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Sharpness of Fréchet-Bounds

Ludger Rüschendorf

Institut für Statistik und Wirtschaftsmathematik, Technical University Aachen,
Wüllnerstr. 3, 5100 Aachen, Federal Republic of Germany

1. Introduction

Let $(\mathscr{X}_i, \mathscr{B}_i)$, $1 \leqq i \leqq n$, be measure spaces let $P_i \in \mathscr{M}^1(\mathscr{X}_i, \mathscr{B}_i)$ – the set of probability measures on $\mathscr{B}_i$ – $1 \leqq i \leqq n$, let $(\mathscr{X}, \mathscr{B}) = \bigotimes_{i=1}^{n} (\mathscr{X}_i, \mathscr{B}_i)$ and define

$$\mathscr{M}(P_1, \ldots, P_n) := \{P \in \mathscr{M}^1(\mathscr{X}, \mathscr{B}); P^{\pi_i} = P_i, 1 \leqq i \leqq n\},$$

where π_i: $\mathscr{X} \to \mathscr{X}_i$ is the i-th projection and P^{π_i} is the image of P under π_i.

The following simple characterization of $\mathscr{M}(P_1, \ldots, P_n)$ is wellknown under the name of Fréchet-bounds:

Let $P \in \mathscr{M}^1(\mathscr{X}, \mathscr{B})$, then $P \in \mathscr{M}(P_1, \ldots, P_n)$ if and only if for all $A_i \in \mathscr{B}_i$, $1 \leqq i \leqq n$,

$$\left(\sum_{i=1}^{n} P_i(A_i) - (n-1)\right)_+ \leqq P(A_1 \times \ldots \times A_n) \leqq \min_{1 \leqq i \leqq n} P_i(A_i) \tag{1.1}$$

where for $a \in R^1$, $a_+ = \max\{a, 0\}$. Though very simple the bounds in (1.1) are useful in many applications (cf. [5, 11, 16, 14]).

In the present paper we prove that for fixed $A_1, \ldots, A_n$ the bounds in (1.1) are attained. Furthermore, we shall derive sharp upper and lower bounds for $\{\int \varphi \, dP; P \in \mathscr{M}(P_1, \ldots, P_n)\}$ for more general functions φ on $\mathscr{X}$.

In the special case that $\mathscr{X}_i = \{0, 1\}$, $1 \leqq i \leqq n$, and $p_i = P_i\{1\}$, $1 \leqq i \leqq n$, the Fréchet-bounds are identical with the Bonferoni-bounds of first order for probabilities $P\left(\bigcap_{i=1}^{n} A_i\right)$ when $p_i = P(A_i)$ are given (Note that $(1_{A_1}, \ldots, 1_{A_n})$ has under P a distribution in $\mathscr{M}(P_1, \ldots, P_n)$ where P_i are binomial $B(1, p_i)$-distributed.) So our result especially implies the sharpness of Bonferoni-bounds of first order which was proved for the first time by Fréchet [4].

In the general case there are only few indications for the solution of the problem of sharpness of Fréchet-bounds. The original problem of Fréchet [5] was to find conditions for the existence of an element $P \in \mathscr{M}(P_1, P_2)$ such that $P \leqq \mu$, where μ is a given measure on $\mathscr{B}_1 \otimes \mathscr{B}_2$. The solution of this problem

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has been given in various generality and by very interesting methods by Fréchet [5], Dall'Aglio [1], Kellerer [9], Strassen [15] and Hansel, Troallic [7]:

There exists an element $P \in \mathscr{M}(P_1, P_2)$ with $P \leqq \mu$ if and only if for all $A_i \in \mathscr{B}_i$, $i=1,2$

$$\mu(A_1 \times A_2) \geqq P_1(A_1) + P_2(A_2) - 1. \tag{1.2}$$

Though indicating in some sense the sharpness of Fréchet's lower bound, the left- and right-hand sides in (1.1) do not define probability measures. In an interesting paper of Dall'Aglio [1], Theorem 3, it was shown that even in the set of distribution functions of elements of $\mathscr{M}(P_1, \ldots, P_n)$ (in the case $(\mathscr{X}_i, \mathscr{B}_i) = (R^1, \mathscr{B}^1)$) for $n \geqq 3$ there is only in very exceptional cases a lower bound.

2. A Generalization of the Fréchet-bounds

Let $B(\mathscr{X}, \mathscr{B})$ denote the set of bounded, $\mathscr{B}$-measurable functions on $(\mathscr{X}, \mathscr{B})$ and define for $\varphi \in B(\mathscr{X}, \mathscr{B})$

$$\begin{aligned} m &= \inf \{\textstyle\int \varphi \, dP;\ P \in \mathscr{M}(P_1, \ldots, P_n)\} \\ M &= \sup \{\textstyle\int \varphi \, dP;\ P \in \mathscr{M}(P_1, \ldots, P_n)\}. \end{aligned} \tag{2.1}$$

The determination of m, M by means of duality theory was given by Gaffke, Rüschendorf [6] in the case where $\mathscr{X}_i$ are compact and φ is continuous. For the application to Fréchet-bounds a generalization of this result is needed. Let $ba(P_1, \ldots, P_n)$ be the set of finitely additive, nonnegative set functions on $\bigotimes_{i=1}^{n} \mathscr{B}_i$ with i-th marginal P_i, $1 \leqq i \leqq n$, and define

$$\begin{aligned} m_0 &= \inf \{\textstyle\int \varphi \, dP;\ P \in ba(P_1, \ldots, P_n)\} \\ M_0 &= \sup \{\textstyle\int \varphi \, dP;\ P \in ba(P_1, \ldots, P_n)\} \end{aligned} \tag{2.2}$$

Proposition 1. *If* $\varphi \in B(\mathscr{X}, \mathscr{B})$, *then*

$$m_0 = \sup \left\{ \sum_{i=1}^{n} \int f_i \, dP_i;\ f_i \in B(\mathscr{X}_i, \mathscr{B}_i),\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq \varphi \right\} \tag{2.3}$$

and there exist solutions of both sides in (2.3).

Proof. The proof of Proposition 1 is similar to that of Theorem 1, Proposition 2 and Corollary 3 of [6]. We only indicate a sketch of the proof.

Let $Z = B(\mathscr{X}, \mathscr{B})$, $X = \prod_{i=1}^{n} B(\mathscr{X}_i, \mathscr{B}_i)$, $F\colon X \to R^1$ defined by

$$F(f_1, \ldots, f_n) = \sum_{i=1}^{n} \int f_i \, dP_i, \qquad \psi\colon X \to Z$$

defined by

$$\psi(f_1, \ldots, f_n) = -\sum_{i=1}^{n} f_i \circ \pi_i, \qquad z_0 = \varphi \tag{2.4}$$

and the cone $\mathscr{E}=\{f\in B(\mathscr{X},\mathscr{B});\ f\geqq 0\}$. Choosing norm-topology on $B(\mathscr{X},\mathscr{B})$ the dual space $(B(\mathscr{X},\mathscr{B}))^*$ equals the set of bounded, additive set functions on $\mathscr{B}$. By this choice similarly to the proof of Theorem 1 in [6] the following duality theorem of Isii [8], Theorem 2, 3 can be applied.

$$\begin{aligned}&\sup\{F(x);\ x\in X,\ \psi(x)+z_0\geqq 0\}\\&\quad=\inf\{z^*(z_0);\ z^*\in Z^*,\ z^*\geqq 0,\ z^*(\psi(x))+F(x)\leqq 0,\ \forall x\in X\}.\end{aligned}\tag{2.5}$$

The left hand side of (2.5) is identical to

$$\sup\left\{\sum_{i=1}^{n}\int f_i\,dP_i;\ f_i\in B(\mathscr{X}_i,\mathscr{B}_i),\ 1\leqq i\leqq n,\ \sum_{i=1}^{n} f_i\circ\pi_i\leqq\varphi\right\}$$

while the right hand side of (2.5) is identical to

$$\inf\{\textstyle\int\varphi\,dP;\ P\in ba(P_1,\ldots,P_n)\}.$$

The proof of existence of a solution of the right hand side of (2.3) is analogously to the proof of Proposition 2 of [6], since only boundedness of φ has been used in this proof. The existence of a solution of the left hand side follows from Theorem 2.1 of [8]. ⌟

Remark. a) Proposition 2 implies that

$$M_0=\inf\left\{\sum_{i=1}^{n}\int f_i\,dP_i;\ f_i\in B(\mathscr{X}_i,\mathscr{B}_i),\ 1\leqq i\leqq n,\ \sum_{i=1}^{n} f_i\circ\pi_i\geqq\varphi\right\}\tag{2.6}$$

and also the existence of solutions.

b) $ba(P_1,\ldots,P_n)$ is by Alaoglu's Theorem (cf. [3], Theorem 2, p. 424) compact in weak*-topology. This again implies the existence of a solution of the left hand side of (2.3). ⌟

We now want to give some conditions which imply that $m_0=m$ and $M_0=M$. We need the following lemmas. Let $\mathscr{R}\left(\prod_{i=1}^{n}\mathscr{B}_i\right)$ be the algebra generated by $\prod_{i=1}^{n}\mathscr{B}_i$.

Lemma 2. *If $(\mathscr{X}_i,\mathscr{B}_i)$, $1\leqq i\leqq n$, are polish spaces (with Borel σ-Algebra $\mathscr{B}_i$) and $P\in ba(P_1,\ldots,P_n)$, then P is σ-additive on $\mathscr{R}\left(\prod_{i=1}^{n}\mathscr{B}_i\right)$.*

Proof. If $A\in\mathscr{R}\left(\prod_{i=1}^{n}\mathscr{B}_i\right)$ then there exist $A_i^j\in\mathscr{B}_i$, $1\leqq j\leqq m$, $1\leqq i\leqq n$, such that $A=\sum_{i=1}^{m} A_1^j\times\ldots\times A_n^j$. For $A_k\in\mathscr{B}_k$, $k\neq i$,

$$\tilde{P}_i=P(A_1\times\ldots\times A_{i-1}\times\cdot\times A_{i+1}\times\ldots\times A_n)$$

considered as map on $\mathscr{B}_i$ is dominated by P_i, $\tilde{P}_i(A')\leqq P_i(A')$, $\forall A'\in\mathscr{B}_i$, and, therefore, is σ-additive on $\mathscr{B}_i$, $1\leqq i\leqq n$. Since $(\mathscr{X}_i,\mathscr{B}_i)$, $1\leqq i\leqq n$, are polish $\tilde{P}_i$ are tight measures on $\mathscr{B}_i$, $1 \leqq i \leqq n$. Therefore, there exist compact subsets $T_i^j \subset A_i^j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$, such that for $\varepsilon > 0$

$$\begin{aligned} P(A_1^j \times \ldots \times A_n^j) &\leqq P(A_1^j \times \ldots \times A_{n-1}^j \times T_n^j) + \frac{\varepsilon}{mn} \\ &\leqq P(A_1^j \times \ldots \times T_{n-1}^j \times T_n^j) + \frac{2\varepsilon}{mn} \leqq \ldots \\ &\leqq P(T_1^j \times \ldots \times T_n^j) + \frac{\varepsilon}{m}. \end{aligned}$$

This implies $P(A) \leqq \sum_{j=1}^{m} P(T_1^j \times \ldots \times T_n^j) + \varepsilon$ and, therefore, by Proposition 1.6.2 of Neveu [12] P is σ-additive on $\mathscr{R}\left(\prod_{i=1}^{n} \mathscr{B}_i\right)$. $\lrcorner$

Let $(Y, \mathscr{C})$ be a topological space, let $\mathscr{R}$ be an algebra on Y. A non-negative content μ on $\mathscr{A}(\mathscr{R})$ is called outer (inner) regular if

$$\mu(U) = \inf\{\mu(0);\ 0 \in \mathscr{C} \cap \mathscr{A}(\mathscr{R}),\ U \subset 0\},$$
$$(\mu(U) = \sup\{\mu(F);\ F \text{ closed},\ F \in \mathscr{A}(\mathscr{R}),\ F \subset U\})$$

for all $U \in \mathscr{A}(\mathscr{R})$. (This is a specialization of Definition 11, p. 137 of [3].) If μ is bounded, then outer regularity of μ is equivalent to inner regularity of μ and μ is called regular in this case.

Lemma 3. *If $\mathscr{R}$ contains a countable base of the topology of Y and if P is a bounded, nonnegative, regular content on $\mathscr{A}(\mathscr{R})$ which is σ-additive on $\mathscr{R}$, then P is σ-additive on $\mathscr{A}(\mathscr{R})$.*

Proof. Let $\tilde{P}$ be the unique extension of $P/\mathscr{R}$ as measure on $\mathscr{A}(\mathscr{R})$ and let $O \in \mathscr{C}$. Then there exist $O_i \in \mathscr{R} \cap \mathscr{C}$ with $O = \bigcup_{i=1}^{\infty} O_i$. This implies $P(O) \geqq P\left(\bigcup_{i=1}^{n} O_i\right) = \tilde{P}\left(\bigcup_{i=1}^{n} O_i\right)$, $\forall n \in \mathbb{N}$ and, therefore,

$$P(0) \geqq \tilde{P}(0), \qquad \forall\, 0 \in \mathscr{C}. \tag{2.7}$$

By outer regularity of P (2.7) implies for all closed sets $F \in \mathscr{A}(\mathscr{R})$

$$\begin{aligned} P(F) &= \inf\{P(0);\ 0 \in \mathscr{C},\ 0 \supset F\} \\ &\geqq \inf\{\tilde{P}(0);\ 0 \in \mathscr{C},\ 0 \supset F\} \geqq \tilde{P}(F) \geqq P(F). \end{aligned}$$

This implies $P(0) = \tilde{P}(0)$ for all $0 \in \mathscr{C}$ and, therefore, for $A \in \mathscr{A}(\mathscr{R})$

$$\begin{aligned} P(A) &= \inf\{P(0);\ 0 \in \mathscr{C},\ 0 \supset A\} \\ &= \inf\{\tilde{P}(0);\ 0 \in \mathscr{C},\ 0 \supset A\} \geqq \tilde{P}(A) \end{aligned}$$

and, similarly, $P(A^c) \geqq \tilde{P}(A^c)$ which implies $P = \tilde{P}$. $\lrcorner$

Let $rba(\mathscr{X}, \mathscr{B})$ denote the set of regular, bounded, nonnegative contents on $(\mathscr{X}, \mathscr{B})$.

Corollary 4. *If* $(\mathscr{X}_i, \mathscr{B}_i)$, $1 \leqq i \leqq n$, *are polish spaces, then*

$$rba(\mathscr{X}, \mathscr{B}) \cap ba(P_1, \ldots, P_n) = \mathscr{M}(P_1, \ldots, P_n). \quad \lrcorner$$

For $P \in ba(P_1, \ldots, P_n)$ let $L^1\left(\mathscr{X}, \mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right), P\right)$ denote the set of P-integrable functions where P is considered as content on $\mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right)$ (cf. Dunford, Schwartz [3], Def. 17, p. 112) and define

$$L^1(ba(P_1, \ldots, P_n)) := \bigcap_{P \in ba(P_1, \ldots, P_n)} L^1\left(\mathscr{X}, \mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right), P\right).$$

$L^1(ba(P_1, \ldots, P_n))$ contains $B\left(\mathscr{X}, \mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right)\right)$ – the closure of all finite linear combinations of characteristic functions of sets in $\mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right)$ w.r.t. uniform metric. Let $C_b(\mathscr{X})$ denote the set of bounded, continuous functions on $\mathscr{X}$. The following theorem is the main result of this section.

Theorem 5. *Let* $(\mathscr{X}_i, \mathscr{B}_i)$, $1 \leqq i \leqq n$, *be polish spaces.*

a) *If* $\varphi \in L^1(ba(P_1, \ldots, P_n)) \cup C_b(\mathscr{X})$, *then for each* $P \in ba(P_1, \ldots, P_n)$ *there exists a* $\tilde{P} \in \mathscr{M}(P_1, \ldots, P_n)$ *with* $\int \varphi \, dP = \int \varphi \, d\tilde{P}$. *Especially,* $m_0 = m$ *and* $M_0 = M$.

b) *If* $P^* \in \mathscr{M}(P_1, \ldots, P_n)$ *and* $\varphi \in L^1(ba(P_1, \ldots, P_n)) \cup C_b(\mathscr{X})$, *then* $\int \varphi \, dP^* = m$ *if and only if there exist* $f_i^* \in B(\mathscr{X}_i, \mathscr{B}_i)$, $1 \leqq i \leqq n$, *with* $\sum_{i=1}^n f_i^* \circ \pi_i \leqq \varphi$ *and* $P^*\left\{\sum_{i=1}^n f_i^* \circ \pi_i = \varphi\right\} = 1$.

Proof. a) Each $P \in ba(P_1, \ldots, P_n)$ considered as content on $\mathscr{R}\left(\prod_{i=1}^n \mathscr{B}_i\right)$ has by Lemma 2 a unique extension to an element $\tilde{P}$ of $\mathscr{M}(P_1, \ldots, P_n)$. Therefore, by Lemma 1, p. 165 of [3] it holds for $\varphi \in L^1(ba(P_1, \ldots, P_n))$ that $\int \varphi \, dP = \int \varphi \, d\tilde{P}$.

If $\varphi \in C_b(\mathscr{X})$ we can replace in the proof of Proposition 1 $B(\mathscr{X}_i, \mathscr{B}_i)$ by $C_b(\mathscr{X}_i)$ and $B(\mathscr{X}, \mathscr{B})$ by $C_b(\mathscr{X})$ and obtain from (2.5) using that $rba(\mathscr{X}, \mathscr{R}(\mathscr{C})) = \{z \in (C_b(\mathscr{X}))^*;\ z \geqq 0\}$

$$\begin{aligned} &\inf\{\textstyle\int \varphi \, dP;\ P \in rba(\mathscr{X}, \mathscr{R}(\mathscr{C})),\ \int f_i \circ \pi_i \, dP = \int f_i \, dP_i,\ \forall f_i \in C_b(\mathscr{X}_i),\ 1 \leqq i \leqq n\} \\ &\quad = \sup\left\{\sum_{i=1}^n \int f_i \, dP_i;\ f_i \in C_b(\mathscr{X}_i),\ 1 \leqq i \leqq n,\ \sum_{i=1}^n f_i \circ \pi_i \leqq \varphi\right\} \end{aligned} \tag{2.8}$$

where $\mathscr{C}$ is the system of open sets in $\mathscr{B}$. Using Lemmas 2, 3 and Corollary 4 each $P \in rba(\mathscr{X}, \mathscr{R}(\mathscr{C}))$ with marginals $P_1, \ldots, P_n$ has a unique σ-additive extension to an element of $rba(\mathscr{X}, \mathscr{B}) \cap ba(P_1, \ldots, P_n) = \mathscr{M}(P_1, \ldots, P_n)$ such that integrals w.r.t. elements of $C_b(\mathscr{X})$ are identical. Therefore, the left hand side of (2.8) equals m. The right hand side of (2.8) is easily shown to be identical to

$$\sup\left\{\sum_{i=1}^n \int f_i \, dP_i;\ f_i \in B(\mathscr{X}_i, \mathscr{B}_i),\ 1 \leqq i \leqq n,\ \sum_{i=1}^n f_i \circ \pi_i \leqq \varphi\right\}.$$

b) is immediate from a) and Proposition 1. $\lrcorner$

Remark. a) A very interesting result of Douglas [2], Theorem 1 implies that an element P of $\mathcal{M}(P_1, \ldots, P_n)$ is an extreme point of $\mathcal{M}(P_1, \ldots, P_n)$ if and only if

$$F = \left\{ \sum_{i=1}^{n} f_i \circ \pi_i;\ f_i \in B(\mathcal{X}_i, \mathcal{B}_i),\ 1 \leqq i \leqq n \right\}$$

is dense in $L^1(P)$. An extension of this result to $ba(P_1, \ldots, P_n)$ is possible by techniques which are used in the proof of Theorem 1 of Plachky [13]. Clearly the inf and sup of (2.1) are attained in extreme points. Theorem 5 and Proposition 1 show that in this case one even can approximate φ by elements of F which are less than or equal to φ (resp. larger than or equal).

b) A somewhat shorter proof of Corollary 4 (without reference to Lemma 3 could have been given by refering to Alexandroff's Theorem (cf. [3], Th. 13, p. 138).

3. Sharpness of Fréchet-Bounds

The aim of this section is to prove that the bounds given in (1.1) are sharp.

Theorem 6. *Let $(\mathcal{X}_i, \mathcal{B}_i)$, $1 \leqq i \leqq n$, be polish spaces, then for all $A_i \in \mathcal{B}_i$, $1 \leqq i \leqq n$,*

a)
$$\begin{aligned} &\max\{P(A_1 \times \ldots \times A_n);\ P \in \mathcal{M}(P_1, \ldots, P_n)\} \\ &\quad = \min\{P_i(A_i);\ 1 \leqq i \leqq n\}, \end{aligned} \tag{3.1}$$

b)
$$\begin{aligned} &\min\{P(A_1 \times \ldots \times A_n);\ P \in \mathcal{M}(P_1, \ldots, P_n)\} \\ &\quad = \left(\sum_{i=1}^{n} P_i(A_i) - (n-1) \right)_+ . \end{aligned} \tag{3.2}$$

Proof. a) Let

$$A = \inf\left\{ \sum_{i=1}^{n} \int f_i \, dP_i;\ f_i \in B(\mathcal{X}_i, \mathcal{B}_i),\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \geqq 1_{A_1 \times \ldots \times A_n} \right\} \tag{3.3}$$

Let (f_i) be admissible for (3.3) and define

$$a_i = \inf\{f_i(x);\ x \in \mathcal{X}_i\}, \qquad 1 \leqq i \leqq n,$$

then $\sum_{i=1}^{n} a_i \geqq 0$. Define

$$J_0 = \{i \leqq n;\ a_i < 0\}, \quad \tilde{f}_i = f_i - a_i, \qquad i \in J_0$$

and

$$\tilde{f}_i = f_i - a_i + \frac{1}{|J - J_0|} \sum_{j=1}^{n} a_j, \qquad i \in J - J_0,\ J = \{1, \ldots, n\}.$$

Then $\tilde{f}_i \geqq 0$, $1 \leqq i \leqq n$ and $\sum_{i=1}^{n} \tilde{f}_i \circ \pi_i = \sum_{i=1}^{n} f_i \circ \pi_i$ such that $(\tilde{f}_i)$ are admissible for (3.3) and $\sum_{i=1}^{n} \int f_i \, dP_i = \sum_{i=1}^{n} \int \tilde{f}_i \, dP_i$. Therefore, w.l.g. we can assume, that $a_i \geqq 0$, $1 \leqq i \leqq n$.

Define $b_i = \inf\{f_i(x);\ x \in A_i\}$, $1 \leqq i \leqq n$, then $b_i \geqq 0$, $1 \leqq i \leqq n$, and $\sum_{i=1}^{n} b_i \geqq 1$. This implies that (f_i^*) are admissible, where

$$f_i^* = b_i 1_{A_i},\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \geqq \sum_{i=1}^{n} (f_i 1_{A_i}) \circ \pi_i$$
$$\geqq \sum_{i=1}^{n} b_i 1_{A_i} \circ \pi_i = \sum_{i=1}^{n} f_i^* \circ \pi_i \quad \text{and} \quad \sum_{i=1}^{n} \int f_i \, dP_i \geqq \sum_{i=1}^{n} b_i P_i(A_i).$$

Therefore,

$$A = \inf\left\{\sum_{i=1}^{n} b_i P_i(A_i);\ b_i \geqq 0,\ 1 \leqq i \leqq n,\ \sum_{i=1}^{n} b_i = 1\right\}$$
$$= \min\{P_i(A_i);\ 1 \leqq i \leqq n\}.$$

Now Theorem 5 and Proposition 1 imply (3.1).

b) Let

$$B = \sup\left\{\sum_{i=1}^{n} \int f_i \, dP_i;\ f_i \in B(\mathscr{X}_i, \mathscr{B}_i),\right. \quad\quad (3.4)$$
$$\left. 1 \leqq i \leqq n,\ \sum_{i=1}^{n} f_i \circ \pi_i \leqq 1_{A_1 \times \ldots \times A_n}\right\}.$$

Let (f_i) be admissible for (3.4) and let $b_i = \inf\{f_i(x);\ x \in A_i^c\}$, $a_i = \inf\{f_i(x);\ x \in A_i\} - b_i$, $1 \leqq i \leqq n$. Then $(\tilde{f}_i)$ is admissible for (3.4), where

$$\tilde{f}_i = a_i 1_{A_i} + b_i \quad \text{and} \quad \tilde{f}_i \leqq f_i, \quad 1 \leqq i \leqq n.$$

Therefore, w.l.g. we may assume that $f_i = a_i 1_{A_i} + b_i$, $1 \leqq i \leqq n$. With $b = \sum_{i=1}^{n} b_i$ admissibility of (f_i) is equivalent to

$$\sum_{i=1}^{n} a_i + b \leqq 1 \quad \text{and} \quad \sum_{j \in J} a_j + b \leqq 0, \quad \forall J \subsetneq \{1, \ldots, n\} \quad\quad (3.5)$$

($\subsetneq$ means strict inclusion) and

$$B = \max\left\{\sum_{i=1}^{n} a_i P_i(A_i) + b;\ (a_i), b \text{ satisfy } (3.5)\right\}.$$

If the max is attained then equality holds in at least one restriction of (3.5).

Case 1. $\sum_{i=1}^{n} a_i + b = 1$, then $a_i \geqq a_i + \sum_{j \neq i} a_j + b = 1$, $1 \leqq i \leqq n$, and

$$\sum_{i=1}^{n} a_i P_i(A_i) + b = \sum_{i=1}^{n} a_i (P_i(A_i) - 1) + 1$$
$$\leqq \sum_{i=1}^{n} P_i(A_i) - (n-1) \quad\quad (3.6)$$

and the right hand side of (3.6) is attained for $a_i=1$, $1\leqq i\leqq n$, $b=-(n-1)$ (which are admissible).

Case 2. There exists $J_0 \subset \{1,\dots,n\}$ with $\sum_{j\in J_0} a_j+b=0$. If $a_i<0$, define

$$\tilde{a}_j=\begin{cases} a_j, & j\neq i \\ 0, & j=i. \end{cases}$$

Then

$$\sum_{j\in J}\tilde{a}_j+b=\sum_{j\in J\setminus\{i\}} a_j+b\leqq 0, \qquad \forall\, J\subset\{1,\dots,n\}.$$

So $(\tilde{a}_j)$, b are admissible and

$$\sum_{i=1}^{n}\tilde{a}_i P_i(A_i)+b\geqq\sum_{i=1}^{n} a_i P_i(A_i)+b.$$

Therefore, w.l.g. we can assume that $a_i\geqq 0$, $1\leqq i\leqq n$.

Case 2. a) Let $|J_0|<n-1$, then $a_i\geqq 0$, $1\leqq i\leqq n$, and $\sum_{j\in J_0} a_j+b=0$ imply that $a_j=0$, $\forall\, j\in J_0^c$ and, therefore,

$$\sum_{i=1}^{n} a_i P_i(A_i)+b=\sum_{i\in J_0} a_i(P_i(A_i)-1)\leqq 0. \tag{3.7}$$

So the max is obtained in this case for $a_i=0$, $1\leqq i\leqq n$.

Case 2. b) Let $|J_0|=n-1$ and $i\notin J_0$ such that $\sum_{j\neq i} a_j+b=0$. This implies that for all $j_0\in\{1,\dots,n\}$, $j_0\neq i$

$$\sum_{j\neq j_0} a_j+b=a_i+\sum_{j\neq j_0,i} a_j-\sum_{j\neq i} a_j=a_i-a_{j_0}\leqq 0,$$

and, therefore, $a_i\leqq\min\{a_j;\, j\neq i\}$. So in case 2b)

$$\begin{aligned}
&\max\left\{\sum_{j=1}^{n} a_j P_j(A_j);\ (a_j), b \text{ admissible}, \sum_{j\neq i} a_j+b=0\right\}\\
&=\max\left\{\sum_{j=1}^{n} a_j P_j(A_j);\ 0\leqq a_i\leqq a_j,\ \forall j\neq 1,\ a_i\leqq 1\right\}\\
&=\max\left\{a_i P_i(A_i)+\sum_{j\neq i} a_j(P_j(A_j)-1);\ 0\leqq a_i\leqq a_j,\ j\neq i,\ a_i\leqq 1\right\}\\
&=\max\left\{a_i P_i(A_i)+a_i\sum_{j\neq i}(P_j(A_j)-1);\ 0\leqq a_i\leqq 1\right\}\\
&=\left(\sum_{j=1}^{n} P_j(A_j)-(n-1)\right)_+.
\end{aligned} \tag{3.8}$$

(3.6), (3.7), (3.8) imply that

$$B=\left(\sum_{j=1}^{n} P_j(A_j)-(n-1)\right)_+$$

and, therefore, Theorem 5 and Proposition 1 imply (3.2). $\square$

From Theorem 5 and Theorem 6 we obtain

Corollary 7. *Let $A_i \in \mathscr{B}_i$, $1 \leqq i \leqq n$, then there exists a $P \in \mathscr{M}(P_1, \ldots, P_n)$ with*

a) *P has support in* $\prod_{i=1}^{n} A_i \cup \mathscr{X}_1 \times \ldots \times \mathscr{X}_{j-1} \times A_j^c \times \ldots \times \mathscr{X}_n$ *if* $P_j(A_j) = \min_{1 \leqq i \leqq n} P_i(A_i)$

b) *P has support in* $\bigcup_{j=1}^{n} \mathscr{X}_1 \times \ldots \times A_j \times \ldots \times \mathscr{X}_n$ *if* $\sum_{i=1}^{n} P_i(A_i) \geqq n-1$

c) *P has support in* $\bigcup_{j=1}^{n} \mathscr{X}_1 \times \ldots \times A_j^c \times \ldots \times \mathscr{X}_n$ *if* $\sum_{i=1}^{n} P_i(A_i) \leqq n-1$.

Remark. a) If $(\mathscr{X}_i, \mathscr{B}_i) = (R^1, \mathscr{B}^1)$, $1 \leqq i \leqq n$, $A_i = (-\infty, x]$, $1 \leqq i \leqq n$, with $x \in R^1$, then for $P \in \mathscr{M}(P_1, \ldots, P_n)$

$$P(A_1 \times \ldots \times A_n) = P(\max_{1 \leqq i \leqq n} x_i \leqq x).$$

For this special case it has been shown in an interesting paper of Lai, Robbins [10] that the bounds (3.2) are attained by an element $P_0 \in \mathscr{M}(P_1, \ldots, P_n)$ simultaneously for all $x \in R^1$, in other words: the distribution of $\max_{1 \leqq i \leqq n} x_i$ is stochastically maximized by P_0 w.r.t. $\mathscr{M}(P_1, \ldots, P_n)$.

b) With $\tilde{A}_i = \mathscr{X}_1 \times \ldots \times \mathscr{X}_{i-1} \times A_i \times \ldots \times \mathscr{X}_n$ and $p_i = P_i(A_i)$, $1 \leqq i \leqq n$, Theorem 6 says that the upper and lower Fréchet-bounds for $A_1 \times \ldots \times A_n$ are identical with the Bonferoni-bounds of first order for $\tilde{A}_1, \ldots, \tilde{A}_n$ $\left(\text{Note that } P(A_1 \times \ldots \times A_n) = P\left(\bigcap_{i=1}^{n} \tilde{A}_i\right)\right)$. This remark has for applications to simultaneous confidence intervals some interesting consequences.

References

1. Dall'Aglio, G.: Fréchet classes and compatibility of distribution functions. Symposia mathematica **9**, 131–150. Academic Press (1972)
2. Douglas, R.G.: On extremal measures and subspace density. Michigan Math. J. **11**, 243–246 (1964)
3. Dunford, N., Schwartz, J.T.: Linear Operators, Part I: General Theory. New York: Interscience publishers, inc. 1957
4. Fréchet, M.: Généralisations du théorème des probabilités totales. Fund. Math., **25** (1935)
5. Fréchet, M.: Sur les tableaux de corrélation dont les marges sont données. Annales de l'Université de Lyon, Sciences **4**, 13–84 (1951)
6. Gaffke, N., Rüschendorf, L.: On a class of extremal problems in statistics. To appear in: Math. Operationsforschung und Statistik. Series Optimization.
7. Hansel, G., Troallic, J.P.: Mesures marginales et théorème de Ford-Fulkerson. Z. Wahrscheinlichkeitstheorie verw. Gebiete **43**, 245–251 (1978)
8. Isii, K.: Inequalities of the types of Chebychev and Cramér-Rao and mathematical programming. Ann. Inst. Statist. Math. **16**, 277–293 (1964)
9. Kellerer, H.: Funktionen auf Produkträumen mit vorgegebenen Marginalfunktionen. Math. Ann. **144**, 323–344 (1961)
10. Lai, T.L., Robbins, H.: A class of dependent random variables and their maxima. Z. Wahrscheinlichkeitstheorie verw. Gebiete **42**, 89–111 (1978)

11. Lehmann, E.L.: Some concepts of dependence. Ann. Math. Statist. **37**, 1137–1153 (1966)
12. Neveu, J.: Bases mathématiques du calcul des probabilités. Paris: Masson, 1970
13. Plachky, D.: Extremal and monogenic additive set functions. Proc. Amer. Math. Soc. **54**, 193–197 (1976)
14. Schaefer, M.: Notes on the k-dimensional Jensen inequality. Ann. Probab. **4**, 502–504 (1976)
15. Strassen, V.: The existence of probability measures with given marginals. Ann. Math. Statist. **36**, 423–439 (1965)
16. Whitt, M.: Bivariate distributions with given marginals. Ann. Statist. **4**, 1280–1289 (1976)

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