Comparison of time-inhomogeneous Markov processes

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Abstract

Comparison results are given for time-inhomogeneous Markov processes with respect to function classes induced stochastic orderings. The main result states comparison of two processes, provided that the comparability of their infinitesimal generators as well as an invariance property of one process is assumed. The corresponding proof is based on a representation result for the solutions of inhomogeneous evolution problems in Banach spaces, which extends previously known results from the literature. Based on this representation, an ordering result for Markov processes induced by bounded and unbounded function classes is established. We give various applications to time-inhomogeneous diffusions, to processes with independent increments and to Lévy driven diffusion processes.

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1 Introduction

Stochastic ordering and comparison results for stochastic models are topics which have undergone an intensive development in various areas of probability and statistics such as decision theory, financial economics, insurance mathematics, risk management, queueing theory and many others. There are several approaches for comparing homogeneous Markov processes in the literature. An approach which investigates the infinitesimal generator of Markov processes in order to derive comparison results was established by Massey (1987). Compare in this context also Chen (2004), Chen and Wang (1993) and Wang (2013). A diffusion equation approach is given for the study of stochastic monotonicity in Herbst and Pitt (1991). Bassan and Scarsini (1991) consider partial orderings for stochastic processes induced by expectations of convex or increasing convex (concave or increasing concave) functionals. For bounded generators and in the case of discrete state spaces Daduna and Szekli (2006) give a comparison result for the stochastic ordering of Markov processes in terms of their generators. Rüschendorf (2008) established a comparison result for homogeneous Markov processes using boundedness conditions on the order defining function classes. Comparison results for homogeneous Markov processes with transition functions defined on general Banach spaces are given in Rüschendorf and Wolf (2011). The results are based on an integral representation of solutions to the inhomogeneous Cauchy problem. The directionally convex ordering of a special system of time-inhomogeneous interacting diffusions was considered in a similar way in Cox et al. (1996) and Greven et al. (2002).

Mainly motivated by financial applications a stochastic analysis approach has been developed in El Karoui et al. (1998), Bellamy and Jeanblanc (2000), Gushchin and Mordecki (2002) and Bergenthum and Rüschendorf (2006, 2007a). In these papers comparison results for d-dimensional semimartingales are established based on the Itô-formula and on the Kolmogorov backward equation (see also Guendouzi (2009)). A coupling approach for diffusion processes and stochastic volatility models has been developed in Hobson (1998). An approximation method is used in Bergenthum and Rüschendorf (2007b) to give some comparison results for Lévy processes and processes with independent increments. Several examples and applications to α-stable processes, *NIG* processes and *GH* processes are discussed. For multidimensional Lévy processes using an analytical formula Bäuerle et al. (2008) investigate dependence properties and establish some comparison results for the supermodular order. They study the question, whether dependence properties and orderings of the distributions of a Lévy process can be characterized by corresponding properties of the Lévy copula.

Comparison results for time-inhomogeneous Markov processes based on the theory of evolution systems on general Banach spaces as used in this paper have not been investigated before. For our main comparison result we establish a representation result for solutions of the evolution problem associated with a family of infinitesimal generators. We do not use for our approach approximation arguments or coupling arguments as in Hobson (1998), Greven et al. (2002) or in Bergenthum and Rüschendorf (2007b). Moreover, the application of the theory of evolution systems on Banach spaces allows us to reduce regularity assumptions necessary in the stochastic analysis approach based on Itô's formula.

Applications of this comparison result are given to processes with independent increments (in the sequel abbreviated as PII), inhomogeneous diffusions and to diffusion models driven by Lévy processes. Therefore, we introduce generators of several interesting orderings for which the conditions of the comparison result do hold. Since we are interested in comparison of Markov processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ w.r.t. an integral stochastic order $\leq_{\mathcal{F}}$, that is $Ef(X_t) \leq Ef(Y_t)$, $t \geq 0$ for all $f \in \mathcal{F}$, an integrability condition like

$$f \in \bigcap_{t\geq 0} \mathcal{L}^1(P^{X_t}) \cap \mathcal{L}^1(P^{Y_t}) \text{ for all } f \in \mathcal{F} \tag{1.1}$$

is indispensable and is made from now on. Also for general state spaces $\mathbb{E}$ let $\mathcal{F}$ be a set of real functions on $\mathbb{E}$ in some Banach function space $\mathbb{B}$ and let $\leq_{\mathcal{F}}$ denote the corresponding stochastic order on $\mathcal{M}^1(\mathbb{E}, \mathscr{E})$, the set of probability measures on $\mathbb{E}$, defined by

$$\mu \leq_{\mathcal{F}} \nu \text{ if } \int f d\mu \leq \int f d\nu, \tag{1.2}$$

for all $f \in \mathcal{F}$ such that the integrals exist. Some interesting examples of stochastic orderings $\leq_{\mathcal{F}}$ are given by the following function classes $\mathcal{F}$ for $\mathbb{E} = \mathbb{R}^d$,

$$\mathcal{F}_{\text{st}} := \{f : \mathbb{R}^d \to \mathbb{R}; f \text{ is increasing}\} \tag{1.3}$$

$$\mathcal{F}_{\text{cx}} := \{f : \mathbb{R}^d \to \mathbb{R}; f \text{ is convex}\} \tag{1.4}$$

$$\mathcal{F}_{\text{dcx}} := \{f : \mathbb{R}^d \to \mathbb{R}; f \text{ is directionally convex}\} \tag{1.5}$$

$$\mathcal{F}_{\text{sm}} := \{f : \mathbb{R}^d \to \mathbb{R}; f \text{ is supermodular}\} \tag{1.6}$$

$$\mathcal{F}_{\text{icx}} := \mathcal{F}_{\text{cx}} \cap \mathcal{F}_{\text{st}},\ \mathcal{F}_{\text{idcx}} = \mathcal{F}_{\text{dcx}} \cap \mathcal{F}_{\text{st}},\ \mathcal{F}_{\text{ism}} = \mathcal{F}_{\text{sm}} \cap \mathcal{F}_{\text{st}}. \tag{1.7}$$

or by subclasses of them. Orderings induced by one of these function classes $\mathcal{F}$ are also generated by $\mathcal{F} \cap C^\infty$, where C^∞ is the set of all infinitely differentiable functions, as well as by many other order generating function classes $\mathcal{F}^0 \subset \mathcal{F}$. For notions, properties and applications on stochastic orders we refer to Tong (1980), Shaked and Shanthikumar (1994) and Müller and Stoyan (2002).

As mentioned before the overall aim is to compare Markov processes w.r.t. stochastic orderings $\leq_{\mathcal{F}}$. For stochastic orderings induced by bounded function classes, the Banach space $B(\mathbb{R}^d)$ of bounded functions respectively $\mathcal{L}^\infty(\nu)$ of measurable and essentially bounded functions, where ν is a suitable measure, respectively $C_{\text{b}}(\mathbb{R}^d)$ is utilized. For processes with translation invariant transition functions comparison results are given for unbounded function classes on modified $\mathcal{L}^p$-spaces, which are introduced for this purpose in Section 2. We establish easy to verify and flexible ordering criteria which allow us to apply these results to general classes of models as for example to PII and to Lévy driven diffusion processes and to general order defining function classes. Lévy driven diffusions, PII, stochastic interest rate models and stochastic volatility models driven by a Lévy process have found considerable recent attention in the financial mathematics literature. Using the general frame of evolution system and Banach function space theory we are able to treat such classes of time-inhomogeneous Markov processes.

In detail our paper is structured as follows: In Section 2 we introduce some relevant notions and results from evolution system theory on general Banach spaces. We establish that transition functions associated to inhomogeneous Markov processes are evolution systems on certain Banach spaces. In particular we introduce $\mathcal{L}^p$-type spaces with modified $\mathcal{L}^p$-norm and prove that the transition operators define evolution systems on these modified $\mathcal{L}^p$-spaces. As a consequence several interesting examples like comparison w.r.t. convex functions can be dealt with in generality. In Section 3 we introduce the weak evolution problem. The main tool of our ordering method is the representation theorem for solutions of the weak evolution problem given in Theorem 3.1. This result extends corresponding results previously known in the literature. As consequence we obtain a general comparison result for inhomogeneous Markov processes.

In Section 4 we discuss several applications that can be dealt with by the generalized approach in this paper. We apply this approach to Lévy driven diffusion on $C_0(\mathbb{R}^d)$ in Section 4.1, to PII's in Section 4.2 and to Lévy driven diffusions on the unbounded function class $\overline{\mathcal{L}}_2^2(\nu)$ in Section 4.3. In Section 5 we consider as particular example the Sobolev Slobodeckii spaces $\mathcal{H}^r(\mathbb{R}^d)$. The transition operators then define pseudo-differential operators on $\mathcal{H}^r(\mathbb{R}^d)$. In this case our representation result is closely related to a representation result in Böttcher (2008).

2 Evolution systems and their infinitesimal generators

In this section we at first recollect some notions and results from evolution system theory which is strongly related to the semigroup theory on general Banach spaces. Our main reference is Friedman (1969); see also Pazy (1983) and Engel and Nagel (2000) for examples and applications.

A two parameter family of bounded linear operators $(T_{s,t})_{s\leq t}$, $s,t \in \mathbb{R}_+$ on a Banach space $(\mathbb{B}, \|\cdot\|)$ is called an *evolution system* (ES) if the following three conditions are satisfied:

(1) $T_{s,s} = \mathrm{id}_{\mathbb{B}} = \mathbb{1}$,

(2) $T_{s,t}\mathbb{1} = \mathbb{1}$,

(3) $T_{s,t} = T_{s,u}T_{u,t}$ for $0 \leq s \leq u \leq t$. (*evolution property*)

The evolution system is strongly continuous if the map $(s,t) \mapsto T_{s,t}$ is *strongly continuous* for $0 \leq s \leq t$. T is called a *contraction*, if

$$\|T_{s,t}f\| \leq \|f\| \text{ for } 0 \leq s \leq t,\ f \in \mathbb{B}. \tag{2.1}$$

For a strongly continuous ES we consider the corresponding family of right derivatives or *infinitesimal generators* $A_s : \mathcal{D}(A_s) \subset \mathbb{B} \to \mathbb{B}$

$$A_s f = \lim_{h\downarrow 0} \frac{1}{h}\Big(T_{s,s+h}f - f\Big) \text{ for } s > 0 \tag{2.2}$$

defined on its *domain*

$$\mathcal{D}(A_s) := \left\{ f \in \mathbb{B} \middle| \lim_{h\downarrow 0} \frac{1}{h}\Big(T_{s,s+h}f - f\Big) \text{ exists} \right\}. \tag{2.3}$$

For the reader's convenience we give a proof for the following elementary result (cf. Gulisashivili and van Casteren (2006), Section 2.3). The corresponding results for semigroups on Banach spaces can be found in Dynkin (1965) or Friedman (1969). Integrals on Banach spaces are meant in Riemann sense (cf. e.g. Ethier and Kurtz (2005), page 8 and Lemma 1.4).

Lemma 2.1 *Let $(A_s, \mathcal{D}(A_s))_{s>0}$ be the family of infinitesimal generators of a strongly continuous evolution system $T = (T_{s,t})_{s\le t}$. Then, it holds:*

(1) *For $f \in \mathbb{B}$ and $0 \le s \le u \le t$,*

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} T_{s,u} f \, du = T_{s,t} f. \tag{2.4}$$

(2) *If $s \mapsto T_{s,t} f$ is right-differentiable for $0 < s < t$, then it holds*

$$\frac{d^+}{ds} T_{s,t} f = -A_s T_{s,t} f. \qquad \textit{(backward equation)} \tag{2.5}$$

In particular $T_{s,t} f \in \mathcal{D}(A_s)$ for $0 < s < t$.

(3) *If $f \in \mathcal{D}(A_s)$ for $s, t \in \mathbb{R}_+$, $s < t$ it holds that*

$$\frac{d^+}{dt} T_{s,t} f = T_{s,t} A_t f. \qquad \textit{(forward equation)} \tag{2.6}$$

Proof: Part (1) follows directly from the continuity of $t \mapsto T_{s,t} f$ for all $s \in [0,t]$ and $f \in \mathbb{B}$. For (2) let $s < t$, then the evolution property and the strong continuity of T lead to

$$\begin{aligned}
\frac{d^+}{ds} T_{s,t} f &= \lim_{h\downarrow 0} T_{s,s+h} \frac{d^+}{ds} T_{s,t} f \\
&= \lim_{h\downarrow 0} T_{s,s+h} \left(\frac{d^+}{ds} T_{s,t} f - \frac{T_{s+h,t} f - T_{s,t} f}{h} \right) + \lim_{h\downarrow 0} T_{s,s+h} \frac{T_{s+h,t} f - T_{s,t} f}{h} \\
&= 0 + \lim_{h\downarrow 0} \frac{T_{s,t} f - T_{s,s+h} T_{s,t} f}{h} \\
&= -\lim_{h\downarrow 0} \frac{T_{s,s+h} - \mathrm{id}_{\mathbb{B}}}{h} T_{s,t} f \\
&= -A_s T_{s,t} f.
\end{aligned}$$

Now let $f \in \mathcal{D}(A_s)$, $s \in [0,t]$ then we obtain again due to the evolution property

$$\begin{aligned}
\frac{d^+}{dt} T_{s,t} f = \lim_{h\downarrow 0} \frac{1}{h} \big(T_{s,t+h} f - T_{s,t} f \big) &= \lim_{h\downarrow 0} T_{s,t} \Big(\frac{1}{h} \big(T_{t,t+h} f - f \big) \Big) \\
&= T_{s,t} \Big(\lim_{h\downarrow 0} \frac{1}{h} \big(T_{t,t+h} f - f \big) \Big) \\
&= T_{s,t} A_t f
\end{aligned}$$

and part (3) is done. $\square$

For a time-inhomogeneous Markov process (X_t) on a measure space $(\mathbb{E}, \mathscr{E})$ let $(P_{s,t})_{s\le t}$ denote the transition kernel or *transition function*

$$P_{s,t}(x, V) = P(X_t \in V \mid X_s = x), \, x \in \mathbb{E}, \, V \in \mathscr{E} \tag{2.7}$$

and $T = (T_{s,t})_{s\le t}$ the corresponding evolution system, i.e. the transition operator

$$T_{s,t} f(x) = \int_{\mathbb{E}} P_{s,t}(x, dy) f(y) \tag{2.8}$$

$$= E(f(X_t) \mid X_s = x) \tag{2.9}$$

for f in a suitable Banach space $\mathbb{B}$ of functions on $\mathbb{E}$. This puts Markov processes in the framework of evolution systems and evolution equations.

As first example we consider the Banach spaces $\mathcal{L}^\infty(\nu) := \{f : \mathbb{R}^d \to \bar{\mathbb{R}} \mid f \text{ is } \nu\text{-measurable}, \|f\|_\infty < \infty\}$ of ν-measurable and essentially bounded functions, where ν is a suitable measure on $\mathbb{R}^d$, $B(\mathbb{R}^d)$ the class of bounded functions on $\mathbb{R}^d$ and $C_\mathrm{b}(\mathbb{R}^d)$ the continuous functions in $B(\mathbb{R}^d)$. Recall that for Markov processes the properties (1),(2) and (3) from the definition of ES always hold true on these Banach spaces. Moreover, observe that uniform continuity of the transition kernels in (2.8) implies strong continuity of the transition operator T:

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |P_{s,t}(x,dy) - P_{s,u}(x,dy)| \to 0 \text{ as } t \to u, \text{ and} \\ &\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |P_{s,t}(x,dy) - P_{u,t}(x,dy)| \to 0 \text{ as } s \to u. \end{aligned} \tag{2.10}$$

Proposition 2.2 (ES-property for time-inhomogeneous Markov processes on bounded function classes)
Let $(P_{s,t})_{s\leq t}$ be the transition kernel of a time-inhomogeneous Markov process $(X_t)_{t\geq 0}$ on $(\mathbb{E},\mathscr{E})$. If the transition kernel $(P_{s,t})_{s\leq t}$ satisfies condition (2.10) *then the operator defined in* (2.8) *is a strongly continuous contraction ES on $\mathbb{B} \in \{B(\mathbb{R}^d), \mathcal{L}^\infty(\nu)\}$. If additionally, $f \in C_b(\mathbb{R}^d)$ implies $T_{s,t}f \in C_b(\mathbb{R}^d)$, then the operator is a strongly continuous contraction ES on $\mathbb{B} = C_b(\mathbb{R}^d)$.*

Proof: Since T is a family of bounded operators it leaves bounded functions invariant, that is, for $f \in C_\mathrm{b}(\mathbb{R}^d)$ we obtain

$$\|T_{s,t}f\|_\infty \leq \sup_x \int_{\mathbb{R}^d} P_{s,t}(x,dy)|f(y)| \leq \|f\|_\infty.$$

Hence, the strong continuity follows due to

$$\|T_{s,t}f - f\| = \sup_x |T_{s,t}f(x) - T_{s,s}f(x)| \leq \|f\|_\infty \cdot \sup_x \int_{\mathbb{R}^d} |P_{s,t}(x,dy) - P_{s,s}(x,dy)|$$

for all $s \leq t$ and the continuity property (2.10).
The proof for $\mathbb{B} \in \{B(\mathbb{R}^d), \mathcal{L}^\infty(\nu)\}$ is very similar and therefore omitted. □

Remark 2.3 **(a)** *For a PII $L = (L_t)_{t\geq 0}$ the transition operators on $\mathbb{B}$ are given by*

$$T_{s,t}f(x) = \int_{\mathbb{R}^d} P^{L_t - L_s}(dy) f(x+y). \tag{2.11}$$

If $\mathbb{B} = C_b(\mathbb{R}^d)$ the map $x \mapsto T_{s,t}f(x)$ is continuous since the continuity of f transfers to $(T_{s,t})$ thus $C_b(\mathbb{R}^d)$ is invariant under $(T_{s,t})_{s\leq t}$. In particular the transition kernels of Lévy process such as Brownian motion, NIG, VG, GH processes have this invariance property.

(b) *Note that the Brownian motion respectively its associated Brownian kernel*

$$P_{s,t}(x,dy) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\pi(t-s)}} \exp\Big(-\frac{|x-y|^2}{2(t-s)}\Big) dy$$

for $x \in \mathbb{R}^d$ and $s \leq t$ is not a strongly continuous ES on $C_b(\mathbb{R}^d)$. Hence, it is vital to consider further Banach spaces with different norms to establish the ES-property for Markov processes. In Proposition 2.4 we consider the Banach space of p-integrable functions on $\mathbb{R}^d$ and show that in case of translation invariant transition function the ES-property does hold.

(c) *In Section 4.1 we will see that Lévy driven diffusion defined via the stochastic differential equation* (4.1) *possesses the ES-property on C_0. Thus, in order to establish stochastic ordering results induced by bounded function classes, the Banach space $(C_0, \|\cdot\|_\infty)$ is used as a reference space.*

For many stochastic orderings the function spaces $C_\mathrm{b}(\mathbb{R}^d)$, $B(\mathbb{R}^d)$ or $\mathcal{L}^\infty(\nu)$ are sufficient; some orderings as for example the convex ordering $\leq_{\mathrm{cx}}$ however do not allow bounded generating classes of the ordering. The following proposition shows that Markov processes with translation invariant transition functions $(P_{s,t})_{s\leq t}$ are strongly continuous ES on $\mathcal{L}^p$-spaces. Hereto, we consider the p-integrable functions on $\mathbb{R}^d$, which we denote by

$$\mathcal{L}^p(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \bar{\mathbb{R}} | f \text{ is measurable, } \|f\|_p < \infty\},\ 1 \leq p < \infty.$$

Proposition 2.4 (ES-property for time-inhomogeneous translation invariant Markov processes on $\mathcal{L}^p(\mathbb{R}^d)$)
Let $(X_t)_{t\geq 0}$ be a time-inhomogeneous Markov process on $\mathbb{R}^d$ with translation invariant transition function $(P_{s,t})_{s\leq t}$. Then the family of transition operators $T = (T_{s,t})_{s\leq t}$ defined by

$$T_{s,t}f(x) = \int\limits_{\mathbb{R}^d} P_{s,t}(x, dy)f(y) = \int\limits_{\mathbb{R}^d} P_{s,t}(0, dy)f(y + x) \tag{2.12}$$

for $f \in \mathcal{L}^p(\mathbb{R}^d)$, is a strongly continuous contraction ES on $\mathcal{L}^p(\mathbb{R}^d)$, $1 \leq p$.

Proof: Let $s \leq t$ and set $P_{s,t}(0, dy) =: P_{s,t}(dy)$. We have for $f \in \mathcal{L}^p(\mathbb{R}^d)$

$$\begin{aligned}\|T_{s,t}f - f\|_p^p &= \int\limits_{\mathbb{R}^d} \Big| \int\limits_{\mathbb{R}^d} P_{s,t}(dy)f(x + y) - f(x)\Big|^p dx \\ &\leq \int\limits_{\mathbb{R}^d} P_{s,t}(dy) \int\limits_{\mathbb{R}^d} |f(x + y) - f(x)|^p dx =: \int\limits_{\mathbb{R}^d} P_{s,t}(dy)h(y),\end{aligned}$$

where h is a bounded and continuous function with $h(0) = 0$ (see Sato (1999, E. 34.10)). For each $\varepsilon > 0$ we can find a $\delta > 0$ such that $\int_{\{|y|\leq\delta\}} P_{s,t}(dy)h(y) < \varepsilon$. By the triangle inequality to show strong continuity of $T_{s.t}$ it is enough to consider for $u \in \mathbb{R}_+$ the case that $s \leq t$ and $s, t \to u$. Then

$$\begin{aligned}\lim_{\substack{s\to u, t\to u,\\ s<t}} \int\limits_{\mathbb{R}^d} P_{s,t}(dy)h(y) &\leq \varepsilon + \lim_{s\to u, t\to u} \int\limits_{\{|y|>\delta\}} P_{s,t}(dy)h(y) \\ &= \varepsilon + \int\limits_{\{|y|>\delta\}} P_{u,u}(dy)h(y) \\ &= \varepsilon + \int\limits_{\{|y|>\delta\}} \delta_0(dy)h(y) = \varepsilon.\end{aligned}$$

Finally, we obtain due to the convexity of $x \mapsto |x|^p$,

$$\begin{aligned}\|T_{s,t}f\|_p^p &= \int \Big| \int P_{s,t}(dy)f(x + y)\Big|^p dx \\ &\leq \int P_{s,t}(dy) \int |f(x + y)|^p dx \end{aligned} \tag{2.13}$$

$$= \int P_{s,t}(dy) \int |f(x)|^p dx = \|f\|_p^p.$$

Thus, the operator norm of $T_{s,t}$ is bounded by one, i.e. $\|T_{s,t}\| \leq 1$ and $T_{s,t}f \in \mathcal{L}^p(\mathbb{R}^d)$ for all $0 \leq s \leq t$, i.e. T is a strongly continuous contraction evolution system on $\mathcal{L}^p(\mathbb{R}^d)$. $\square$

For some classes of models it is possible to establish the ES-property by comparison to the translation invariant case.

Corollary 2.5 *Let $(X_t)_{t\geq 0}$ be a time-inhomogeneous Markov process on $\mathbb{R}^d$ with transition function $(P_{s,t})_{s\leq t}$ and transition operator $T = (T_{s,t})_{s\leq t}$ defined by (2.8). If there exists a positive constant c and a translation invariant transition function $(Q_{s,t})_{s\leq t}$ such that*

$$\int\limits_{\mathbb{R}^d} f(y) P_{s,t}(x, dy) \leq c \int\limits_{\mathbb{R}^d} f(y) Q_{s,t}(x, dy), \tag{2.14}$$

for all $x \in \mathbb{R}^d$, all positive functions f on $\mathbb{R}^d$ and all $s, t \in \mathbb{R}_+$ with $s \leq t$, then T is a strongly continuous ES on $\mathcal{L}^p(\mathbb{R}^d)$, $1 \leq p < \infty$.

As we see in Proposition 2.4 the translation invariance property of the Lebesgue measure and the translation invariance of the associated transition function are crucial for T to be a strongly continuous contraction ES on $\mathcal{L}^p(\mathbb{R}^d)$. For a σ-finite measure ν we circumvent the lack of the invariance property by introducing a suitable weighted sup-norm on a sufficiently large subspace of $\mathcal{L}^p(\nu)$:

$$\bar{\mathcal{L}}^p_\varrho(\nu) := \Big\{ f \in \mathcal{L}^p(\nu) \,\Big|\, \|f\|^*_{p,\varrho} := \sup_{y\in\mathbb{R}^d} \frac{1}{(1+\|y\|)^{\frac{\varrho}{p}}} \Big(\int\limits_{\mathbb{R}^d} |f(x+y)|^p \nu(dx) \Big)^{\frac{1}{p}} < \infty \Big\}$$

for $1 \leq p < \infty$ and $\varrho \geq 0$. For $p = \infty$ we define $\|\cdot\|_{\infty,\varrho} = \|\cdot\|_\infty = \|\cdot\|_{\infty,\nu}$. Thus, the space $\bar{\mathcal{L}}^\infty_\varrho(\nu)$ equals $\mathcal{L}^\infty(\nu)$. The $\mathcal{L}^p$-type space $(\bar{\mathcal{L}}^p_\varrho(\nu), \|\cdot\|^*_{p,\varrho})$, $\varrho \geq 0$, $1 \leq p \leq \infty$ is a Banach space.

Lemma 2.6 $(\bar{\mathcal{L}}^p_\varrho(\nu), \|\cdot\|^*_{p,\varrho})$, $\varrho \geq 0$, $1 \leq p \leq \infty$ *is a Banach space.*

Proof: By definition $\|\cdot\|^*_{p,\varrho}$ is a norm and therefore, $\bar{\mathcal{L}}^p_\varrho(\nu)$ is a vector space. It remains to show that $\bar{\mathcal{L}}^p_\varrho(\nu)$ is complete. This is done similarly to the proof of the *Riesz–Fischer* Theorem.

Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\bar{\mathcal{L}}^p_\varrho(\nu)$, that is

$$\|f_j - f_k\|^*_{p,\varrho} = \sup_{y\in\mathbb{R}^d} \frac{1}{(1+\|y\|)^{\varrho/p}} \Big(\int\limits_{\mathbb{R}^d} \big|f_j(x+y) - f_k(x+y)\big|^p \nu(dx) \Big)^{1/p} \to 0 \tag{2.15}$$

as j, k approaches infinity. Thus, we have to show that there exists $f \in \bar{\mathcal{L}}^p_\varrho(\nu)$ such that $f_j \to f$ for $j \to \infty$ in $\bar{\mathcal{L}}^p_\varrho(\nu)$. Due to

$$\|f - f_j\|^*_{p,\varrho} \leq \|f - f_{j_i}\|^*_{p,\varrho} + \|f_{j_i} - f_j\|^*_{p,\varrho}$$

it remains to verify this fact for a subsequence of $(f_j)_{j\in\mathbb{N}}$. We choose the subsequence $(f_{j_m})_{m\in\mathbb{N}}$ such that (f_{j_m}) converges a.s. and

$$\sum_{m=1}^\infty \|f_{j_{m+1}} - f_{j_m}\|^*_{p,\varrho} < \infty. \tag{2.16}$$

and denote (f_{j_m}) by (f_m) again. Since

$$\frac{1}{(1+\|0\|)^{\varrho/p}} \Big(\int\limits_{\mathbb{R}^d} |g(x+0)|^p \nu(dx) \Big)^{1/p} \leq \sup_y \frac{1}{(1+\|y\|)^{\varrho/p}} \Big(\int\limits_{\mathbb{R}^d} \big|g(x+y)\big|^p \nu(dx) \Big)^{1/p}$$

for all $g \in \bar{\mathcal{L}}^p_\varrho(\nu)$, we have $\|f_m\|_p \le \|f_m\|^*_{p,\varrho}$. Due to (2.15) the sequence $(f_m)_{m\in\mathbb{N}}$ is convergent in $\mathcal{L}^p(\nu)$ and there exists a limit $f \in \mathcal{L}^p(\nu)$. By the Lemma of *Fatou* we obtain for $y \in \mathbb{R}^d$:

$$\begin{aligned}
&\frac{1}{(1+\|y\|)^\varrho}\int |f(x+y)-f_m(x+y)|^p\nu(dx)\\
&\quad\le \liminf_{l\to\infty}\frac{1}{(1+\|y\|)^\varrho}\int |f_l(x+y)-f_m(x+y)|^p\nu(dx)\\
&\quad\le \liminf_{l\to\infty}\Big(\|f_l-f_m\|^*_{p,\varrho}\Big)^p\\
&\quad\le \lim_{l\to\infty}\Big(\sum_{k=m}^{l-1}\|f_{k+1}-f_k\|^*_{p,\varrho}\Big)^p=\Big(\sum_{k=m}^{\infty}\|f_{k+1}-f_k\|^*_{p,\varrho}\Big)^p<\infty.
\end{aligned}$$

Thus, we have

$$\sup_{y\in\mathbb{R}^d}\frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int |f(x+y)-f_m(x+y)|^p\nu(dx)\Big)^{1/p}\le\Big(\sum_{k=m}^{\infty}\|f_{k+1}-f_k\|^*_{p,\varrho}\Big).$$

By (2.16) the last term tends to zero as m approaches infinity. This implies the statement of the lemma. □

For time-inhomogeneous translation invariant Markov process $(X_t)_{t\ge 0}$ on $\mathbb{R}^d$ with transition function $(P_{s,t})_{s\le t}$ satisfying a certain integrability condition the ES-property holds for $\bar{\mathcal{L}}^p_{c,\varrho}(\nu)=\bar{\mathcal{L}}^p_\varrho(\nu)\cap C(\mathbb{R}^d)$, the set of continuous functions in $\bar{\mathcal{L}}^p_\varrho(\nu)$, with respect to $\|\cdot\|^*_{p,\varrho}$-norm as well.

Proposition 2.7 (ES-property for time-inhomogeneous translation invariant Markov processes) *Assume that $\|z\|^\varrho$ is uniformly integrable w.r.t. $P_{s,t}$ for some $\varrho>0$ with $s\le t$, i.e.,*

$$\sup_{s\le t}\int_{\|z\|\ge K}\|z\|^\varrho P_{s,t}(dz)\underset{K\to\infty}{\longrightarrow}0.$$

Then, the family of transition operators $T=(T_{s,t})_{s\le t}$ defined for $f\in\bar{\mathcal{L}}^p_{c,\varrho}(\nu)$ by (2.12), is a strongly continuous ES on $\bar{\mathcal{L}}^p_{c,\varrho}(\nu)$ for $1\le p<\infty$. If additionally, ν is absolutely continuous with respect to the Lebesgue measure $\lambda\!\!\!\lambda$, then T is a strongly continuous ES on $\bar{\mathcal{L}}^p_\varrho(\nu)$.

Proof: The proof for the strong continuity is analogous to the proof of Proposition 2.4. Hereto, assume ν to be absolutely continuous with respect to $\lambda\!\!\!\lambda$. For $f\in\bar{\mathcal{L}}^p_\varrho(\nu)$ note that the function

$$h^*(z)=\sup_y\frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int |f(x+y+z)-f(x+y)|^p\nu(dx)\Big)^{1/p}$$

is bounded by $c(1+\|z\|^\varrho)^{1/p}$ and continuous with $h^*(0)=0$ where $c=c_f$ is constant in $\mathbb{R}_+$. In the case where ν is not absolutely continuous w.r.t. $\lambda\!\!\!\lambda$, then simply restrict the function class to $f\in\bar{\mathcal{L}}^p_{c,\varrho}(\nu)$. Consequently, h^* becomes continuous again. For the boundedness statement let $f\in\bar{\mathcal{L}}^p_\varrho(\nu)$ and observe that the expression

$$\Big(\frac{1}{(1+\|y\|)^\varrho}(1+\|y+z\|)^\varrho\Big)^{1/p}$$

assumes its maximum in $y=0$. Thus,

$$h^*(z)\le c\Big(\sup_y\frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int |f(x+y+z)|^p\nu(dx)\Big)^{1/p}+\|f\|^*_{p,\varrho}\Big)$$

$$
\begin{aligned}
&\le c\Big(\sup_y \frac{(1+\|y+z\|)^{\varrho/p}}{(1+\|y\|)^{\varrho/p}}\\
&\quad\cdot \sup_{y+z} \frac{1}{(1+\|y+z\|)^{\varrho/p}}\Big(\int |f(x+y+z)|^p \nu(dx)\Big)^{1/p} + \|f\|^*_{p,\varrho}\Big)\\
&\le c\|f\|^*_{p,\varrho}(1+(1+\|z\|^\varrho)^{1/p})
\end{aligned}
$$

for a sufficiently large $c>0$. Then we have

$$
\begin{aligned}
\|T_{s,t}f-f\|^{*p}_{p,\varrho} &= \sup_y \frac{1}{(1+\|y\|)^\varrho}\Big(\int |T_{s,t}f(x+y)-f(x+y)|^p\nu(dx)\Big)\\
&\le \int P_{s,t}(dz)h^{*p}(z).
\end{aligned}
$$

Note that for each $\varepsilon>0$ we can find a $\delta>0$ such that $\int_{\{|z|<\delta\}} P_{s,t}(dz)h^{*p}(z)<\varepsilon$ due to the continuity of h^*. Since h^{*p} is bounded by the uniformly integrable function $z\mapsto 1+\|z\|^\varrho$ w.r.t. $P_{s,t}$ separating the integral and letting $s,t\to u$, $u\in\mathbb{R}_+$, $s<t$ we obtain for $M\in\mathbb{R}_+$ that

$$
\begin{aligned}
\lim_{\substack{s\to u,t\to u,\\ s<t}} \int_{\mathbb{R}^d} P_{s,t}(dz)h^{*p}(z) &\le \varepsilon + \lim_{s\to u,\, t\to u} \int_{\{|z|>\delta\}\cap[-M,M]} P_{s,t}(dz)h^{*p}(z)\\
&\quad + \lim_{s\to u,\, t\to u} \int_{\{|z|>\delta\}\cap[-M,M]^c} P_{s,t}(dz)h^{*p}(z)\\
&\le \varepsilon + \int_{\{|z|>\delta\}\cap[-M,M]} P_{u,u}(dz)h^{*p}(z)\\
&\quad + \lim_{s\to u,\, t\to u} \int_{\{|z|>\delta\}\cap[-M,M]^c} P_{s,t}(dz)h^{*p}(z)\\
&\le \varepsilon + \int_{\{|z|>\delta\}\cap[-M,M]} \delta_0(dz)h^{*p}(z)+\varepsilon = 2\varepsilon.
\end{aligned}
$$

Moreover, recall that $\int_{\mathbb{R}^d}\|z\|^\varrho P_{s,t}(dz)<\infty$, thus

$$
\begin{aligned}
\|T_{s,t}f\|^*_{p,\varrho} &= \sup_y \frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int_{\mathbb{R}^d} |T_{s,t}f(x+y)|^p\nu(dx)\Big)^{1/p}\\
&= \sup_y \frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} f(x+y+z)P_{s,t}(dz)\Big|^p\nu(dx)\Big)^{1/p}\\
&\le \sup_y \frac{1}{(1+\|y\|)^{\varrho/p}}\Big(\int_{\mathbb{R}^d} P_{s,t}(dz)\Big(\int_{\mathbb{R}^d} |f(x+y+z)|^p\nu(dx)\Big)\Big)^{1/p}.
\end{aligned}
$$

Consequently, we arrive at

$$
\begin{aligned}
\|T_{s,t}f\|^*_{p,\varrho} &\le \|f\|^*_{p,\varrho}\sup_y\Big(\int_{\mathbb{R}^d} P_{s,t}(dz)\Big(\frac{1}{(1+\|y\|)^\varrho}(1+\|y+z\|)^\varrho\Big)\Big)^{1/p}\\
&\le \|f\|^*_{p,\varrho}\Big(\int_{\mathbb{R}^d} P_{s,t}(dz)(1+\|z\|)^\varrho\Big)^{1/p}\\
&\le c\cdot\|f\|^*_{p,\varrho}\Big(\int_{\mathbb{R}^d} \|z\|^\varrho\, P_{s,t}(dz)\Big)^{1/p} \le c'\|f\|^*_{p,\varrho}.
\end{aligned}
$$

Hence, $\|T_{s,t}|_{\bar{\mathcal{L}}^p_\varrho(\nu)}\| \le c'$ and $T_{s,t}f \in \bar{\mathcal{L}}^p_\varrho(\nu)$ for all $0 \le s \le t$, i.e. T is a strongly continuous ES on $\bar{\mathcal{L}}^p_\varrho(\nu)$. $\square$

Remark 2.8 ***(b−bounded functions)***
The norm modified $\boldsymbol{\mathcal{L}^p}$-space, $\boldsymbol{\bar{\mathcal{L}}^p_\varrho(\nu)}$ allows us to deal with orderings generated by unbounded function classes. Let $\boldsymbol{b} : \mathbb{R}^{\boldsymbol{d}} \to [\boldsymbol{1}, \infty)$ be a weight function and define

$$\boldsymbol{B}_b := \{\boldsymbol{f} : \mathbb{R}^{\boldsymbol{d}} \to \mathbb{R} \mid \exists\, \boldsymbol{c} \in \mathbb{R} : |\boldsymbol{f}(\boldsymbol{x})| \le \boldsymbol{c} \cdot \boldsymbol{b}(\boldsymbol{x})\}$$

the class of $\boldsymbol{b}$-bounded functions. In particular, to deal with the convex ordering we choose $\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{1} + \|\boldsymbol{x}\|$ for $\|\cdot\|$ any norm on $\mathbb{R}^{\boldsymbol{d}}$. Then the class of $\boldsymbol{b}$-bounded convex functions is a generator of the convex ordering $\le_{cx}$ and is a subset of $\boldsymbol{\bar{\mathcal{L}}^p_\varrho(\nu)}$, $\boldsymbol{p} \le \boldsymbol{\varrho}$ for any $\boldsymbol{\nu}$ which integrates $(\boldsymbol{1} + \|\boldsymbol{x}\|)^{\boldsymbol{p}}$.

Thus in the case of $\bar{\mathcal{L}}^p_\varrho(\nu)$ the ES-property is implied by the translation invariance property and a integrability condition of the transition function $(P_{s,t})_{s\le t}$. Under an additional assumption on the transition kernel we extend in the following the ES-property to the case of not necessarily translation invariant time-inhomogeneous Markov processes for unbounded function classes. More specifically we consider the class $\bar{\mathcal{L}}^2_2(\nu)$ and impose the following assumption (K), which is a strengthening of the notion of the Hilbert–Schmidt operator.

We assume that the transition kernel $P_{s,t}(x, dz) = k_{s,t}(x, x+z)\nu(dz)$ corresponding to a time-inhomogeneous Markov process has a density w.r.t. ν fulfills

(K)
$$K^*_{s,t} := \sup_{y\in\mathbb{R}^d} \frac{1}{1+\|y\|}\left(\int_{\mathbb{R}^d} |k_{s,t}(y, y+z)|^2 \nu(dz)\right)^{1/2} < \infty$$

and satisfies the continuity assumption

(C)
$$\text{For } 0 < s < t \text{ holds } \lim_{s'\to s, t'\to t} K^*_{s',t'} = K^*_{s,t}.$$

For the continuity in (s, s) we make the following local domination assumption.

(D) For any $s \ge 0$ there exists $\delta = \delta(s) > 0, C > 0,$ and a translation invariant transition function $(Q_{s,t})$ such that $\|z\|^2$ is uniformly integrable w.r.t. $(Q_{s,t})$ and $P_{s,t}f \le CQ_{s,t}f$, $\forall f \in \overline{\mathcal{L}}^2_2(\nu), f \ge 0, t - s \le \vartheta$.

Note that conditions (K), (C) are fulfilled under boundedness and continuity assumptions on k. The domination condition (D) ensuring continuity of $T_{s,t}$ in (s, s) might be verified in several applications but could also be replaced by further ad hoc assumptions.

Proposition 2.9 (ES-property for time-inhomogeneous Markov processes on $\boldsymbol{\bar{\mathcal{L}}^2_2(\nu)}$)
Let ν be a finite measure with finite second moments on $\mathbb{R}^d$ and let $(X_t)_{t\ge0}$ be a time inhomogeneous Markov process with transition kernel $(P_{s,t})_{s\le t}$ which satisfies the assumptions (K), (C) and (D). Then the corresponding family of transition operators $(T_{s,t})_{s\le t}$ is a strongly continuous ES on $\overline{\mathcal{L}}^2_2(\nu)$.

Proof: For $f \in \bar{\mathcal{L}}^2_2(\nu)$ and $x, y \in \mathbb{R}^d$ holds that

$$\begin{aligned}
A_{s,t} &:= \sup_y \frac{1}{1+\|y\|}\left(\int (T_{s,t}f(x+y))^2 \nu(dx)\right)^{1/2} \\
&\le \sup_y \left(\int \left(\int \frac{k_{s,t}(x+y, x+y+z)}{1+\|y\|} \frac{f(x+y+z)}{1+\|y\|}\, d\nu(z)\right)^2 \nu(dx)\right)^{1/2} \\
&\le \sup_y \left(\int \left(\int \frac{k^2_{s,t}(x+y, x+y+z)}{(1+\|y\|)^2}\, d\nu(z) \int \frac{f^2(x+y+z)}{(1+\|y\|)^2}\, d\nu(z)\right) \nu(dx)\right)^{1/2}.
\end{aligned}$$

Further we obtain

$$
\begin{aligned}
&\frac{1}{(1+\|y\|)^2}\int f^2(x+y+z)\,d\nu(z)\\
&\quad=\left(\frac{1+\|x+y\|}{1+\|y\|}\right)^2\frac{1}{(1+\|x+y\|)^2}\int f^2(x+y+z)\,d\nu(z)\\
&\quad\le(1+\|x\|)^2(\|f\|_{2,2}^*)^2
\end{aligned}
$$

and, similary,

$$
\sup_y\frac{1}{(1+\|y\|)^2}\int k_{s,t}^2(x+y,x+y+z)\,d\nu(z)\le(1+\|x\|)^2(K_{s,t}^*)^2.
$$

As consequence we obtain

$$
A_{s,t}\le\left(\int(1+\|x\|)^2\,\nu(dx)\right)^{1/2}\|f\|_{2,2}^*\,K_{s,t}^*<\infty.
$$

Thus by assumption (K) $T_{s,t}f\in\bar{\mathcal{L}}_2^2(\nu)$ for $f\in\bar{\mathcal{L}}_2^2(\nu)$ and by the continuity assumption (C) $T_{s,t}$ is strongly continuous in (s,t) for $s<t$. By the domination assumption (D) we obtain from Proposition 2.7 that $T_{s,t}$ is also continuous in (s,s), $0\le s$. $\square$

Several modifications of the domination assumption ensuring continuity in (s,s) could be given.

Example: PII and their infinitesimal generators
Let $L=(L_t)_{t\ge0}$ be a PII with continuity property (2.10) and with characteristic function given by

$$
Ee^{i\langle\xi,L_t\rangle}=e^{\int_0^t\theta_s(i\xi)ds},\tag{2.17}
$$

where for $s\ge0$ the cumulant function $\theta=(\theta_s)_{s\ge0}$ equals

$$
\theta_s(i\xi)=-\frac{1}{2}\langle\xi,\sigma_s\xi\rangle+i\langle\xi,b_s\rangle+\int\left(e^{i\langle\xi,y\rangle}-1-i\langle\xi,\chi_{\mathtt{c}}(y)\rangle\right)F_s(dy)\tag{2.18}
$$

for a *cut-off* function $\chi_{\mathtt{c}}$, that is a bounded, measurable real function on $\mathbb{R}^d$ with compact support and which equals the identity in a neighbourhood of zero. Here for each $s>0$ the *covariance matrix* σ_s is a symmetric, positive semi-definite $d\times d$ matrix, the *drift* b_s is in $\mathbb{R}^d$ and F_s is a Lévy-measure, i.e. a Borel-measure on $\mathbb{R}^d$ which integrates $(1\wedge|x|^2)$ with $F(\{0\})=0$.

For a triplet (b_s,σ_s,F_s) defined as above we consider the operator $G_sf(x)=G_s^Df(x)+G_s^Jf(x)$, $s\in\mathbb{R}_+$ with

$$
\begin{aligned}
G_s^Df(x)&:=\frac{1}{2}\sum_{j,k=1}^d\sigma_s^{j,k}\frac{\partial^2f}{\partial x_j\partial x_k}(x)+\sum_{j=1}^db_s^j\frac{\partial f}{\partial x_j}(x),\ \text{and}\\
G_s^Jf(x)&:=\int_{\mathbb{R}^d}\Big(f(x+y)-f(x)-\sum_{j=1}^d\frac{\partial f}{\partial x_j}(x)(\chi_{\mathtt{c}}(y)^j)\Big)F_s(dy)
\end{aligned}\tag{2.19}
$$

on $\bar{\mathcal{W}}^2(\nu)=\left\{f\in C^2(\mathbb{R}^d)\cap\bar{\mathcal{L}}_2^2(\nu)|\frac{\partial f}{\partial x_i},\frac{\partial^2f}{\partial x_i\partial x_j}\in\bar{\mathcal{L}}_2^2(\nu)\,\forall\,1\le i,j\le d\right\}\subset\bar{\mathcal{L}}_2^2(\nu)$.

For upcoming results it is crucial to know that G_sf belongs to $\bar{\mathcal{L}}_2^2(\nu)$ and whether such an operator can be linked to a strongly continuous ES on $\bar{\mathcal{L}}_2^2(\nu)$ corresponding to a time-inhomogeneous Markov process. Obviously, for $f\in\bar{\mathcal{W}}^2(\nu)$ the expression $G_s^Df\in\bar{\mathcal{L}}_2^2(\nu)$ for every $s\in\mathbb{R}_+$. Moreover, under some mild regularity conditions it can be shown that $G_s^J:\bar{\mathcal{W}}^2(\nu)\to\bar{\mathcal{L}}_2^2(\nu)$, where $\bar{\mathcal{W}}^2(\nu)$ is a subspace of $\bar{\mathcal{L}}_2^2(\nu)$ with norm $\|\cdot\|_{2,2}^*$.

Lemma 2.10 *For the operator $G_s^D f$ introduced in equation (2.19) it holds that $G_s^D|_{\bar{\mathcal{W}}^2(\nu)} : \bar{\mathcal{W}}^2(\nu) \to \bar{\mathcal{L}}_2^2(\nu)$. If additionally*

$$\int_0^t \int_{\{|y|\geq 1\}} |y|^2 F_s(dy)ds < \infty,$$

then the same holds true for the operator $G_s|_{\bar{\mathcal{W}}^2(\nu)}$.

Proof: It remains to show that for $f \in \bar{\mathcal{W}}^2(\nu)$ the jump part $G_s^J f$ in (2.19) belongs to $\bar{\mathcal{L}}_2^2(\nu)$. We only cover the one-dimensional case. The multivariate version is similar. Let $f \in \bar{\mathcal{W}}^2(\nu)$, $s \in \mathbb{R}_+$, and choose the cut-off function $\chi_{\mathsf{C}}(y) = y\mathbb{1}_{\{|y|<1\}}$. Using the Taylor expansion for $x' \in \mathbb{R}$ we arrive at

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} f(x'+y) - f(x') - f'(x')y\mathbb{1}_{\{|y|<1\}}F_s(dy)\Big| \\
&\qquad \leq \Big|\int_{\mathbb{R}} \big(f(x'+y) - f(x') - f'(x')y\big)\mathbb{1}_{\{|y|<1\}}F_s(dy)\Big| \\
&\qquad\quad + \int_{\{|y|\geq 1\}} |f(x'+y) - f(x')|F_s(dy) \\
&\qquad \leq \underbrace{\int_{\{|y|<1\}} \int_0^1 (1-\vartheta)f''(x'+\vartheta y)\, d\vartheta F_s(dy)}_{=:I_1} + \underbrace{\int_{\{|y|\geq 1\}} |f(x'+y)|F_s(dy)}_{=:I_2} \\
&\qquad\quad \underbrace{+ |f(x')| \int_{\{|y|\geq 1\}} (1\wedge |y|^2)F_s(dy)}_{=:I_3},
\end{aligned}$$

where ϑ is a suitable value in $[0,1]$. For $x, z \in \mathbb{R}$, we choose $x' = x+z$, then, squaring and integrating each term successively w.r.t. $\nu(dx)$. For I_3 we then have

$$\begin{aligned}
&\Big(\int_{\{|y|\geq 1\}} (1\wedge |y|^2)F_s(dy)\Big)^2 \cdot \int |f(x+z)|^2\nu(dx) \\
&\qquad \leq c_3(1+|z|)^2 \cdot \sup_{z'} \frac{1}{(1+|z'|)^2}\Big(\int |f(x+z')|^2\nu(dx)\Big) \\
&\qquad \leq c_3(1+|z|)^2 \cdot (\|f\|_{2,2}^*)^2
\end{aligned}$$

for a suitable non-negative constant c_3. Since

$$\int_0^t \int_{\{|y|\geq 1\}} |y|^2 F_s(dy)ds < \infty,$$

for the middle term I_2 we observe that

$$\begin{aligned}
&\int \int_{\{|y|\geq 1\}} |f(x+z+y)|^2 F_s(dy)\nu(dx) \\
&\quad = \int_{\{|y+z|\geq 1\}} (1+|y+z|)^2\Big(\frac{1}{(1+|y+z|)^2}\int |f(x+y+z)|^2\nu(dx)\Big)F_s(dy)
\end{aligned}$$

$$\leq \int_{\{|y|\geq 1\}} (1+|z+y|)^2 \cdot \sup_{z'} \frac{1}{(1+|z'+y|)^2} \Big(\int |f(x+z'+y)|^2 \nu(dx) \Big) F_s(dy)$$
$$\leq (\|f\|^*_{2,2})^2 \Big(\int_{\{|y|\geq 1\}} (1+|z|)^2 + |y|^2) F_s(dy) \Big)$$
$$\leq c_{21} (\|f\|^*_{2,2})^2 ((1+|z|)^2 + c_{22})$$
$$\leq c_2 (1+|z|)^2 \cdot (\|f\|^*_{2,2})^2,$$

where in the first equality Fubini's theorem is used, and again c_2, c_{21} and c_{22} are suitable non-negative constants. For the term I_1 we obtain similarly after integrating by ν that it is bounded by

$$\int_{|y|<1} \int_0^1 (1+|z+\vartheta y|)|y| \left(\frac{1}{1+|z+\vartheta y|} \int |f''(x+z+\vartheta y)| d\nu(x) \right) F_s(dy)$$
$$\leq \|f''\|^*_{1,1} \int_{|y|<1} (1+|z|)(1+\vartheta|y|)|y| F_s(dy)$$
$$\leq c_1 \|f''\|^*_{1,1} (1+|z|) \leq c_1 \|f''\|^*_{2,2} (1+|z|)$$

where ϑ is again in $[0,1]$. In summary this yields

$$\int_{\mathbb{R}} |G^J_s f(x+z)|^2 \nu(dx)$$
$$= \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} f(x+z+y) - f(x+z) - f'(x+z) y \mathbb{1}_{\{|y|<1\}} F_s(dy) \Big|^2 \nu(dx)$$
$$\leq (1+|z|) \Big(c_1 \|f''\|^*_{2,2} + (c_2+c_3)(\|f\|^*_{2,2}) \Big) \leq (1+|z|)^2 \cdot \Big(c_1 \|f''\|^*_{2,2} + (c_2+c_3)(\|f\|^*_{2,2}) \Big),$$

which implies

$$\sup_z \frac{1}{1+|z|} \Big(\int_{\mathbb{R}} |G^J_s f(x+z)|^2 \nu(dx) \Big)^{1/2} \leq \Big(c_1 \|f''\|^*_{2,2} + (c_2+c_3)(\|f\|^*_{2,2}) \Big)^{1/2} < \infty.$$

Consequently, we proved $\|G^J_s f\|^*_{2,2} < \infty$, hence, the statement holds true. □

Proposition 2.11 (Infinitesimal generator for PII on $\bar{\mathcal{L}}^2_2(\nu)$)
Let $(L_t)_{t\geq 0}$ be a PII such that $E|L_t|^2 < \infty$ for all $t \in \mathbb{R}_+$. Denote by $T = (T_{s,t})_{s\leq t}$ the corresponding transition operator on $\bar{\mathcal{L}}^2_2(\nu)$, and its infinitesimal generator $A_s f$ defined via equation (2.2) for $f \in \mathcal{D}(A_s)$. Then, $A_s f = G_s f$ on $\bar{\mathcal{W}}^2(\nu)$. In particular $\bar{\mathcal{W}}^2(\nu) \subset \mathcal{D}(A_s)$ for each $s \in \mathbb{R}_+$.

Proof: From Proposition 2.7 we know that $T_{s,t}|_{\bar{\mathcal{L}}^2_{c,2}(\nu)}$ is a strongly continuous ES on the Banach space $\bar{\mathcal{L}}^2_2(\nu)$. Hence, the limit $A_s f = \lim_{h\to 0} \frac{1}{h}(T_{s,s+h} f - f)$ can be understood in the strong sense on $\mathcal{D}(A_s)$. Thus, $A_s : \mathcal{D}(A_s) \to \bar{\mathcal{L}}^2_2(\nu)$. Now, let $f \in \bar{\mathcal{W}}^2(\nu)$ and consider a smooth approximation sequence[1] $f_N \in C^2_c(\mathbb{R}^d), N \in \mathbb{N}$ such that $\partial f_N, \partial^2 f_N \in C^2_b(\mathbb{R}^d)$ with $\|f_N - f\|^*_{2,2} \to 0$. Recall that ν integrates $(1+\|x\|^2)$, thus $f_N \in \bar{\mathcal{L}}^2_2(\nu)$ as well as $\partial f_N, \partial^2 f_N \in \bar{\mathcal{L}}^2_2(\nu)$, or equivalently, $f_N \in \bar{\mathcal{W}}^2(\nu)$. From standard literature[2], e.g.

[1] Utilize standard approximation by a mollifier φ which is smooth up to the boundary.

[2] In Dynkin (1965) the infinitesimal generator is given for $g \in C^2_c(\mathbb{R}^d)$ for time-inhomogeneous Lévy processes and in Jacob (2001) for $g \in C^2_b(\mathbb{R}^d)$ for Markov processes.

Dynkin (1965) and Jacob (2001), we know that $A_s g = G_s g$ for all $g \in C_c^2(\mathbb{R}^d)$, where G_s is the operator as defined in equation (2.19). Consequently, we obtain $A_s f_N = G_s f_N$. The moment condition on $(L_t)_{t\geq 0}$ yields

$$\int_0^t \int_{\{|y|\geq 1\}} |y|^2 F_s(dy)ds < \infty,$$

Lemma 2.10 then implies

$$\|A_s f_N\|_{2,2}^* = \|G_s f_N\|_{2,2}^* \leq \|G_s^D f_N\|_{2,2}^* + \|G_s^J f_N\|_{2,2}^* < \infty.$$

Since $G_s f_N$ and $G_s f \in \bar{\mathcal{L}}_2^2(\nu)$ for all $N \in \mathbb{N}$ it follows by majorization

$$A_s f := \lim_{N\to\infty} A_s f_N = \lim_{N\to\infty} G_s f_N = G_s(\lim_{N\to\infty} f_N) = G_s f.$$

In particular, it holds that $\|A_s f\|_{2,2}^* < \infty$ and, thus, $\bar{\mathcal{W}}^2(\nu) \subset \mathcal{D}(A_s)$ for each $s \in \mathbb{R}_+$. □

Similar results as Lemma 2.10 and Proposition 2.11 hold true under some mild regularity conditions, also for time-inhomogeneous pure diffusion processes.

3 Comparison of Markov processes

The main goal is to prove a general comparison result for time-inhomogeneous Markov process. Hereto, we need a representation result for strongly continuous ES on general Banach spaces. We start by recalling the weak evolution problem.

3.1 Representation result for solutions of the weak evolution problem

The *weak evolution problem* is crucial for the comparison result in the next section. Let $\mathbb{B}$ be a Banach space. For a strongly continuous evolution system $T = (T_{s,t})_{s\leq t}$ on $\mathbb{B}$ with corresponding family of infinitesimal generators $(A_s)_{s>0}$ define $F_t(s) := T_{s,t}f$, $s \leq t$, $f \in \mathcal{D}(A_s)$. Then, by Lemma 2.1 for fixed t, F_t is a solution of the homogeneous *weak evolution equation*

$$\begin{aligned} \frac{d^+u(s)}{ds} &= -A_s u(s) \text{ for } s < t, \\ u(t) &= f. \end{aligned} \tag{3.1}$$

As usual, F_t is called the fundamental solution of (3.1). In the autonomous case where A_s is independent of s, equation (3.1) becomes the homogeneous Cauchy problem.

Fundamental solutions and the Cauchy problem are studied in Friedman (1969) imposing conditions on the resolvent of the generator. In order to deal with inhomogeneous Markov processes we need to study solutions of the evolution problem in the following form which is related to Friedman (1969, Part 2, Chapter 3) (see also Pazy (1983, Chapter 5)). This extension is the foundation of the comparison results in the next section.

For every $r \in [s,t]$, $s,t \in \mathbb{R}_+$ let $A_r : \mathcal{D}(A_r) \subset \mathbb{B} \to \mathbb{B}$ be a linear operator on $\mathbb{B}$ and let $G(r)$ be a $\mathbb{B}$-valued function on $[s,t]$. We consider for a $\mathbb{B}$-valued function u on $[s,t]$, which is right differentiable on (s,t), $u(r) \in \mathcal{D}(A_r)$ for $s < r \leq t$, the *weak initial value problem* or *weak evolution problem*

$$\begin{aligned} \frac{d^+u(r)}{dr} &= -A_r u(r) + G(r) \text{ for } s < r \leq t, \\ u(t) &= f. \end{aligned} \tag{3.2}$$

A $\mathbb{B}$-valued function $u : [s,t] \to \mathbb{B}$ is a *classical solution* of (3.2) if u solves (3.2) and is continuous on $[s,t]$.

Note that the autonomous case of the weak evolution problem is a weakening of the inhomogeneous Cauchy problem. The basic representation result for solutions of the inhomogeneous Cauchy equation in a Banach space was used in Rüschendorf and Wolf (2011) to establish comparison results for homogeneous Markov processes. The following is an extension of this representation result to the weak evolution problem.

Theorem 3.1 (Representation result)
Let $T = (T_{s,t})_{s\le t}$ be a strongly continuous ES on $\mathbb{B}$ with corresponding family of infinitesimal generators $(A_s)_{s\ge 0}$. Further, let F_t, $G : [0,t] \to \mathbb{B}$ for $t \in \mathbb{R}_+$ be functions such that

(1) *the map $r \mapsto T_{s,r}G(r)$ is right continuous.*

(2) *$\int_s^t T_{s,r}G(r)dr$ exists for all $s, t \in \mathbb{R}_+$ with $s \le t$.*

(3) *F_t solves the weak evolution problem (3.2), i.e. for $s, t \in \mathbb{R}_+$ such that $s \le t$, it holds that $r \mapsto F_t(r)$ is continuous on $[s,t]$, right differentiable on (s,t), $F_t(r) \in \mathcal{D}(A_r)$ on $(s,t]$ and*
$$\frac{d^+F_t(s)}{ds} = -A_sF_t(s) + G(s) \text{ for } s \le t. \tag{3.3}$$

Then,
$$F_t(s) = T_{s,t}F_t(t) - \int_s^t T_{s,r}G(r)dr. \tag{3.4}$$

Proof: Let $s, t \in \mathbb{R}_+$ with $s < t$. The right derivatives $\frac{d^+}{dr}T_{s,r}$ and $\frac{d^+}{dr}F_t(r)$ exist due to the assumption on T and F_t. Using (2.6) and (3.3) we obtain
$$\begin{aligned}\frac{d^+}{dr}\Big(T_{s,r}F_t(r)\Big) &= \lim_{h\downarrow 0} T_{s,r+h}\frac{1}{h}\Big(F_t(r+h) - F_t(r)\Big) + \lim_{h\downarrow 0}\frac{1}{h}\Big(T_{s,r+h} - T_{s,r}\Big)F_t(r)\\ &= T_{s,r}\Big(\frac{d^+}{dr}F_t(r)\Big) + \Big(\frac{d^+}{dr}T_{s,r}\Big)F_t(r)\\ &= T_{s,r}\Big(-A_rF_t(r) + G(r)\Big) + T_{s,r}A_rF_t(r)\\ &= T_{s,r}G(r).\end{aligned} \tag{3.5}$$

Thus $\frac{d^+}{dr}\big(T_{s,r}F_t(r)\big)$ is integrable on $[s,t]$ due to assumption 3.1-(2). Since it is right continuous we can integrate (3.5) and obtain
$$\int_s^t T_{s,r}G(r)dr = \int_s^t \frac{d^+}{dr}\big(T_{s,r}F_t(r)\big)dr = T_{s,t}F_t(t) - T_{s,s}F_t(s).$$

Thus, we obtain the representation of the solution of the weak evolution problem
$$F_t(s) = T_{s,t}F_t(t) - \int_s^t T_{s,r}G(r)dr. \qquad \square$$

Remark 3.2 **(a)** *Using the theory of pseudo-differential operators, Böttcher (2008) developed an integral representation result on certain subspaces of C_0 for solutions of an evolution problem related to* (3.2). *See also our Section 5.*

(b) *For a special class of infinitesimal generators of strongly continuous semigroups Pazy (1983, Chapter 5, Theorem 4.2) derived a similar representation formula for evolution systems on Banach spaces.*

3.2 General comparison result for Markov processes

We assume that X and Y are two time-inhomogeneous Markov processes with values in $(\mathbb{E}, \mathscr{E})$ and let $S = (S_{s,t})_{s \le t}$ and $T = (T_{s,t})_{s \le t}$ denote their strongly continuous evolution systems on some Banach function space $\mathbb{B}$ on $\mathbb{E}$. Let by $A = (A_s)_{s \ge 0}$ and $B = (B_s)_{s \ge 0}$ denote the corresponding families of infinitesimal generator of S and T respectively. Further, we assume that for each $s > 0$

$$\mathcal{F} \subset \mathcal{D}(A_s) \cap \mathcal{D}(B_s). \tag{3.6}$$

Theorem 3.3 (Conditional comparison result) *Assume that*

(1) $f \in \mathcal{F}$ *implies* $s \mapsto T_{s,t} f$ *is right-differentiable for* $0 < s < t$,

(2) $f \in \mathcal{F}$ *implies* $S_{s,t} f \in \mathcal{F}$ *for* $0 \le s \le t$, (stochastic monotonicity of S)

(3)
$$A_s f \le B_s f \ [P^{X_0}] \text{ for all } f \in \mathcal{F} \text{ and } s \le t. \tag{3.7}$$

Then

$$S_{s,t} f \le T_{s,t} f \ [P^{X_0}] \text{ for all } f \in \mathcal{F},\ s \le t. \tag{3.8}$$

Proof: Define for $f \in \mathcal{F}$ and $t \in \mathbb{R}_+$ the function $F_t : [0,t] \to \mathbb{B}$ by $F_t(s) := T_{s,t} f - S_{s,t} f$. Then due to Lemma 2.1 F_t satisfies the differential equation

$$\frac{d^+ F_t(s)}{ds} = -B_s T_{s,t} f + A_s S_{s,t} f,\ F_t(t) = 0, \text{ on } [0,t].$$

As consequence we obtain

$$\frac{d^+ F_t(s)}{ds} = -B_s (T_{s,t} f - S_{s,t} f) + \Big(A_s - B_s\Big) S_{s,t} f = -B_s F_t(s) + G(s),$$

where by (3.6) and assumption 3.3-(2) $G(s) := \big(A_s - B_s\big) S_{s,t} f$ is well defined. Further it is non-positive due to assumption (3.7). Moreover we obtain that $\big(-B_s F_t(s)\big)$ is also well defined due to assumptions 3.3-(1), Lemma 2.1 and 3.3-(2). The continuity of S and T transfers to $F_t(s)$. Furthermore, the strong continuity of $T_{s,t}$ and $S_{s,t}$ transfers to the map $r \mapsto T_{s,r} G(r)$, hence, it is right continuous, too. Thus, $F_t(s)$ solves the weak evolution problem. Since $-G(r) \ge 0$ for $s \le r \le t$ it follows that $-\int_s^t T_{s,r} G(r) dr$ exists and is finite for all $s, t \in \mathbb{R}_+$. Thus, the assumptions of Theorem 3.1 are satisfied and imply that the solution $F_t(s)$ has an integral representation of the form

$$F_t(s) = T_{s,t} F_t(t) - \int_s^t T_{s,r} G(r) dr = \int_s^t T_{s,r}\big(-G(r)\big) dr \ge 0,$$

as $F_t(t) = 0$ and $-G(r) \ge 0$ for all $s \le t$. $\square$

Remark 3.4 (a) *The comparison result for homogeneous Markov processes in Rüschendorf and Wolf (2011, Theorem 3.1) is implied by Theorem 3.3 since in the case of homogeneous Markov processes the corresponding semigroups fulfill condition 3.3-(1) (see for example Theorem 2.4 in Pazy (1983)).*

(b) *Let* $\mathcal{F}$ *be any function class in* (1.3)–(1.7). *If the Markov process* X *on* $\mathbb{R}^d$ *has a translation invariant transition function* $(P_{s,t})_{s \le t}$, *then* X *is stochastically monotone w.r.t. to* $\mathcal{F}$ *(see Lemma 2.8 in Bergenthum and Rüschendorf (2007a)) and thus condition 3.3-(2) in Theorem 3.3 holds.*

(c) (Smooth operator) *Let* A_s *be an operator as in Proposition 2.11, then, for transition operators of PII on* $\bar{\mathcal{L}}^2_{c,2}(\nu)$ *resp.* $\bar{\mathcal{L}}^2_2(\nu)$ *the condition 3.3-(1) reduces to show that* T *is a* smooth operator. *For instance, let* $f \in \mathcal{F}^0 = \mathcal{F} \cap \bar{\mathcal{W}}^2(\nu)$ *and let the assumptions of Proposition 2.7 hold true. Then, for a smooth operator it holds that* $T_{s,t} f \in C^2(\mathbb{R}^d)$. *Further, due to Proposition 2.7, it holds that* $T_{s,t} f \in \bar{\mathcal{L}}^2_2(\nu)$. *Since,* ∂f, $\partial^2 f \in \bar{\mathcal{L}}^2_2(\nu)$ *we also have* $T_{s,t}(\partial f)$ *and* $T_{s,t}(\partial^2 f) \in \bar{\mathcal{L}}^2_2(\nu)$. *Hence,* $T_{s,t} f \in \bar{\mathcal{W}}^2(\nu) \subset \mathcal{D}(A_s)$.

4 Applications

4.1 Lévy driven diffusion processes on $C_0(\mathbb{R}^d)$

In the recent financial mathematics literature Lévy driven diffusion processes have found considerable attention as stochastic interest rate models or stochastic volatility models. For a classical survey see Heston (1993). Recent results and developments in this field can be found in Poulsen et al. (2009) and the references therein.

Let $(L_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process with local characteristics (b, σ, F). We consider the Lévy driven diffusion processes defined as solutions of the following stochastic differential equation:

$$\begin{aligned} dX_t &= \Phi(X_{t-}, t) dL_t \\ X_0 &= x,\, x \in \mathbb{R}^d, \end{aligned} \tag{4.1}$$

where $\Phi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ is in $C_{\mathrm{b}}^{1,1}(\mathbb{R}^d \times \mathbb{R}_+)$.

In the present section we will make use of the (probabilistic) symbol. This opens a neat way to derive the structure of the generator of Lévy driven diffusions and, more general, to the class of Feller evolution processes.

Definition 4.1 *Let X be a Markov process with right-continuous paths (a.s) and*

$$T_{s,t}u(x) := E^{x,s}u(X_t) = E(u(X_t)|X_s = x)$$

for $s \leq t$. If $T_{s,t}$ is a strongly continuous ES on the Banach space C_0 of continuous functions vanishing at infinity, then X is called a C_0-Feller evolution process. *The process is called* rich, *if*

$$C_c^\infty \subseteq \mathcal{D}(A_s) \text{ for every } s \geq 0.$$

Let us mention that in some parts of the literature C_b is used instead of C_0. This larger space is then endowed usually with the strict topology (cf. Van Casteren (2011), Section 2.1). Here, it is convenient to use C_0 as reference space. We will see in the proof of Lemma 4.6 below that in the setting (4.1) we obtain a C_0-Feller evolution process for *every* Lévy process. If we used C_b instead, we would either loose the Banach space structure (using the strict topology which is defined via a family of semi-norms) or we would loose strong continuity of the semigroup in most cases (using the sup-norm). The following concept of a probabilistic symbol has proved to be useful in the time-homogeneous case already. We introduce this notion for the time-inhomogeneous case in order to derive path properties of the process. In the present article we will use it to derive the structure of the generator.

Definition 4.2 *Let X be an $\mathbb{R}^d$-valued Markov process, which is conservative and normal, that is $P^x(X_0 = x) = 1$, and having right-continuous paths (a.s.). Fix a starting point x and a starting time $t \geq 0$ and define $\tau = \tau_R^x$ to be the first exit time from the ball of radius $R > 0$ after time t:*

$$\tau := \tau_R^x := \inf\big\{h \geq 0 : \big\|X_{t+h}^x - x\big\| > R\big\}.$$

The function $p : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$p(t, x, \xi) := -\lim_{h\downarrow 0} E^{x,t}\left(\frac{e^{i\langle X_{t+h}^\tau - x, \xi\rangle} - 1}{h}\right) \tag{4.2}$$

is called the (time-dependent) probabilistic symbol *of the process, if the limit exists for every $t \geq 0$ and $x, \xi \in \mathbb{R}^d$ independently of the choice of $R > 0$.*

Let us recall that in the above definition $X^{\tau}_{t+h} = X_{\min\{t+h,\tau\}}$ denotes the process (which is started at time t in x) stopped at time τ.

At first sight it might be a bit surprising that it is possible to demand that the limit does not depend on the choice of R respectively τ. Intuitively speaking, this is due to the fact that we are dealing with right-continuous processes. The symbol describes only the local dynamics of the process and this is not changed by using the stopping time.

We need two auxiliary results in order to prove Theorem 4.5. The first one is the time-inhomogeneous version of Dynkin's formula. It follows from the well-known fact that $(M_h^{[x,t,u]})_{h\geq 0}$ given by

$$M_h^{[x,t,u]} := u(X_{t+h}, t+h) - u(x,t) - \int_t^{t+h} (A_s + \partial_s) u(X_s, s)\, ds$$

is a martingale for every $u \in \bigcap_{t\leq s\leq t+h} \mathcal{D}(A_s) \cap \mathcal{D}(\partial_s)$ with respect to every $P^{x,t}$, $x \in \mathbb{R}^d$, $t \geq 0$. To our knowledge the most general version of this result can be found in (Casteren, 2011, Theorem 2.11). There it is formulated for the C_b case but it can be easily adapted to the C_0 case. For a function u which depends on the time-inhomogeneous process, but not on time itself, we obtain:

Lemma 4.3 *Let X be a Feller evolution process (C_b or C_0) and τ a stopping time. Then we have*

$$E^{x,t}\Big[\int_t^{\tau\wedge(t+h)} A_s u(X_s)\, ds\Big] = E^{x,t}\big[u(X_{\tau\wedge(t+h)})\big] - u(x) \tag{4.3}$$

for all $t > 0$ and $u \in \bigcap_{t\leq s\leq t+h} \mathcal{D}(A_s)$.

In the previous lemma we have used that the class of martingales is stable under stopping. The following result has been established in Schilling and Schnurr (2010, Lemma 2.5).

Lemma 4.4 *Let $K \subset \mathbb{R}^d$ be a compact set. Let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a smooth truncation function, i.e., $\chi \in C_c^\infty(\mathbb{R}^d)$ with*

$$\mathbb{1}_{B_1(0)}(y) \leq \chi(y) \leq \mathbb{1}_{B_2(0)}(y)$$

for $y \in \mathbb{R}^d$. Furthermore we define $\chi_n^x(y) := \chi((y-x)/n)$ and for $\xi \in \mathbb{R}^d$ $u_n^x(y) := \chi_n^x(y) e^{i\langle y,\xi\rangle}$. Then for sufficiently large $c > 0$ we have for all $z \in K$

$$\left|u_n^x(z+y) - u_n^x(z) - \langle y, \nabla u_n^x(z)\rangle \mathbb{1}_{B_1(0)}(y)\right| \leq c \cdot \left(1 \wedge |y|^2\right).$$

Now we are in a position to show that the probabilistic symbol (which is easy to calculate) and the functional analytic symbol (which we will use below) coincide.

Theorem 4.5 *Let $X = (X_t)_{t\geq 0}$ be a conservative càdlàg Feller evolution process such that $C_c^\infty \subset D(A_s)$ ($s \geq 0$). Then the generator $A_s|_{C_c^\infty}$ is a pseudo-differential operator (cf. Section 5 below) with symbol $-q(s,x,\xi)$, that is*

$$A_s f(x) = -(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} q(s,x,\xi) \hat{f}(\xi) d\xi \tag{4.4}$$

where the symbol $q : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ has the following properties for fixed $s \geq 0$:

- *$q(s,\cdot,\cdot)$ is locally bounded in x, ξ.*
- *$q(s,\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^d$.*
- *$q(s,x,\cdot)$ is a continuous negative definite function in the sense of Schoenberg for every $x \in \mathbb{R}^d$.*

Let

$$\tau := \tau_R^x := \inf\{h \geq 0 : \|X_{s+h}^x - x\| > R\}. \tag{4.5}$$

If $x \mapsto q(s,x,\xi)$ is continuous, then we have

$$\lim_{h\downarrow 0} E^{x,s}\left(\frac{e^{i\langle X_{s+h}^\tau - x,\xi\rangle} - 1}{h}\right) = -q(s,x,\xi), \tag{4.6}$$

independently of the choice of R respectively τ, that is, the probabilistic symbol of the process exists and coincides with the symbol of the generator.

Proof: As in the time homogeneous case, it is easily seen, that the operators A_s fulfill the positive maximum principle. Therefore, they are pseudodifferential operators by Courrège (1966, Théorèm 1.5). This is the functional analytic approach; now we complement this with the probabilistic approach. Let $(\chi_n^x)_{n\in\mathbb{N}}$ be the sequence of cut-off functions of Lemma 4.4 and we write $e_\xi(x) := e^{i\langle x,\xi\rangle}$ for $x,\xi \in \mathbb{R}^d$. By the bounded convergence theorem and formula (4.3) we see

$$\begin{aligned} E^{x,t}\left(e^{i\langle X_{t+h}^\tau - x,\xi\rangle} - 1\right) &= \lim_{n\to\infty}\left(E^{x,t}\left[\chi_n^x(X_{t+h}^\tau)e_\xi(X_{t+h}^\tau)e_{-\xi}(x) - 1\right]\right) \\ &= e_{-\xi}(x)\lim_{n\to\infty} E^{x,t}\left(\chi_n^x(X_{t+h}^\tau)e_\xi(X_{t+h}^\tau) - \chi_n^x(x)e_\xi(x)\right) \\ &= e_{-\xi}(x)\lim_{n\to\infty} E^{x,t}\int_t^{\tau\wedge(t+h)} A_s(\chi_n^x e_\xi)(X_s)\,ds \\ &= e_{-\xi}(x)\lim_{n\to\infty} E^{x,t}\int_t^{\tau\wedge(t+h)} A_s(\chi_n^x e_\xi)(X_{s-})\,ds. \end{aligned}$$

The last equality follows since a càdlàg process has a.s. a countable number of jumps and since we are integrating with respect to Lebesgue measure. Using our Lemma 4.4 and writing the operator A_s in integro-differential form, we obtain for all $z \in \overline{B_R(x)}$

$$\begin{aligned} &A_s(\chi_n e_\xi)(z) \\ &\quad\leq c_\chi\Big(|b(s,z)| + \frac{1}{2}\sum_{j,k=1}^d |\sigma^{jk}(s,z)| + \int_{y\neq 0}(1\wedge|y|^2)\,F(s,z,dy)\Big)(1+|\xi|^2) \\ &\quad\leq c_\chi' \sup_{z\in\overline{B_R(x)}}\sup_{|\xi|\leq 1}|p(s,z,\xi)| \end{aligned}$$

where c_χ and c_χ' are positive constants only depending on χ. The last estimate follows with techniques described in the appendix of Schilling and Schnurr (2010). The function $p(t,x,\xi)$ is locally bounded since it is continuous. By definition of the stopping time τ we know that for all $t \leq s \leq \tau\wedge(t+h)$ we have $X_{s-} \in \overline{B_R(x)}$. Therefore, the integrand $A_s(\chi_n^x e_\xi)(X_{s-})$, $t \leq s \leq \tau\wedge(t+h)$ appearing in the above integral is bounded and we may use the dominated convergence theorem again. This yields

$$\begin{aligned} E^{x,t}\left(e^{i\langle X_{t+h}^\tau - x,\xi\rangle} - 1\right) &= e_{-\xi}(x)E^{x,t}\int_t^{\tau\wedge(t+h)}\lim_{n\to\infty} A_s(\chi_n^x e_\xi)(z)|_{z=X_{s-}}\,ds \\ &= -e_{-\xi}(x)E^{x,t}\int_t^{\tau\wedge(t+h)} e_\xi(z)p(s,z,\xi)|_{z=X_{s-}}\,ds. \end{aligned}$$

The second equality follows from classical results due to Courrège (1966, Sections 3.3 and 3.4). Therefore,

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{E^{x,t}\left(e^{i\langle X^\tau_{t+h}-x,\xi\rangle}-1\right)}{h}\\
&\quad= -e_{-\xi}(x)\lim_{h\downarrow 0} E^{x,t}\left(\frac{1}{t}\int_t^{t+h} e_\xi(X^\tau_{s-})p(s,X^\tau_{s-},\xi)1\!\!1_{[[t,\tau[[}(s)\,ds\right)\\
&\quad= -e_{-\xi}(x)\lim_{h\downarrow 0} E^{x,t}\left(\frac{1}{t}\int_t^{t+h} e_\xi(X^\tau_{s})p(s,X^\tau_{s},\xi)1\!\!1_{[[t,\tau[[}(s)\,ds\right)
\end{aligned}$$

since we are integrating with respect to Lebesgue measure. The process X^τ is bounded on the stochastic interval $[[0,\tau[[$ and $(s,x)\mapsto p(s,x,\xi)$ is continuous for every $\xi\in\mathbb{R}^d$. Thus, again by dominated convergence,

$$\lim_{h\downarrow 0} \frac{E^{x,t}\left(e^{i\langle X^\tau_{t+h}-x,\xi\rangle}-1\right)}{h} = -e_{-\xi}(x)e_\xi(x)p(t,x,\xi) = -p(t,x,\xi). \qquad \square$$

In fact, above we have been more general then needed in the context of equation (4.1). For bounded coefficients (and hence a bounded symbol) it is not necessary to introduce a stopping time. In this case the limit without stopping time, that is,

$$-\lim_{h\downarrow 0} E^{x,t}\left(\frac{e^{i\langle X_{t+h}-x,\xi\rangle}-1}{t}\right)$$

would have been sufficient and coincides with $p(t,x,\xi)$ and $q(t,x,\xi)$ above. For future reference, however, we have included the general case. Now we use the above result on Lévy driven diffusions.

Lemma 4.6 *The unique strong solution of the SDE* (4.1) $X^x_t(\omega)$ *has the symbol* $q:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ *given by*

$$q(t,x,\xi)=p(t,x,\xi)=\psi(\Phi^\top(x,t)\xi)$$

where Φ *is the coefficient of the SDE and* ψ *the symbol of the driving Lévy process. Hence, the (family of) generators on* $\mathcal{D}(A_s)\supseteq C_0^2(\mathbb{R}^d)$ *can be written as*

$$\begin{aligned}
A_s f(x) &= \frac{1}{2}\sum_{j,k=1}^d \big(\sigma\Phi(x,s)\big)^{j,k}\frac{\partial^2 f}{\partial x_j\partial x_k}(x) + \sum_{j=1}^d \big(b\Phi(x,s)\big)^j\frac{\partial f}{\partial x_j}(x)\\
&\quad+\int\Big(f(x+\Phi(x,s)y)-f(x)-\sum_{j=1}^d\frac{\partial f}{\partial x_j}(x)\big(\Phi(x,s)\chi_{\mathfrak{c}}(y)\big)^j\Big)F(dy),
\end{aligned}\tag{4.7}$$

where F *is the Lévy measure corresponding to* L *which integrates* $(1\wedge|y|^2)$ *and* $\chi_{\mathfrak{c}}$ *is a cut-off function.*

Proof: It is a well-known fact that the unique solution of a Lévy driven SDE with Lipschitz coefficient is a Markov process (cf. Protter (1977, Theorem 5.9). In order to carry the solution, the stochastic basis on which the Lévy process is defined has to be enlarged in a canonical way (cf. e.g. Protter (2005, Section V.6)). Next, one has to show that the solution is a universal time-inhomogeneous Markov process, that is, the transition function does not depend on the starting point. This can be done similarly to Schnurr (2009, Theorem 2.47). The C_0-Feller property follows as in Schnurr (2009, Theorem 2.49).

The calculation of the symbol works as in the proof of Schilling and Schnurr (2010, Theorem 3.1). We sketch the proof here in the case $d=n=1$ in order to emphasize analogies and differences: let τ be the stopping time given by (4.5). Fix $x, \xi \in \mathbb{R}$. We apply Itô's formula for general semimartingales to the function $e_\xi(\cdot - x) = \exp(i(\cdot - x)\xi)$:

$$\begin{aligned}
&\frac{1}{h} E^{x,t}\big(e^{i(X^\tau_{t+h}-x)\xi} - 1\big) \\
&\qquad = \frac{1}{h} E^{x,t}\Bigg(\int_{t+}^{t+h} i\xi\, e^{i(X^\tau_{s-}-x)\xi}\, dX^\tau_s - \frac{1}{2}\int_{t+}^{t+h} \xi^2\, e^{i(X^\tau_{s-}-x)\xi}\, d[X^\tau, X^\tau]^c_s \\
&\qquad\quad + e^{-ix\xi} \sum_{t<s\le t+h} \Big(e^{iX^\tau_s\xi} - e^{iX^\tau_{s-}\xi} - i\xi e^{iX^\tau_{s-}\xi}\Delta X^\tau_s \Big) \Bigg).
\end{aligned} \tag{4.8}$$

For the first term we get

$$\begin{aligned}
&\frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi\, e^{i(X^\tau_{s-}-x)\xi} \Big)\, dX^\tau_s \\
&= \frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi\, e^{i(X^\tau_{s-}-x)\xi} \Big)\, d\left(\int_0^s \Phi(X_{r-}, r-)\mathbb{1}_{[[0,\tau]]}(\cdot, r)\, dL_r \right) \\
&= \frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi\, e^{i(X^\tau_{s-}-x)\xi} \Phi(X_{s-}, s-)\mathbb{1}_{[[0,\tau]]}(\cdot, s) \Big)\, dL_s \\
&= \frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi\, e^{i(X^\tau_{s-}-x)\xi} \Phi(X_{s-}, s-)\mathbb{1}_{[[0,\tau]]}(\cdot, s) \Big)\, d(bs) \qquad (4.9) \\
&\quad + \frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi\, e^{i(X^\tau_{s-}-x)\xi} \Phi(X_{s-}, s-)\mathbb{1}_{[[0,\tau]]}(\cdot, s) \Big)\, d\left(\sum_{0<r\le s} \Delta L_r \mathbb{1}_{\{|\Delta L_r|\ge 1\}} \right)
\end{aligned}$$

where we have used the well-known Lévy–Itô decomposition of the driving process. Since the integrand is bounded, the martingale terms of the Lévy process yield again martingales whose expected value is zero.

For the drift part (4.9) we obtain

$$\frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi \cdot e^{i(X^\tau_{s-}-x)\xi} \Phi(X_{s-}, s-)\mathbb{1}_{[[0,\tau[[}(\cdot, s) b \Big)\, ds \xrightarrow{h\downarrow 0} i\xi b \Phi(x,t),$$

since

$$\begin{aligned}
&\frac{1}{h} E^{x,t} \int_{t+}^{t+h} \Big(i\xi \cdot e^{i(X^\tau_{s-}-x)\xi} \Phi(X_{s-}, s-)\mathbb{1}_{[[0,\tau[[}(\cdot, s) b \Big)\, ds \\
&\qquad = i\xi b \cdot E^{x,t} \int_{t}^{t+1} \Big(e^{i(X^\tau_{sh}-x)\xi} \Phi(X_{sh}, sh)\mathbb{1}_{[[0,\tau[[}(\cdot, sh) \Big)\, ds
\end{aligned}$$

The other parts work alike. In the end we obtain

$$\begin{aligned}
p(t,x,\xi) &= -ib(\Phi(x,t)\xi) + \frac{1}{2}(\Phi(x,t)\xi)\sigma(\Phi(x,t)\xi) \\
&\quad - \int_{y\ne 0} \Big(e^{i(\Phi(x,t)\xi)y} - 1 - i(\Phi(x,t)\xi)y \cdot \mathbb{1}_{\{|y|<1\}}(y) \Big)\, F(dy)
\end{aligned}$$

$$= \psi(\Phi(x,t)\xi). \tag{4.10}$$

Note that in the multi-dimensional case the matrix $\Phi(x,t)$ has to be transposed, that is, the symbol of the solution is $\psi(\Phi^\top(x,t)\xi)$. We already know that the probabilistic and the classical symbol coincide. Hence, some Fourier analysis gives us the structure of the generator. $\square$

Remark 4.1 *We need the boundedness of Φ in order to derive the Feller property. The calculation of the symbol works even for unbounded coefficients (e.g. Lipschitz continuous Φ).*

Now we can prove the comparison result for Lévy driven diffusion processes. We consider $\mathcal{F}$ to be a subset of $C_0^2(\mathbb{R}^d)$ and $(L_t^{(i)})_{t\geq 0}$, $i = 1,2$ to be a Lévy process with local characteristics $(b^{(i)}, \sigma^{(i)}, F^{(i)})$. Further, let X and Y be Lévy driven diffusions defined by the stochastic differential equation (4.1) associated to $(\Phi^{(1)}, L^{(1)})$ and $(\Phi^{(2)}, L^{(2)})$ with corresponding infinitesimal generator A_s and B_s respectively. Denote their evolution systems by $S = (S_{s,t})_{t\geq 0}$ and $T = (T_{s,t})_{t\geq 0}$, respectively.

Corollary 4.7 (Comparison of time-inhomogeneous Lévy driven diffusion processes)
Assume that $X_0 \stackrel{d}{=} Y_0$ and

(1) *$f \in \mathcal{F}$ implies $s \mapsto T_{s,t}f$ is right-differentiable for $0 < s < t$,*

(2) *X is stochastically monotone w.r.t. $\mathcal{F}$ i.e. $f \in \mathcal{F}$ implies $S_{s,t}f \in \mathcal{F}$ for all $s \leq t$, and*

(3) $$A_s f \leq B_s f \ [P^{X_0}] \quad \textit{for all } f \in \mathcal{F} \textit{ and } s \leq t. \tag{4.11}$$

Then
$$X_t \leq_{\mathcal{F}} Y_t, \quad t \geq 0. \tag{4.12}$$

Proof: We have seen above already that for every choice of $\Phi^{(i)}$ and every Lévy process $L^{(i)}$ the solution of (4.1) is a C_0-Feller evolution process. Since $(C_0, \|\cdot\|_\infty)$ is a Banach space we can directly use Theorem 3.3 and obtain $S_{s,t}f \leq T_{s,t}f$ for all $f \in \mathcal{F}$, $s \leq t$. By using $X_0 \stackrel{d}{=} Y_0$ this yields for $f \in \mathcal{F}$ and $t \geq 0$

$$\begin{aligned} Ef(X_t) &= E_0[E[f(X_t)|X_0 = x]] \\ &= E_0 S_{0,t} f(x) \\ &\leq E_0 T_{0,t} f(x) \\ &= E_0[E[f(Y_t)|Y_0 = x]] = Ef(Y_t), \end{aligned}$$

where E_0 is the expectation with respect to P^{X_0}. $\square$

Corollary 4.7 shows that ordering results for Lévy driven diffusion processes for $\mathcal{F} \subset C_0^2$ are implied by a stochastic monotonicity assumption of one process and comparability of the local characteristics. In Section 4.3 we consider Lévy driven diffusion processes as inhomogeneous Markov processes on $\bar{\mathcal{L}}_2^2(\nu)$ in order to derive comparison results w.r.t. unbounded function classes. Some further applications of these ordering results derived in a similar way are listed in the following remark.

4.2 Processes with independent increments

Let $(L_t)_{t\geq 0}$ be a PII such that $E|L_t|^2 < \infty$ for all $t \in \mathbb{R}_+$. Its characteristic function is given by $Ee^{i\langle \xi, L_t\rangle} = \exp(\int_0^t \theta_u(i\xi)du)$ with cumulant function given in (2.18). We consider the case of PII with Lévy measures $F_s^{(i)}, i = 1,2$ which integrates $(|y| \wedge |y|^2)$. We choose the cut-off function χ_{C} as the identity. Denote for $s \geq 0$ the local characteristics of $L^{(i)}$ by $(b_s, \sigma_s^{(i)}, F_s^{(i)})$. The corresponding infinitesimal generators are A_s and B_s

respectively. If the assumptions of Proposition 2.7 are fulfilled the family $(T_{s,t})_{s\le t}$ of operators defined in (2.11) is a strongly continuous ES on $\bar{\mathcal{L}}_2^2(\nu)$. The corresponding infinitesimal generator $(A_s)_{s\ge 0}$ is given in (2.19) defined on $\mathcal{D}(A_s)$. For this instance we state a comparison result for the convex order $\le_{\mathrm{cx}}$ which is generated by

$$\mathcal{F}_{\mathrm{cx}}^0 = \mathcal{F}_{\mathrm{cx}} \cap B_{\mathrm{b}} \cap \bar{\mathcal{W}}^2(\nu),$$

where B_{b} is the function class of b-bounded function and b is chosen as in Remark 2.8. Similar results hold for $\le_{\mathrm{dcx}}, \le_{\mathrm{sm}}, \le_{\mathrm{st}}, \le_{\mathrm{ism}}$ as well.

Corollary 4.8 (Comparison of PII w.r.t. $\mathcal{F}_{\mathrm{cx}}$)
Assume $L_r^{(1)} \stackrel{d}{=} L_r^{(2)}$ for some $r \le t$. If $(S_{s,t})$ and $(T_{s,t})$ are smooth operators and if $(S_{s,t})$ leaves B_b invariant, then the comparison of the local characteristics

$$\sigma_s^{(1)} \le_{psd} \sigma_s^{(2)} \tag{4.13}$$

$$\int\limits_{\mathbb{R}^d} f(x) F_s^{(1)}(dx) \le \int\limits_{\mathbb{R}^d} f(x) F_s^{(2)}(dx) \tag{4.14}$$

for all $s \le t$ and all $f \in \mathcal{F}_{cx}$ with $f(0) = 0$ such that the integrals exist, implies that

$$L_t^{(1)} \le_{cx} L_t^{(2)},\ t \ge 0. \tag{4.15}$$

Proof: Let $f \in \mathcal{F}_{\mathrm{cx}}^0$. Since $(S_{s,t})$ leaves B_{b} invariant we obtain by Remark 3.4-(c) and the smoothness of $(S_{s,t})$ that $S_{s,t}f \in \bar{\mathcal{W}}^2(\nu)$. The invariance in $\mathcal{F}_{\mathrm{cx}}$ follows by Remark 3.4-(b). Similarly the smoothness of $T_{s,t}$ and Proposition 2.11 ensure that $T_{s,t}f \in \mathcal{D}(B_s)$ for all $s \ge 0$ (see Remark 3.4-(c)). We have choosen above the cut-off function $\chi_{\mathtt{C}}$ as the identity, since the Levy measures fulfill $\int_{\mathbb{R}^d} (|y| \wedge |y|^2) F_s^{(i)}(dy) < \infty$, $i = 1, 2$. Thus the jump part has the form

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} f(x+y) - f(x) - \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x)(y)^{(j)} F_s^{(i)}(dy) \\
&= \int\limits_{\mathbb{R}^d} f(x+y) - f(x) - \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x)(y)^{(j)} \mathbb{1}_{\{|y|<1\}} F_s^{(i)}(dy) \\
&\quad - \int\limits_{\mathbb{R}^d} \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x)(y)^{(j)} \mathbb{1}_{\{|y|\ge 1\}} F_s^{(i)}(dy).
\end{aligned}$$

From this representation we note that both terms belong to $\bar{\mathcal{L}}_2^2(\nu)$ which can be seen by using similar arguments as in the proof of Lemma 2.10. Thus, the infinitesimal generator has the form as in (2.19). Hence, for $f \in \mathcal{F}_{\mathrm{cx}}^0$ and $s \le t$ we obtain

$$\begin{aligned}
(B_s - A_s)f(x) &= \frac{1}{2} \sum_{j,k=1}^d \frac{\partial^2 f}{\partial x_j \partial x_k}(x) \Big({\sigma_s^{(2)}}^{j,k} - {\sigma_s^{(1)}}^{j,k} \Big) \\
&\quad + \int\limits_{\mathbb{R}^d} \Big(f(x+y) - f(x) - \sum_{j=1}^d \frac{\partial f}{\partial x_j}(x)(y)^j \Big) \Big(F_s^{(2)} - F_s^{(1)} \Big)(dy).
\end{aligned} \tag{4.16}$$

Due to the positive semidefiniteness of $({\sigma_s^{(2)}}^{j,k} - {\sigma_s^{(1)}}^{j,k})_{j,k\le d}$ for fixed $s \ge 0$, its spectral decomposition is given by $\big(\sum_{i\le d} \lambda_i \mathrm{e}_i^j \mathrm{e}_i^k\big)$, where the eigenvalues λ_i are non-negative and $\mathrm{e} = (\mathrm{e}_i^1, \ldots, \mathrm{e}_i^d)$ denote the eigenvectors. Convexity of f implies

$$\frac{1}{2} \sum_{i=1}^d \lambda_i \sum_{j,k=1}^d \frac{\partial^2 f}{\partial x_j \partial x_k}(x) \mathrm{e}_i^j \mathrm{e}_i^k \ge 0.$$

Moreover, the function $y \mapsto f(x+y) - f(x) - \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(x) y^{(j)}$ is convex, since it is a sum of a convex function and a linear function. Now using the comparison of the Lévy measures in (4.14) we get $(B_s - A_s)f(x) \geq 0$. Thus, the assumptions of Theorem 3.3 are satisfied and we obtain

$$Ef(L_t^{(1)}) = E_r E\big(f(L_t^{(1)})|L_r^{(1)} = x\big) \leq E_r E\big(f(L_t^{(2)})|L_r^{(2)} = x\big) = Ef(L_t^{(2)}),$$

where E_r is the expectation with respect to $P^{L_r^{(1)}}$. □

Remark 4.9 *The following gives for some function classes comparison conditions of the local characteristics which imply stochastic ordering of the Lévy driven diffusion processes in the case of Lévy measures, $F^{(i)}$, $i = 1, 2$ which integrate $(|y| \wedge |y|^2)$. For $d \times d$ matrices A, B with real entries, the positive semidefinite order $A \leq_{psd} B$ is defined by $x^\top (B - A)x \geq 0$ for all $x \in \mathbb{R}^d$, where $x^\top$ is the transpose of x. Here, we make use of the explicit representation of the generator given in (4.7).*

If $\Phi^{(i)} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}_+^{d\times d}$, $i = 1, 2$ and $\Phi^{(1)} \leq_{psd} \Phi^{(2)}$, then sufficient conditions for (4.11) are:

Ordering	**Drift**	**Diffusion**	**Jump**
$\mathcal{F}_{st}$	$b^{(1)} \leq b^{(2)}$	$\sigma^{(1)} = \sigma^{(2)}$	$F^{(1)} = F^{(2)}$
$\mathcal{F}_{sm}$	$b^{(1)} = b^{(2)}$	$\sigma^{(1)} \leq_d \sigma^{(2)}$	$F^{(1)} \leq_{sm} F^{(2)}$
$\mathcal{F}_{ism}$	$b^{(1)} \leq b^{(2)}$	$\sigma^{(1)} \leq_d \sigma^{(2)}$	$F^{(1)} \leq_{sm} F^{(2)}$

where $\sigma^{(1)} \leq_d \sigma^{(2)}$ means pointwise ordering $\sigma^{(1)} \leq \sigma^{(2)}$, ${\sigma^{(1)}}^{j,j} = {\sigma^{(2)}}^{j,j}$, $j \leq d$. For every generating function class, the latter fact can be justified as follows: Let, for instance, $f \in \mathcal{F}_{st}$. Then, the first order derivatives of f are non-negative. Moreover, let $b^{(1)} \leq b^{(2)}\, \sigma^{(1)} = \sigma^{(2)}$ and $F^{(1)} = F^{(2)}$, hence the corresponding generators in (4.7) are ordered, too. Then, the comparison result follows from Corollary 4.10.

4.3 Lévy driven diffusion processes on $\bar{\mathcal{L}}_2^2(\nu)$

We reconsider Lévy driven diffusion processes as inhomogeneous Markov processes. In comparison to Corollary 4.7 where these processes were considered as strongly continuous evolution systems on $C_0(\mathbb{R}^d)$ we now consider them as strongly continuous evolution systems on $\bar{\mathcal{L}}_2^2(\nu)$ assuming for both processes the kernel assumption (K) as well as (C) and (D). We state a comparison result for Lévy driven diffusion w.r.t. $\leq_{\mathcal{F}^0}$, where $\mathcal{F}$ is from (1.3) - (1.7) and

$$\mathcal{F}^0 = \mathcal{F} \cap B_{\mathrm{b}} \cap \bar{\mathcal{W}}^2(\nu).$$

Now let $(L_t^{(i)})_{t\geq 0}$, $i = 1, 2$ be a Lévy process such that $E|L_t^{(i)}|^2 < \infty$, $t > 0$, with local characteristics $(b^{(i)}, \sigma^{(i)}, F^{(i)})$ such that $F^{(i)}$, $i = 1, 2$ integrates $(|y| \wedge |y|^2)$. Further, let X and Y be Lévy driven diffusions defined by the stochastic differential equation (4.1) associated to $(\Phi^{(1)}, L^{(1)})$ and $(\Phi^{(2)}, L^{(2)})$ respectively. Denote their infinitesimal generator by A_s and B_s respectively defined in (4.7) and their evolution systems on $\bar{\mathcal{L}}_2^2(\nu)$ by $S = (S_{s,t})_{t\geq 0}$ and $T = (T_{s,t})_{t\geq 0}$.

Extending the ES of X and Y onto $\bar{\mathcal{L}}_2^2(\nu)$ by Proposition 2.9 it is clear that the form and the mapping behaviour of the associated infinitesimal generators (see equation (4.7)) for $f \in \bar{\mathcal{W}}^2(\nu)$ is deduced similarly as in Lemma 2.10 and Proposition 2.11.

Corollary 4.10 (Comparison of Lévy driven diffusion processes)
Assume that X and Y possess the transition kernels $(P_t^X)_{t\geq 0}$ and $(P_t^Y)_{t\geq 0}$, which satisfy (K), (C) and (D). If $X_0 \overset{d}{=} Y_0$ and

(1) *$f \in \mathcal{F}$ implies $s \mapsto T_{s,t} f$ is right-differentiable for $0 < s < t$,*

(2) *X is stochastically monotone w.r.t. $\mathcal{F}^0$, and*

(3) $$A_s f \leq B_s f \quad a.s.\ for\ all\ f \in \mathcal{F}\ and\ s \leq t, \tag{4.17}$$

then

$$X_t \leq_{\mathcal{F}} Y_t,\, t \geq 0. \tag{4.18}$$

5 Pseudo-differential operators on Sobolev Slobodeckii spaces

Finally, we make some remarks on a method to compare general Markov processes based on *Sobolev Slobodeckii* spaces and the theory of pseudo-differential operators. We relate this method to strongly continuous evolution systems as used in this paper. The main reference for this final section is Böttcher (2008) who gave a representation result for pseudo-differential operators on Sobolev Slobodeckii spaces similar as in Theorem 3.1. His result is based on developments on pseudo-differential operators in Eskin (1981), Hoh (1998) and Jacob (2001, 2002).

For $r \in \mathbb{R}$ the *Sobolev-Slobodeckii* space is defined as

$$\mathcal{H}^r(\mathbb{R}^d) := \left\{u \in \mathcal{S}'(\mathbb{R}^d) \middle| \hat{u} \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^d) \text{ s.t. } \|u\|_r < \infty\right\} \tag{5.1}$$

with

$$\|u\|_r^2 = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2(1+|\xi|)^{2r} d\xi, \tag{5.2}$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the dual space of the *Schwartz* space $\mathcal{S}(\mathbb{R}^d)$ and $\hat{u}$ denotes the Fourier transform of u. The space $\mathcal{H}^r(\mathbb{R}^d)$ endowed with the scalar product $\langle\cdot,\cdot\rangle_r$,

$$\langle u, v\rangle_r := \int \hat{u}(\xi)\overline{\hat{v}(\xi)}(1+|\xi|)^{2r} d\xi,\ u, v \in \mathcal{H}^r(\mathbb{R}^d)$$

is a separable Hilbert space (see Eskin (1981, Theorem 4.1)). Moreover for $r \geq 0$ $(\mathcal{H}^r(\mathbb{R}^d), \langle\cdot,\cdot\rangle_r)$ is isomorph to

$$\left\{u \in \mathcal{L}^2(\mathbb{R}^d, \mathbb{C}) \middle| \int |\hat{u}(\xi)|^2(1+|\xi|)^{2r} d\xi < \infty\right\}$$

endowed with the same scalar product. The space $\mathcal{H}^r(\mathbb{R}^d)$ is an interesting example of a Banach function class which uses regularity conditions on the elements but allows unbounded functions.

Elements u of $\mathcal{H}^r(\mathbb{R}^d)$ fulfill certain integrability and regularity assumptions since the existence of higher moments of $\hat{u}$ in (5.2) corresponds to higher order regularity of the function. For instance, $u \in \mathcal{L}^1(\mathbb{R})$ is continuously differentiable if the first moment of $\hat{u}$ is integrable, that is if $\int |\xi||\hat{u}(\xi)| d\xi < \infty$, and in this case we have

$$\begin{aligned}\frac{\partial}{\partial x} u(x) &= \frac{\partial}{\partial x}\frac{1}{(2\pi)^{1/2}}\int e^{-ix\xi}\hat{u}(\xi)d\xi \\ &= \frac{1}{(2\pi)^{1/2}}\int (-i)e^{-ix\xi}\xi\hat{u}(\xi)d\xi.\end{aligned}$$

A pseudo-differential operator $q(t, x, D)$ is defined on a suitable space by

$$q(t, x, D)f(x) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} q(t, x, \xi)\hat{f}(\xi)d\xi \tag{5.3}$$

if the right-hand side exists. The function q which defines the operator is called the *symbol* of the pseudo-differential operator.

In Böttcher (2008, Theorem 2.3) a representation result for solutions of evolution equations on $\mathcal{H}^r(\mathbb{R}^d)$ is established. For an application of this result it has to be checked that the transition functions $T_{s,t}$ and the infinitesimal generator A_s (corresponding to a time-inhomogeneous Markov process) are pseudo-differential operators with symbols in suitable 'symbol classes', i.e. the symbols have sufficient regularity and growth behaviour. Under this condition on the symbol it is verified in Theorem 1.8 in Böttcher (2008) that the transition function $T_{s,t}$ maps $\mathcal{H}^r(\mathbb{R}^d)$ into itself. This mapping property is based on general theory of pseudo-differential operators as described in Eskin (1981), Hoh (1998) and Jacob (2001, 2002).

Our representation result in Theorem 3.1 can be applied to this context if the transition operator $T_{s,t}$ is a strongly continuous ES on $\mathcal{H}^r(\mathbb{R}^d)$, that is, it maps $\mathcal{H}^r(\mathbb{R}^d)$ into $\mathcal{H}^r(\mathbb{R}^d)$. In other words, it is sufficient to prove an analogous result to Proposition 2.4 for $\mathcal{H}^r(\mathbb{R}^d)$.

We can apply our representation Theorem 3.1 and get as corollary a result related to Böttchers representation result without any growth and regularity conditions on the symbol of the infinitesimal generator.

Corollary 5.1 *Under the assumptions of Theorem 3.1, let $(T_{s,t})_{s\leq t}$ be a strongly continuous ES on $\mathcal{H}^r(\mathbb{R}^d)$, then the solutions F_t of the weak evolution problem (3.2) on $\mathcal{H}^r(\mathbb{R}^d)$ have the representation*

$$F_t(s) = T_{s,t}F_t(t) - \int_s^t T_{s,r}G(r)dr. \tag{5.4}$$

For the application of Corollary 5.1 first note that any transition operator $(T_{s,t})$ of a Markov process $(X_t)_{t\geq 0}$ has a representation as a pseudo-differential operator.

For the proof let $f \in \mathcal{H}^r(\mathbb{R}^d)$, $r \geq 0$ and consider a time-inhomogeneous Markov process $X = (X_t)_{t\geq 0}$ with transition function $(P_{s,t})_{s\leq t}$. Denote by $F(f)$ the Fourier transform of f and by F^{-1} the inverse Fourier transform. Then

$$\begin{aligned} T_{s,t}f(x) &= E_{s,x}\Big(F^{-1}\big(F(f)\big)(X_t)\Big) \\ &= E_{s,x}\bigg(\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\langle X_t,\xi\rangle}F(f)(\xi)d\xi\bigg) \\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}q_s(t,x,\xi)\hat{f}(\xi)d\xi, \end{aligned} \tag{5.5}$$

where $q_s(t,x,\xi) := E\big(e^{-i\langle X_t - x,\xi\rangle}|X_s = x\big)$. Thus $(T_{s,t})$ is a pseudo-differential operator with symbol $q_s(t,x,\xi)$.

For applications of Corollary 5.1 to comparison results we have to establish that $T_{s,t} : \mathcal{H}^r(\mathbb{R}^d) \to \mathcal{H}^r(\mathbb{R}^d)$. This can be done in the case of PII, which are particularly suitable for the pseudo-differential operator approach.

Example 5.1 (Transition functions of PII on $\mathcal{H}^r(\mathbb{R}^d)$)
Let $X = (X_t)_{t\geq 0}$ be PII with characteristic function $E\big(e^{-i\langle X_t - X_s,\xi\rangle}\big) = \exp(\int_s^t \theta_u(-i\xi)du) =: \hat{\mu}_{s,t}(-\xi)$. Then from the representation in equation (5.5) we obtain for $f \in \mathcal{H}^r(\mathbb{R}^d)$, that

$$T_{s,t}f(x) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle}\hat{\mu}_{s,t}(-\xi)\hat{f}(\xi)d\xi. \tag{5.6}$$

Hence the characteristic function $\hat{\mu}_{s,t}(-\xi)$ is the symbol of $T_{s,t}$. It is not difficult to see that the transition operator of a PII is a pseudo-differential operator with bounded C^∞-symbol, if all absolute moments of the Lévy measure exists. Thus the symbol lies in the symbol class S_0^0. In consequence the conditions on the symbol for the mapping property $T_{s,t} : \mathcal{H}^r(\mathbb{R}^d) \to \mathcal{H}^r(\mathbb{R}^d)$ in Theorem 1.8 of Böttcher (2008) are fulfilled and Corollary 5.1 gives the necessary representation result. result Corollary 4.8.

As consequence of Corollary 5.1 the comparison theorem (Theorem 3.3) allows us to state a comparison result for PII for function classes $\mathcal{F} \subset \mathcal{H}^r(\mathbb{R}^d)$. Note however that by this approach, based on $\mathcal{H}^r(\mathbb{R}^d)$, we *need the strong condition of existence of all absolute moments of the Lévy measure, while our comparison result based on modified $\bar{\mathcal{L}}_2^2(\nu)$ spaces as in Corollary 4.8 is established under much weaker conditions.*

For general Markov processes with symbol $q_s(t,x,\xi)$ dependent on x it is not easy to check that its symbol has suitable regularity and growth behaviour. Hence this method is difficult to apply for time-inhomogeneous Markov processes as for example for time-inhomogeneous diffusions. Thus it seems that the approach in this paper based on our representation result for strongly continuous evolution system in general Banach spaces is more flexible and easier to apply in examples.

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