) - K\sum_{i=1}^n B^0(T,T_i)\Big)^+\Big].$$

8.5. Credit Spread Options. A *credit spread option* is an option which depends on the credit spread, that is the difference between the yield of the underlying defaultable bond and the yield of a reference bond, which is usually assumed to be default-free. For example, a credit spread call with strike (yield) K at maturity T has the payoff

$$\Big(\bar{B}(T,T') - e^{-K(T'-T)}B(T,T')\Big)^+,$$

where $T' > T$ is the maturity of the underlying defaultable bond.

Thus the call is in the money if the yield of the defaultable bond is higher than the yield of the riskless bond plus the strike (yield) K. We use continuous compounding[24] of the yield rate, and note that this represents an annual yield, if the time scale is denoted in entities of 1 year.

Schmid (2002) discusses credit spread options with a *knock-out* feature. In this case a credit spread call option with maturity T on an underlying defaultable bond with maturity T' and strike K, knocked out at default, has the payoff

$$1_{\{\tau>T\}}\Big(\bar{B}(T,T') - e^{-K(T'-T)}B(T,T')\Big)^+.$$

In contrast to the option-specific payoff, a credit spread swap with strike K and maturity T has the payoff

$$\bar{B}(T,T') - e^{-K(T'-T)}B(T,T').$$

[24] The relation to the discrete time value of money concept is the following. The discounting factor for a time period of T years are

$$\frac{1}{(1+y)^{nT}} = e^{-K\cdot T},$$

if the yield y is paid n times a year. This yields the relation

$$y = (\ln K)^{\frac{1}{n}}.$$

To replicate the payoff of the credit spread swap, the seller buys a portfolio at time t, which consists of the defaultable bond with maturity T' and sells $(1+K\cdot B(t,T))$ risk free bonds with maturity T'. A replicating argument yields the value at time t of the above payoff to be $\bar{B}(t,T')-B(t,T')\exp[-K(T'-T)]$. Consequently, the credit spread swap premium, which has to be paid at times $t_1,\dots,t_n$, equals

$$S=\frac{\bar{B}(t,T')-e^{-K(T'-T)}B(t,T')}{\sum_{i=1}^{n}B(t_i,T)}.$$

If the credit spread swap is knocked out at default of the underlying, the premium relates to zero recovery bonds $B^0(\cdot,T')$, which promise the par value, 1, if the reference bond $\bar{B}(\cdot,T')$ did not default until its maturity T' and zero otherwise. Then the premium equals

$$S=\frac{\bar{B}(t,T')-e^{-K(T'-T)}B(t,T')}{\sum_{i=1}^{n}B^0(t_i,T)}.$$

8.6. kth-to-default Options. Derivatives with a *kth-to-default* feature are quite common in the market. For example, a first-to-default put covers the loss of the first defaulted asset in a considered portfolio. These types of products offer a cheaper protection against losses, if one considers more than k assets to default in a certain time interval as unlikely, and therefore offer tailor-made credit risk profiles, which may be used to redistribute credit risk or release regulatory capital.

Once a price for a kth-to-default put is obtained, the premium of a kth-to-default swap can be calculated via formula (8.1). See Section 4 for applications, where we already obtained the following formula for the premium of a first-to-default swap

$$S_{1\mathrm{st}} \quad = \quad \frac{(1-\delta)\mathbb{E}\big[\exp(-\int_t^{\tau^{1\mathrm{st}}} r_u\,du)1_{\{\tau^{1\mathrm{st}}\le T\}}\big]}{\sum_{i=1}^{m}\mathbb{E}\big[\exp(-\int_0^{T_i} r_u\,du)1_{\{\tau^{1\mathrm{st}}>T_i\}}\big]}.$$

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