PROPAGATION OF CHAOS AND CONTRACTION OF STOCHASTIC MAPPINGS*

S. T. Rachev
University of California
Santa Barbara

L. Rüschendorf
University of Freiburg

Abstract

In this paper we use contraction properties of stochastic mappings with respect to suitable chosen metrics in order to establish some new examples of propagation of chaos. In particular systems of SDE's with mean field type interactions and the corresponding nonlinear SDE's of Mc Kean-Vlasov type for the limiting cases are considered. We also study the rate of convergence to the limit. Assumptions on the smoothness and growth properties of the coefficients of the SDE's are to be reflected in the choice of the probabiliy metric in order to obtain contraction properties. This allows us in particular to investigate some new kinds of interactions as well as to consider systems with weakened Lipschitz assumptions.

1 Introduction

The idea of propagation of chaos due to Kac was to study the relation between simple markovian models of interacting particles and nonlinear Boltzmann type equations. For a detailed introduction to the propagation theory we refer to Sznitman (1989). A formal definition is the following. Let (u_N) be a sequence of symmetric probability measures on E^N, E a separable metric space, and let u be a probability on E, then (u_N) is called <u>u-chaotic</u>, if $\pi_k u_n \xrightarrow{w} u^{(k)}$, π_k the k-marginal distribution, $u^{(k)}$ the k-fold product, and $\xrightarrow{w}$ denotes the weak convergence.

A basic example for chaotic sequences is *McKean's Interacting Diffusion* (cf. the laboratory example in Sznitman (1989), p. 172), cf. for this and related examples also [16], [2], [13], [14], [15], [5], [6]. Consider a system of interacting diffusions:

$$\begin{aligned} dX_t^{i,N} &= dW_t^i + \frac{1}{N}\sum_{j=1}^N b(X_t^{i,N}, X_t^{j,N})dt,\ i = 1, \dots, N \qquad (1.1) \\ X_o^{i,N} &= x_o^i, \end{aligned}$$

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where W^i are independent Brownian motions and b satisfies a bounded Lipschitz condition. Let u_N denote the distribution of $(X^{1,N}, \dots, X^{N,N})$. The nonlinear limiting equation is given by the Mc Kean-Vlasov equation

$$dX_t = dB_t + \int b(X_t, y) u_t(dy) dt, \tag{1.2}$$

B_t a Brownian motion, u_t the distribution of X_t. Then u_N is u-chaotic, where u is the distribution of X on $C(\mathbb{R}_+, \mathbb{R}^d)$.

An alternative example of chaotic behavior of particles, not described by SDE's, are uniform distributions on p-spheres. Let u_N denote the uniform distribution on the p-sphere of radius N in $\mathbb{R}_+^N$ i.e. on $S_{p,N} := \{x \in \mathbb{R}_+^N;\ \Sigma x_i^p = N\}$ and let u denote the probability measure on $\mathbb{R}_+$ with density $f_p(x) = \frac{p^{1-1/p}}{\Gamma(1/p)} e^{-x^p/p},\ x \geq 0$. Then for $N > k + p$, and k and N big enough

$$\|\pi_k u_N - u^{(k)}\| \leq \frac{2(k+p)+1}{N-k-p}, \tag{1.3}$$

where $\| \ \|$ denotes the total variation distance (cf. [9]). In particular we obtain that u_N is u-chaotic. This example has its origin in Poincare's theorem on the asymptotic behaviour of particle systems. More general examples of this kind have been developed in statistical physics in connection with the "equivalence of ensembles" in many papers but typically without a quantitative estimate as in (1.3).

The main goal of this paper is the study of the propagation of chaos of several modifications of McKean's example concerning the form of the interaction and the regularity assumptions on the coefficients. To this end we introduce suitable probability metrics allowing to derive contraction properties of the stochastic equations defined by the corresponding Liouville type equatons. Dobrushin (1979) introduced the use of the Kantorovich metric for the interacting diffusions in example (1.1), (1.2). The success of this metric is based on a coupling argument inherent in its definition. This metric has been applied since then in several other papers (cf. [14], [15], [5], [6]). For our modifications of this example we shall need some variants of the Kantorovich metric giving the suitable regularity and ideality properties for the equations considered. In particular we need metrics which are of higher order ideal when relaxing the Lipschitz conditions in equations (1.1), (1.2). Our modifications of the form of interactions allow to treat much more complicated forms of interactions as in McKean's example. In particular we consider nonlinear interactions via some general energy function as e.g. the p-norm of the vector of all pair interactions. We also consider interactions with the other particles over the whole past (history) of the process, describing some non-Markovian systems. We demonstrate the flexibility of the approach based on suitable probability metrics to cope also with nonstandard forms of interactions and develop in some examples the tools to analyse these models indicating the applicability of this method also to more complicated real physical systems.

2 Equations With p-Norm Interacting Drifts

Consider the system of N interacting diffusions with p-th norm interacting drifts, i.e. the drift is given by the p-th norm of the vector of all pair interactions, which can be considered as driving force in the system.

$$\begin{aligned} dX_t^{i,N} &= dW_t^i + \{\frac{1}{N}\sum_{j=1}^N b^p(X_t^{i,N}, X_t^{j,N})\}^{1/p} dt \\ X_o^{i,N} &= X_o^i,\ 1 \le i \le N, \end{aligned} \tag{2.1}$$

$b \ge 0,\ p \ge 1$. $((W_t^i), X_o^i)$ are independent identically distributed for all i.) We shall establish that each $X^{i,N}$ has a natural limit $\bar{X}^i$, where $(\bar{X}^i)$ are independent copies of the solutions of a nonlinear equation

$$\begin{cases} dX_t = dB_t + (\int b(X_t, y)^p\, u_t(dy))^{1/p} dt \\ X_{t=0} = X \end{cases} \tag{2.2}$$

with $B \stackrel{d}{=} W^1$ a process on C_T, $u_t = P^{X_t}$. In order to obtain the necessary contraction properties of these equations we consider the L_p^* resp. the minimal L_p^*-metric ℓ_p^* defined for processes X, Y (resp. probability measures $m_1, m_2 \in M^1(C_T)$, here and in the following $M^1(C_T)$ denotes the class of all probability distributions on C_T, by

$$L_{p,t}^*(X,Y) = (E \sup_{s \le t} |X_s - Y_s|^p)^{1/p} \tag{2.3}$$

and

$$\ell_{p,t}^*(m_1, m_2) = \inf\{L_{p,t}^*(X,Y);\ X \stackrel{d}{=} m_1,\ Y \stackrel{d}{=} m_2\}. \tag{2.4}$$

In (2.4) we tacitly assumed that the probability space is rich enough to support all possible couplings of m_1, m_2, which is true, for example, in case of atomless spaces. Define for $m_o \in M^1(C_T)$

$$M_p(C_T, m_o) := \{m_1 \in M^1(C_T);\ \ell_{p,T}^*(m_o, m_1) < \infty\}, \tag{2.5}$$

$\mathcal{X}_p(C_T, m_o)$ the class of processes on C_T with distribution $m \in M_p(C_T, m_o)$.

For $m_o = \delta_a$ the one-point measure in $a \in C_T$, this is the class of all distributions on C_T with finite p-th moment of the norm. For $m \in M_p(C_T, m_o)$ consider the Liouville type equation corresponding to (2.2)

$$X_t = B_t + \int_0^t (\int_{C_T} b(X_s, y_s)^p\, dm(y))^{1/p} ds, \tag{2.6}$$

where y_s is the value of y at time s. Let (B_t) be a real valued process on $C_T = C[0,T]$ with finite p-th absolute moment ($E \sup_{s \le T} |B_s|^p < \infty$) and let $b \ge 0$ be a Lipschitz function in x

$$|b(x_1, y) - b(x_2, y)| \le c|x_1 - x_2|. \tag{2.7}$$

A strong solution of the SDE (2.6) means as usual a solution measurable w.r.t. the augmented filtration of the process (B_t); in constrast a weak solution of (2.6) is a solution on a suitable filtered space in distribution.

Lemma 2.1 *Assume (2.7) and let* $\int(\int b(0,y_s)^p\, dm(y))^{1/p} ds < \infty$, *then:*

(a) Equation (2.6) has a unique strong solution X.

(b) If $\Phi(m)$ *is the law of* X, *then* $\Phi(m) \in M_p(C_T, m_o)$, *that is,* $\Phi : M_p(C_T, m_o) \to M_p(C_T, m_o)$.

Proof: Let $X \in \mathcal{X}_p(C_T, m_o)$ and define

$$(SX)_t := B_t + \int_0^t (\int b(X_s, y_s)^p\, dm(y))^{1/p} ds.$$

Then for $Y \in \mathcal{X}_p(C_T, m_o)$

$$\begin{aligned}
|(SX)_t - (SY)_t| &\leq \int_0^t ds |(\int b(X_s, y_s)^p\, dm(y))^{1/p} - (\int b(Y_s, y)^p\, dm(y))^{1/p}| \\
&\leq \int_0^t (\int |b(X_s, y_s) - b(Y_s, y_s)|^p\, dm(y))^{1/p}\, ds \\
&\leq c \int_0^t |X_s - Y_s| ds.
\end{aligned}$$

This implies

$$\sup_{s \leq t} |(SX)_s - (SY)_s| \leq c \int_0^t \sup_{u \leq s} |X_u - Y_u| ds$$

and, furthermore,

$$\begin{aligned}
L^*_{p,t}(SX, SY) &= (E \sup_{s \leq t} |(SX)_s - (SY)_s|^p)^{1/p} \\
&\leq c(E(\int_0^t \sup_{u \leq s} |X_u - Y_u| ds)^p)^{1/p} \\
&\leq c \int_0^t L^*_{p,s}(X, Y) ds.
\end{aligned}$$

Define inductively, $X^0 := B,\ X^n := SX^{n-1}$, then by iteration

$$L^*_{p,T}(X^n, X^{n-1}) \leq c^n \frac{T^n}{n!} L^*_{p,T}(X^1, X^0).$$

Since

$$\begin{aligned}
L^*_{p,T}(X^1, X^0) &\leq c' \int_0^T [E|B_s|^p + \int b(0, y_s)^p\, dm(y)]^{1/p}\, ds \\
&\leq c' \int_0^T (E \sup_{u \leq s} |B_u|^p)^{1/p}\, ds + c' \int_0^T (\int b(0, y_s)^p\, dm(y))^{1/p}\, ds \\
&\leq c' T (E \sup_{s \leq T} |B_s|^p)^{1/p} + c' \int_0^T (\int b(0, y_s)^p\, dm(y))^{1/p}\, ds,
\end{aligned}$$

we obtain from the assumptions on B and b that $L^*_{p,T}(X^1, X^0) < \infty$. Consequently, $\sum_{n=1}^{\infty} L^*_{p,T}(X^n, X^{n-1}) \leq e^{cT} L^*_{p,T}(X^1, X^0) < \infty$ by the Gronwall Lemma. This results in $\sum_{n=1}^{\infty} \sup_{s \leq T} |X^n_s - X^{n-1}_s| < \infty$ a.s. and, therefore, X^n converges to some process X a.s., uniformly on bounded intervals. X is a.s. continuous, has finite p-th moments (i.e. $\|X\|^*_{p,t} := E \sup_{s \leq t} |X_s|^p < \infty$) and is a fixed point of S. So, $\Phi(m) = P^X \in M_p(C_T, m_o)$; this holds as $\|B\|^*_{p,T} < \infty$ and $L^*_{p,T}(X, B) < \infty$. □

In addition suppose that b is Lipschitz in both arguments,

$$|b(x_1, y_1) - b(x_2, y_2)| \leq c[|x_1 - y_1| + |x_2 - y_2|] \tag{2.8}$$

and consider the map $\Phi : M_p(C_T, m_o) \to M_p(C_T, m_o)$.

Lemma 2.2 *(Contraction of Φ w.r.t. $\ell^*_{p,t}$)*
Under (2.8) and the assumptions of Lemma 2.1, for $t \leq T$ and $m_1, m_2 \in M_p(C_T, m_o)$, it holds:

$$\ell^*_{p,t}(\Phi(m_1), \Phi(m_2)) \leq c e^{ct} \int_0^t \ell^*_{p,u}(m_1, m_2) du. \tag{2.9}$$

Proof: Let for $i = 1, 2$ and $t \leq T$

$$X^{(i)}_t = B_t + \int_0^t \Big(\int_{C_T} b(X^{(i)}_s, y_s)^p \, dm_i(y) \Big)^{1/p} ds$$

and let $m \in M^1(m_1, m_2)$, the class of probability measures on $C_T \times C_T$ with marginals m_1, m_2. Then

$$\begin{aligned}
\sup_{s \leq t} |X^{(1)}_s - X^{(2)}_s| &\leq \int_0^t ds \Big| \Big[\int_{C_T} b(X^{(1)}_s, y^{(1)}_s)^p \, dm_1(y^{(1)}) \Big]^{1/p} \\
&\qquad - \Big[\int_{C_T} b(X^{(2)}_s, y^{(2)}_s)^p \, dm_2(y^{(2)}) \Big]^{1/p} \Big| \\
&\leq \int_0^t ds \Big[\int_{C_T \times C_T} |b(X^{(1)}_s, y^{(1)}_s) - b(X^{(2)}_s, y^{(2)}_s)|^p \, dm(y^{(1)}, y^{(2)}) \Big]^{1/p} \\
&\leq c \int_0^t ds \Big\{ |X^{(1)}_s - X^{(2)}_s| + \Big[\int |y^{(1)}_s - y^{(2)}_s|^p \, dm(y^{(1)}, y^{(2)}) \Big]^{1/p} \Big\}.
\end{aligned}$$

Minimizing the RHS with respect to the coupling m we obtain

$$\sup_{s \leq t} |X^{(1)}_s - X^{(2)}_s| \leq c \int_0^t ds \sup_{u \leq s} |X^{(1)}_u - X^{(2)}_u| + c \int_0^t ds \ell^*_{p,s}(m_1, m_2). \tag{2.10}$$

Consequently, by Gronwall's lemma,

$$\sup_{s \leq t} |X^{(1)}_s - X^{(2)}_s| \leq c \, e^{ct} \int_0^t \ell^*_{p,s}(m_1, m_2) ds, \tag{2.11}$$

which proves the lemma, passing to the p-th norm in the LHS of (2.11), and then to $\ell^*_{p,t}$. □

Theorem 2.3 *Under (2.8) and* $\int_0^T (\int b(0,y_s)^p \, dm_o(y))^{1/p} \, ds < \infty$*, equation (2.2) has a unique weak and strong solution in* $\mathcal{X}_p(C_T, m_o)$.

Proof: From Lemma 2.2 we obtain for $m \in M_p(C_T, m_o)$

$$\begin{aligned} \ell^*_{p,T}(\Phi^{k+1}(m), \Phi^k(m)) &\leq c_T^k \frac{T^k}{k!} \ell^*_{p,T}(\Phi(m), m) \ \ (c_T = c\, e^{cT}) \\ &\leq c_T^k \frac{T^k}{k!} (\ell^*_{p,T}(\Phi(m), m_o) + \ell^*_{p,T}(m, m_o)) < \infty. \end{aligned}$$

Consequently, $(\Phi^k(m))$ is a Cauchy-sequence in $(C_T, \ell^*_{p,T})$ and converges to a fixed point of Φ. Let X^{k+1}, X^k denotes the couplings of $\Phi^{k+1}(m)$, $\Phi^k(m)$ corresponding to the iteration, then by (2.9) we have that $L^*_{p,T}(X^{k+1}, X^k) \leq c_T^k \frac{T^k}{k!} \ell^*_{p,T}(\Phi(m), m)$ and, therefore, we get a unique strong solution with finite p-th moment. □

Remark 1.

(a) While the Liouville equation in Lemma 2.1 can be handled with the L_1-metric, in Lemma 2.2 we only obtain a contraction w.r.t. the minimal ℓ_p-metric $\ell^*_{p,T}$ (cf. equation (2.9) in this respect).

(b) The result of Theorem 2.3 can be extended to the case $p = \infty$, using the metric

$$L^*_{\infty,T}(X,Y) = \operatorname{ess\,sup} \sup_{s \leq T} |X_s - Y_s| \tag{2.12}$$

and the corresponding minimal metric

$$\ell^*_{\infty,T}(m_1, m_2) = \inf\{L^*_{\infty,T}(X,Y);\ X \overset{d}{=} m_1, Y \overset{d}{=} m_2\}. \tag{2.13}$$

Again the equation

$$X_t = B_t + \int_0^t \operatorname*{ess\,sup}_{u_s(dy)} b(X_s, y) ds \tag{2.14}$$

has a unique solution in $M_\infty(C_T, m_o)$ if B is a.s. bounded, i.e. $\operatorname{ess\,sup}_{s \leq T} |B_s| < \infty$.

(c) Several extensions of equation (2.2) can be handled in a similar way, as for example

$$X_t = B_t + \int_0^t (\int b(X_s, y)^p \, u_s^{(k)}(dy))^{1/p} \, ds, \tag{2.15}$$

where $u_s^{(k)}) = \bigotimes_{i=1}^k P^{X_s}$ stands for the k-fold product of u_s and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$. More generally $b = b(s,x,y)$ could depend upon s and the past of the process $y = (y_u)_{u \leq s}$. In this case u_s has to be replaced by $u_{(s)} := P^{(X_u)_{u \leq s}}$ the distribution of the past, and we have to assume a functional Lipschitz condition on b. In a similar way we can handle also the d-dimensional case. □

Based on Theorem 2.3 we next investigate the system of interacting equations in (2.1). The following theorem asserts that, as N goes to infinity, each $X^{i,N}$ has a natural limit $\bar{X}^i$. $(\bar{X}^i)$ are independent copies of the solutions of the nonlinear equation (2.2).

Theorem 2.4 *Let b satisfy the Lipschitz condition (2.8) and suppose that $\int |b(\bar{X}_s^1, y_s)|^{2p}\, u_s(dy_s) < \infty$, a.s.; then*

$$\begin{aligned} &\sup_N \sqrt{N}\, E^{1/p} \sup_{t\le T} |X_t^{i,N} - \bar{X}_t^i|^p < \infty \quad \textit{for} \quad p \ge 2 \quad \textit{and} \\ &N^{p-1}\, E \sup_{t\le T} |X_t^{i,N} - \bar{X}_t^i|^p = o(1) \quad \textit{for} \quad 1 \le p \le 2. \end{aligned} \tag{2.16}$$

Proof: For notational convenience we drop the superscript N; then

$$\begin{aligned} X_t^i - \bar{X}_t^i &= \int_0^t (\{\frac{1}{N}\sum_{j=1}^N b(X_s^i, X_s^j)^p\}^{1/p} - \{\int b(\bar{X}_s^i, y)^p\, u_s(dy)\}^{1/p}) ds \\ &= \int_0^t ds\{[(\frac{1}{N}\sum_j b(X_s^i, X_s^j)^p)^{1/p} - (\frac{1}{N}\sum_j b(\bar{X}_s^i, X_s^j)^p]^{1/p} \\ &\quad +[(\frac{1}{N}\sum_j b(\bar{X}_s^i, X_s^j)^p)^{1/p} - (\frac{1}{N}\sum_j b^p(\bar{X}_s^i, \bar{X}_s^j))^{1/p}] \\ &\quad +[(\frac{1}{N}\sum_j b(\bar{X}_s^i, \bar{X}_s^j)^p)^{1/p} - (\int b(\bar{X}_s^i, y)^p\, u_s(dy)^{1/p}]\}. \end{aligned}$$

By the Minkowski inequality and the Lipschitz condition on b, the above equality implies ($|X|_T := \sup_{s\le T} |X_s|$)

$$\begin{aligned} \|X^i - \bar{X}^i\|_{T,p} : &= (E|X^i - \bar{X}^i|_T^p)^{1/p} \\ &\le \int_0^T ds\{c\|X_s^i - \bar{X}_s^i\|_p + c\frac{1}{N}\sum_{j=1}^N \|X_s^j - \bar{X}_s^j\|_p \\ &\quad +(E|(\frac{1}{N}\sum_j b(\bar{X}_s^i, \bar{X}_s^j)^p)^{1/p} - (\int b(\bar{X}_s^i, y)^p\, u_s(dy))^{1/p}|^p)^{1/p}\}. \end{aligned}$$

Summing up over i and using the symmetry, we find

$$\begin{aligned} &N\|X^1 - \bar{X}^1\|_{T,p} = \sum_{i=1}^N \|X^i - \bar{X}^i\|_{T,p} \\ \le\ & 2c\int_0^T ds\{\sum_{i=1}^N \|X_s^i - \bar{X}_s^i\|_p + \sum_{i=1}^N E|(\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j)^p)^{1/p} \\ &-(\int b(\bar{X}_s^i, y)^p\, u_s(dy))^{1/p}|^p\}. \end{aligned}$$

This amounts to

$$\|X^i - \bar{X}^i\|_{T,p} \le 2c \int_0^T ds\{\|X^i - \bar{X}^i\|_{s,p}$$
$$+ \quad \frac{1}{N}\sum_{i=1}^N [E|(\frac{1}{N}\sum_j b(\bar{X}_s^i, \bar{X}_s^j)^p)^{1/p} - (\int b(\bar{X}_s^i, y)^p \, u_s(dy))^{1/p}|^p\},$$

and, consequently, by the Gronwall lemma

$$\begin{aligned}\|X^i - \bar{X}^i\|_{T,p} &\le 2c\, e^{2cT}\int_0^T ds \frac{1}{N}\sum_{i=1}^N [E|(\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j)^p)^{1/p} \\ &\quad -(\int b(\bar{X}_s^i, y)^p \, u_s(dy))^{1/p}|^p] \\ &= 2c\, e^{2cT}\int_0^T ds\, E|(\frac{1}{N}\sum_j b(\bar{X}_s^1, \bar{X}_s^j)^p)^{1/p} \\ &\quad -(\int b(\bar{X}_s^1, y)^p \, u_s(dy))^{1/p}|^p.\end{aligned}$$

By Taylor-expansion and with $Y_j := b(\bar{X}_s^i, \bar{X}_s^j)^p$ (conditionally on $\bar{X}_s^i$) we obtain

$$E|(\frac{S_N}{N} + a)^{1/p} - a^{1/p}|^p \le \frac{1}{p^p} a^{1-p} E|\frac{S_N}{N}|^p, \tag{2.17}$$

where $S_N = \sum(Y_j - a)$, $a = EY_j > 0$. Therefore, from the Marcinkiewicz-Zygmund inequality (cf. Chow-Teicher, p. 357) we conclude

$$\sqrt{N} E^{1/p}|(\frac{S_N}{N} + a)^{1/p} - a^{1/p}|^p \le \text{ const. } E^{1/p}|\frac{S_N}{\sqrt{N}}|^p = 0(1),$$

which yields (2.16) for $p \ge 2$. For $1 \le p < 2$ the claim follows from the moment bounds of Pyke and Root giving $E|\frac{S_N}{N^{1/p}}|^p = 0(1)$ and, therefore,

$$N^{p-1} E|(\frac{S_N}{N} + a)^{1/p} - a^{1/p}|^p = o(1). \tag{2.18}$$

□

We next interprete Theorem 2.4 as a chaotic property of the diffusions governed by (2.1). Recall that by Proposition 2.2 in Sznitman (1989) a sequence (u_N) of symmetric probability measures on $E^{(N)}$ is u-chaotic, $u \in M^1(E)$, if for $(X_1, \ldots, X_N)$ distributed as u_N, it holds

$$\frac{1}{N}\sum_{i=1}^N \delta_{X_i} \xrightarrow{w} u. \tag{2.19}$$

For $\bar{X}_N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ we obtain from Theorem 2.4

$$\bar{X}_N \xrightarrow{w} \bar{X}, \tag{2.20}$$

where $\bar{X}$ is the solution of equation (2.2). Therefore, with m denoting the law of $\bar{X}$ and m_N denoting the law of $(X^{1,N}, \ldots, X^{N,N})$ we obtain from (2.19)

Corollary 2.5 *Under the assumptions of Theorems 2.3 and 2.4,* (m_N) *is* m*-chaotic.*

Remark 2.

(a) For $p = \infty$ (see (2.14)) the propagation of chaos property does not hold. Also the case $0 < p < 1$ does not lead to propagation of chaos and there does not exist a unique strong solution of

$$X_t = B_t + \int_0^t \int b(X_s, y_s)^p \, dm(y) ds. \tag{2.21}$$

(b) An example leading to Burger's type equation
In our example

$$dX_t^i = dW_t^i + (\frac{1}{N} \sum_{j=1}^N b(X_t^i, X_t^j)^p)^{1/p} \, dt, \ i = 1, \dots, N, \tag{2.22}$$

with $b(\cdot, \cdot)$ Lipschitz, the instantaneous drift term seen by particle i, is

$$\Delta_i = (\frac{1}{N} \sum_{j=1}^N b(X_t^i, X_t^j)^p)^{1/p}.$$

Under the assumptions of Theorem 2.4,

$$\lim_{N \to \infty} E[\frac{1}{N} \sum_{i=1}^N (\Delta_i^p - (\int b^p(X_t^i, y) u_t(dy))^{1/p})^2] = 0,$$

as well as

$$\lim_{N \to \infty} E[\frac{1}{N} \sum_{i=1}^N (\Delta_i^p - \int b^p(X_t^i, y) u_t(dy))]^2 = 0.$$

Similar to the above limit relation we would like to examine the average behavior of the "pseudo drift" $\frac{1}{N} \sum_{i=1}^n Z_i^p$. Here, $Z_i^p := \frac{1}{N-1} \sum_{j \neq i} \Phi_{N,a}^p (X_t^i - X_t^j)$, and $\Phi_{N,a}(x - y) = N^{ad/p} \Phi(N^a(x - y))$, where $\Phi(\cdot) \geq 0$ is smooth, compactly supported on $\mathbb{R}^d$ and $\int \Phi(x) dx = 1$ (we consider the vector valued case here). Note that

$$\begin{aligned} Z_1^p &= \frac{1}{N-1} \sum_{j=2}^N \Phi_{N,a}^p (X_t^1 - X_t^j) \\ &= \frac{1}{N-1} \sum_{j=1}^N N^{ad} \Phi^p (N^a (X_t^i - X_t^j)), \end{aligned}$$

and consequently,

$$\begin{aligned} EZ_1^p &= N^{ad} (E\Phi^p (N^a (X_t^1 - X_t^2))) \\ &= (\frac{N^{ad} E\Phi^p (N^a (X_t^1 - X_t^2))}{\int \Phi^p}) \int \Phi^p \\ &\underset{N \to \infty}{\longrightarrow} \|u_t\|_{L^2}^2 \int \Phi^p =: u_{t,p}(X_t). \end{aligned}$$

Consider

$$
\begin{aligned}
a_n &= E[\frac{1}{N}\sum_{i=1}^{N}(Z_i^p - u_{t,p}(X_t^1))]^2 \\
&= E[\frac{1}{N}\sum_{i=1}^{N}(\frac{1}{N-1}\sum_{j\neq i}\Phi_{N,a}^p(X_t^i - X_t^j)) - u_{t,p}(X_t^1))]^2 \\
&= E[\frac{1}{N-1}\sum_{j=1}^{N}\Phi_{N,a}^p(X_t^1 - X_t^j)) - u_{t,p}(X_t^1)]^2.
\end{aligned}
$$

Arguing as Sznitman (1989), p. 196, we find that

$$
a_N \to \begin{cases} 0 & \text{if } 0 < a < \frac{1}{d} \\ (\int \Phi^{2p}\,dx)\|u_t\|_{L^2}^2 \frac{(\int \Phi^p)^2}{\int \Phi^{2p}} & \text{if } a = \frac{1}{d} \\ \infty & \text{if } a > \frac{1}{d} \end{cases} .
$$

Therefore, only in the case of moderate interaction we obtain Burger's equation in the limit. □

3 A Random Number of Particles

Let $(W^i)_{i\in\mathbb{N}}$ be an iid system of real-valued processes (as in (2.1)) with p-th moments and let $(N_n)_{n\geq 1}$ be an iid integer valued sequence of r.v.'s independent of (W^i). Consider the following system of SDE with a random number of particles and interactions:

$$
dX_t^{i,n} = dW_t^i + (\frac{1}{n}\sum_{j=1}^{N_n} b(X_t^{i,n}, X_t^{j,n})^p)^{1/p}\,dt,\ i = 1,\ldots,N_n. \tag{3.1}
$$

We assume the asymptotic stability condition

$$
\frac{N_n}{n} \to Y \ \text{ a.s. as } \ n\to\infty. \tag{3.2}
$$

As in Section 2 it turns out that $X^{i,N}$ have a natural limit $\bar{X}^i$, a solution of the following nonlinear SDE

$$
dX_t = dB_t + Y^{1/p}(\int b(X_t, y)^p\, u_t(dy))^{1/p}, \tag{3.3}
$$

where $B \stackrel{d}{=} W^1$ and Y is assumed to be independent of B.

For $m_o \in M^1(C_T)$ let $M_p(C_T, m_o)$, $L_{p,T}^*$, $\ell_{p,T}^*$ be defined as in Section 2.

Lemma 3.1 *Let $\int_0^t |B_s|ds < \infty$ a.s., then for any $m \in M_p(C_T, m_o)$ there exists a unique strong solution of the equation*

$$
X_t = B_t + Y^{1/p}\int_0^t (\int_{C_T} b(X_s, y_s)^p\, dm(y))^{1/p}\, ds. \tag{3.4}
$$

Proof: With $(SX)_t := Y^{1/p} \int_0^t (\int_{C_T} b(X_s, y_s)^p \, dm(y))^{1/p} \, ds$ we obtain as in the proof of Lemma 2.1

$$\sup_{s \le t} |(SX)_s - (SY)_s| \le cY^{1/p} \int_0^t \sup_{0 \le u \le s} |X_u - Y_u| du.$$

Defining inductively $X^0 := B, \; X^n := SX^{n-1}$, we have

$$\begin{aligned}
\sup_{s \le t} |X_s^n - X_s^{n-1}| &\le c^n Y^{n/p} \frac{t^n}{n!} \sup_{s \le t} |X_s^1 - X_s^0| \\
&\le c^n Y^{(n+1)/p} \frac{t^n}{n!} [\int_0^t |B_s \; ds + \int_0^t (\int |y_s|^p \, dm(y))^{1/p} \, ds] < \infty.
\end{aligned}$$

This implies the existence of a unique strong solution. □

Let for $m \in M_p(C_T, m_o)$, $\Phi(m)$ denote the distribution of the solution of (3.4), then

Lemma 3.2 *Let $A_p := \|cY^{1/p} \, e^{cY^{1/p}}\|_p < \infty$, then for $t \le T, \; m_1, m_2 \in M_p(C_T, m_o)$:*

$$\ell_{p,t}^*(\Phi(m_1), \Phi(m_2)) \le A_p \int_0^t \ell_{p,s}^*(m_1, m_2) ds. \tag{3.5}$$

Proof: Let $X^{(i)}$ be solutions of

$$X_t^{(i)} = B_t + Y^{1/p} \int_0^t (\int_{C_T} b(X_s^{(i)}, y_s)^p \, dm_i(y))^{1/p} ds,$$

then as in the proof of Lemma 2.2

$$\sup_{s \le t} |X_s^{(1)} - X_s^{(2)}| \le cY^{1/p} \int_0^t \sup_{u \le s} |X_u^{(1)} - X_u^{(2)}| + c \int_0^t ds \, \ell_{p,s}(m_1, m_2).$$

By Gronwall's lemma

$$\sup_{s \le t} |X_s^1 - X_s^2| \le cY^{1/p} \, e^{cY^{1/p}} \int_0^t \ell_{p,s}(m_1, m_2) ds$$

implying that

$$\ell_{p,t}^*(\Phi(m_1), \Phi(m_2)) \le \|cY^{1/p} \, e^{cY^{1/p}}\|_p \int_0^t \ell_{p,s}^*(m_1, m_2) ds.$$

□

From Lemmas 3.1 and 3.2 we conclude that (3.3) has a unique solution. The proof is similar to that of Theorem 2.3.

Theorem 3.3 *Under the assumptions of Lemmas 3.1 and 3.2, equation (3.3) has a unique solution if*

$$\|B\|_{p,T}^* < \infty \quad \textit{and} \quad \int (\int b(0, y_s)^p \, dm_o(y))^{1/p} \, ds < \infty.$$

4 p-th Mean Interaction in Time; A Non-Markovian Case

Let $(X_t^{i,N})_{i=1,\ldots,N}$ describe a system of N particles and let $b(X_s^{i,N},\cdot) := (b(X_s^{i,N},X_s^{j,N}))_{1\le i\le N}$ denote the interaction vector. While in Section 2 we considered in equation (2.1) a drift of the form $\|b(X_s^{i,N},\cdot)\|_p$ – the p-th norm of the interaction vector – in this section we study mean interactions in time.

Let

$$F_i(s) := |\frac{1}{N}\sum_{j=1}^N b(X_s^{i,N},X_s^{j,N})| \tag{4.1}$$

be the average of the interaction vector and consider the equations:

$$\begin{aligned} X_t^{i,N} &= W_t^i + (\int_0^t |F_i(s)|^p\, ds)^{1/p} \\ X_o^{i,N} &= X_o^i,\ 1\le i\le N, \ \text{ for } \ 1\le p<\infty; \end{aligned} \tag{4.2}$$

$$\begin{aligned} X_t^{i,N} &= W_t^i + \operatorname*{ess\,sup}_{s\le t,\lambda\!\!\lambda} |F_i(s)| \\ X_o^{i,N} &= X_o^i,\ 1\le i\le N, \ \text{ for } \ p=\infty; \end{aligned} \tag{4.3}$$

$$\begin{aligned} X_t^{i,N} &= W_t^i + \int_0^t |F_i(s)|^p\, ds \\ X_o^{i,N} &= X_o^i\ 1\le i\le N, \ \text{ for } \ 0<p<1, \end{aligned} \tag{4.4}$$

i.e. we consider a drift, resulting from the p-th mean in time of the average of the interaction vector. It is clear from the definition that this describes a system which no longer behaves as a Markovian one but the instantaneous drift $|F_i(t)|^p$ is weighted by the mean interaction $\frac{1}{p}(\int_0^t |F_i(s)|^p\, ds)^{1/p-1}$ over the whole past of the process. From this point of view the propagation of chaos property seems to be not so obvious in this case.

First we consider the case $\underline{1\le p<\infty.}$ The nonlinear limiting equation is given by

$$X_t = B_t + (\int_0^t |\int b(X_s,y)u_s(dy)|^p\, ds)^{1/p},\ u_s = P^{X_s}, \tag{4.5}$$

where $X_t,\ B_t,\ b$ are real-valued, B_t is a process in $C_T = C[0,T]$ and

$$|b(x_1,y)-b(x_2,y)| \le c|x_1-x_2| \quad \text{for some} \quad c>0. \tag{4.6}$$

Define for $m_o\in M^1(\ell_T)$

$$M_p(C_T,m_o) = \{m_1\in M^1(C_T) : \ell^*_{p,t}(m_1,m_o)<\infty\}. \tag{4.7}$$

For $m\in M_p(C_T,m_o)$ consider the Liouville type equation

$$X_t = B_t + (\int_0^t |\int_{C_T} b(X_s,y_s)dm(y)|^p\, ds)^{1/p}, \tag{4.8}$$

where y_s is the value of y at time s.

Lemma 4.1 *Assume (4.6) and let*

$$\int_0^T |\int_{C_T} b(0, y_s) m_s(dy)|^p \, ds < \infty,$$

m_s the distribution at time s under m. Then

(a) Equation (4.8) has a unique strong solution X.

(b) If $\Phi(m)$ is the law of X, then $\Phi(m) \in M_p(C_T, m_o)$, that is $\Phi : M_p(C_T, m_o) \to M_p(C_T, m_o)$.

Proof: Let $X \in \mathcal{X}_p(C_T, m_o)$ and define

$$(SX)_t := B_t + (\int_0^t |\int b(X_s, y) m_s(dy)|^p \, ds)^{1/p}. \tag{4.9}$$

Then

$$\begin{aligned} |(SX)_t - (SY)_t|^p &= |(\int_0^t |\int b(X_s, y) m_s(dy)|^p \, ds)^{1/p} \\ &- (\int_0^t |\int b(Y_s, y) m_s(dy)|^p \, ds)^{1/p})^p \\ &\le (\int_0^t |\int (b(X_s, y) - b(Y_s, y)) m_s(dy)|^p \, ds)^{1/p} \\ &\quad \text{(by the Minkowski inequality)} \\ &\le \int_0^t c^p |X_s - Y_s|^p \, ds \quad \text{(by the Lipschitz condition (4.6))}. \end{aligned}$$

This implies

$$\sup_{s \le t} |(SX)_s - (SY)_s|^p \le c^p \int_0^t \sup_{u \le s} |X_u - Y_u|^p \, ds, \tag{4.10}$$

and, furthermore,

$$L^{*p}_{p,t}(SX, SY) \le c^p \int_0^t L^{*p}_{p,s}(X, Y) ds.$$

Define, inductively,

$$X^0 := B, \ X^n := SX^{n-1},$$

then

$$L^{*p}_{p,t}(X^n, X^{n-1}) \le c^{pn} \frac{T^n}{n!} L^{*p}_{p,T}(X^1, X^0).$$

By (4.6), the integral $\int_{C_T} b(X_s, y_s) m(dy)$ is a Lipschitz function of X_s and

$$\begin{aligned} L^{*p}_{p,T}(X^1, X^0) &= E \sup_{t \le T} \int_0^t |\int b(B_s, y) m_s(dy)|^p \, ds \\ &\le E \int_0^T (\int (|b(0, y)| + c|B_s|) m_s(dy))^p \, ds \\ &\le c' \int_0^T (\int |b(0, y)| m_s(dy))^p \, ds \\ &\quad + c' E \int_0^T |B_s|^p \, ds < \infty \end{aligned}$$

as by the assumptions the integrals on the RHS are finite. Therefore,

$$\sum_{n\geq 1} L^*_{p,T}(X^n, X^{n-1}) \leq \sum_{n\geq 1} c^n (\frac{T^n}{n!})^{1/p} L^*_{p,T}(X^1, X^0) < \infty.$$

This implies $\sum_{n\geq 1} L^*_{p,T}(X^n, X^{n-1}) < \infty$. Then

$$\sum_{n\geq 1} L^*_{1,T}(X^n, X^{n-1}) < \infty.$$

In consequence X^n converges to some process X a.s. uniformly on bounded intervals. X is a.s. continuous, and $E\sup_{s\leq t}|X_s|^p < \infty$, since $E\sup_{s\leq t}|B_s|^p < \infty$; so $\Phi(m) \in M_p(C_T, m_o)$. □

In addition, suppose that b is Lipschitz in both arguments,

$$|b(x_1, y_1) - b(x_2, y_2)| \leq c[|x_1 - x_2| + |y_1 - y_2|] \tag{4.11}$$

and consider the map $\Phi : M_p(X_T, m_o) \to M_p(C_T, m_o)$.

Lemma 4.2 *(Contraction of Φ w.r.t. $\ell^*_{p,t}$)*
Under (4.11) and the assumption of Lemma 4.1, for $t < T$ and $m_1, m_2 \in M_p(C_T, m_o)$, it holds:

$$\ell^*_{p,t}(\Phi(m_1), \Phi(m_2)) \leq c_p\, e^{c_p t} \int_0^t \ell^{*p}_{p,s}(m_1, m_2) ds, \tag{4.12}$$

where $c_p := c\, 2^{p-1}$.

Proof: Let for $i = 1, 2$, and $t \leq T$,

$$X^{(i)}_t = B_t(\int_0^t |\int_{C_T} b(X^{(i)}_s, y_s) dm_i(y)|^p\, ds)^{1/p}$$

and let $m \in M^1(m_1, m_2)$, the class of probabilities on $C_T \times C_T$ with marginals m_1 and m_2. Then

$$\begin{aligned}
\sup_{s\leq t} |X^{(1)}_s - X^{(2)}_s|^p &= |(\int_0^t |\int_{C_T} b(X^{(1)}_s, y^{(1)}_s) dm_1(y^{(1)})|^p\, ds)^{1/p} \\
&\quad - |\int_0^t |\int_{C_T} b(X^{(2)}_s, y^{(2)}_s) dm_2(y^{(2)})|^p\, ds)^{1/p}|^p \\
&\leq \int_0^t ds[\int_{C_T\times C_T} |b(X^{(1)}_s, y^{(1)}_s) - b(X^{(2)}_s, y^{(2)}_s)| dm(y^{(1)}, y^{(2)})]^p \\
&\leq \int_0^t ds[c|X^{(1)}_s - X^{(2)}_s| + \int |y^{(1)}_s - y^{(2)}_s| dm(y^{(1)}, y^{(2)})]^p.
\end{aligned}$$

Minimizing the RHS w.r.t. all couplings we get

$$\sup_{s\leq t} |X^{(1)}_s - X^{(2)}_s|^p \leq \underbrace{c\cdot 2^{p-1}}_{=:c_p} \int_0^t ds \sup_{u\leq s} |X^{(1)}_u - X^{(2)}_u|^p + \underbrace{c\cdot 2^{p-1}}_{=:c_p} \int_0^t ds \ell^{*p}_{1,s}(m_1, m_2).$$

Consequently, for $p \geq 1$ by the Gronwall lemma as $\ell^*_{1,s} \leq \ell^*_{p,s}$

$$\sup_{s<t} |X_s^{(1)} - X_s^{(2)}|^p \leq c_p\, e^{c_p T} \int_0^t ds \ell^{*p}_{p,s}(m_1, m_2),$$

which implies

$$\ell^{*p}_{p,t}(\Phi(m_1), \Phi(m_2)) \leq c_p\, e^{c_p t} \int_0^t \ell^{*p}_{p,s}(m_1, m_2) ds.$$

□

Theorem 4.3 *Under (4.11) and* $\int_0^T (\int_{C_T} b(0, y_s) dm_o(y))^p\, ds < \infty$, *equation (4.8) has a unique weak and strong solution in* $\mathcal{X}_p(C_T, m_o)$.

Proof: From Lemma 4.2 we obtain for $m \in M_p(C_T, m_o)$

$$\begin{aligned} \ell^{*p}_{p,T}(\Phi^{k+1}(m), \Phi^k(m)) &\leq C_T \frac{T^k}{k!} \ell^{*p}_{p,T}(\Phi(m), m) \\ &\leq 2^{p-1} C_T \frac{T^k}{k!} [\ell^{*p}_{p,T}(\Phi(m), m_o) + \ell^{*p}_{p,T}(m, m_o)] < \infty. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 2.3. □

In the next step we now turn to the system of interacting particles defined in (4.2), where $((W^i_t), X^i_o)$ are independent processes identically distributed for all i. The next theorem asserts that as $N \to \infty$ each $X^{i,N}$ has a natural limit $\bar{X}^i$. $(\bar{X}^i)$ are independent copies of the solution of the nonlinear equation of Mc Kean-Vlasov type

$$\left| \begin{array}{l} X_t = B_t + (\int_0^t | \int_{C_T} b(X_s, y) u_s(dy)|^p\, ds)^{1/p}, \\ X_{t=0} = X_o. \end{array} \right. \tag{4.13}$$

Considered in Theorem 4.3 with $B \stackrel{d}{=} W^{(1)}$. Let b satisfy the Lipschitz condition (4.6).

Theorem 4.4 *Suppose that*

$$\int b(\bar{X}^1_s, y)^p\, u_s(dy) < \infty \quad a.s., \tag{4.14}$$

then for any $i \geq 1,\ T > 0$

$$\begin{aligned} &\sup_N \sqrt{N} (E \sup_{t \leq T} |X^{i,N}_t - \bar{X}^i_t|^p)^{1/p} < \infty \quad \textit{for} \quad p \geq 2 \quad \textit{and} \\ &N^{(1/p)-1} (E \sup_{t \leq T} |X^{i,N}_t - \bar{X}^i_t|^p)^{1/p} = o(1) \quad \textit{for} \quad 1 \leq p < 2. \end{aligned} \tag{4.15}$$

Proof: Drop the superscript N. Then

$$\begin{aligned}
X_t^i - \bar{X}_t^i &= (\int_0^t |\frac{1}{N}\sum_{j=1}^N b(X_s^i, X_s^j)|^p \, ds)^{1/p} \\
&\quad -(\int_0^t |\int b(\bar{X}_s^i, y) u_s(dy)|^p \, ds)^{1/p} \\
&= [(\int_0^t |\frac{1}{N}\sum_{j=1}^N b(X_s^i, X_s^j)|^p \, ds)^{1/p} \\
&\quad -(\int_0^t |\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, X_s^j)|^p \, ds)^{1/p}] \\
&\quad +[(\int_0^t |\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, X_s^j)|^p \, ds)^{1/p} \\
&\quad -(\int_0^t |\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j)|^p \, ds)^{1/p}] \\
&\quad +[(\int_0^t |\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j)|^p \, ds)^{1/p} \\
&\quad -(\int_0^t |\int b(\bar{X}_s^i, y) u_s(dy)|^p \, ds)^{1/p}].
\end{aligned}$$

By the Minkowski inequality, with $\|X\|_T = \sup_{s\le T}|X_s|$,

$$\begin{aligned}
&\|X^i - \bar{X}^i\|_{T,p}^p = E\|X^i - \bar{X}^i\|_T^p \\
\le\ & 4^{p-1}[E\int_0^t ds|\frac{1}{N}\sum_{j=1}^N [b(X_s^i, X_s^j) - b(\bar{X}_s^i, X_s^j)]|^p \\
&+E\int_0^T ds|\frac{1}{N}\sum_{j=1}^N [b(\bar{X}_s^i, X_s^j) - b(\bar{X}_s^i, \bar{X}_s^j)]|^p \\
&+E\int_0^T ds|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j) - \int b(\bar{X}_s^i, y) u_s(dy)|^p] \\
\le\ & 4^{p-1}\int_0^T ds\{c^p\, E|X_s^i - \bar{X}_s^i|^p + c^p\, E[\frac{1}{N}\sum_{j=1}^N |X_s^j - \bar{X}_s^j|]^p \\
&+E\int_0^T ds|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \bar{X}_s^j) - \int b(\bar{X}_s^i, y) u_s(dy)|^p\}.
\end{aligned}$$

Summing up over i and using the symmetry, we find

$$N\|X^i - \bar{X}^i\|_{T,p}^p = \sum_{i=1}^N \|X^i - \bar{X}^i\|_{T,p}^p$$

$$\begin{aligned} &\leq 4^{p-1}\int_0^T ds\{c^p\sum_{i=1}^N E\|X_s^i-\bar{X}_s^i\|_p^p \\ &+c^p\,N\,E[\frac{1}{N}(\sum_{j=1}^N |X_s^j-\bar{X}_s^j|^p)^{1/p}]^p \\ &+c^p\sum_{i=1}^N E|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i,\bar{X}_s^j)-\int b(\bar{X}_s^i,y)u_s(dy)|^p\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|X^i-\bar{X}^i\|_{T,p}^p &\leq 4^{p-1}c^p\int_0^T ds\{\|X^i-\bar{X}^i\|_{s,p}^p+c^p\|X^i-\bar{X}^i\|_{s,p}^p \\ &\quad +\frac{1}{N}\sum_{i=1}^N E|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i,\bar{X}_s^j)-\int b(\bar{X}_s^i,y)u_s(dy)|^p\}. \end{aligned}$$

Consequently, by the Gronwall lemma, $C_p=2\cdot 4^{p-1}c^p$

$$\begin{aligned} \|X^i-\bar{X}^i\|_{T,p}^p &\leq C_p\,e^{C_pT}\int_0^T ds[\frac{1}{N}\sum_{i=1}^N E|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i,\bar{X}_s^j)-\int b(\bar{X}_s^i,y)u_s(dy)|^p] \\ &\leq C_p\,e^{C_pT}\int_0^T ds\,E|\frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i,\bar{X}_s^j)-\int b(\bar{X}_s^i,y)u_s(dy)|^p \\ &= C_p\,e^{C_pT}T\cdot E[0(\frac{1}{\sqrt{N}})]^p \end{aligned}$$

by the Marcinkiewicz-Zygmund inequality (cf. Chow-Teicher, p. 357) for $p\geq 2$ respectively the Pyke and Root (1968) inequality for $1\leq p<2$. □

Corollary 4.5 *Let m denote the law of $\bar{X}$ satisfying (4.13), and let m_N denote the law of $(X^{1,N},\dots,X^{N,N})$, then under the assumptions of Theorems 4.3 and 4.4 m_N is m-chaotic.*

We next consider the limiting case $\underline{p=\infty}$ (cf. (4.3)). In contrast to the limiting case in Section 4 of p-th norm interaction, we obtain the propagation of chaos property for p-th mean interaction in time under a stronger Lipschitz condition. Consider for $m\in M^1(C_T)$

$$X_t=B_t+\operatorname*{ess\,sup}_{s\leq t}|\int_{C_T} b(X_s,y)m(dy)|, \tag{4.16}$$

where X_t, B_t and b are real-valued, B_t is a process on C_T, and with $E\,\mathrm{ess\,sup}_{s\leq T}|B_s|^p<\infty$ and

$$|b(x_1,y)-b(x_2,y)|\leq c|x_1-x_2|, \quad \text{with} \quad 0<c<1. \tag{4.17}$$

We use the metric (for $p \geq 1$)

$$\tilde{L}^*_{p,t}(X,Y) := (E \operatorname*{ess\,sup}_{s\leq t} |X_s - Y_s|^p)^{1/p} \quad \text{in} \quad \mathcal{X}(C_T). \tag{4.18}$$

Let

$$\tilde{\ell}^*_{p,t}(m_1,m_2) = \hat{\tilde{L}}^*_{p,t}(m_1,m_2) \tag{4.19}$$

be the corresponding minimal metric and let

$$\tilde{M}_p(C_T,m_o) = \{m_1 \in M^1(C_T);\ \tilde{\ell}^*_{p,T}(m_1,m_o) < \infty\}, \tag{4.20}$$

$\tilde{\mathcal{X}}_p(C_T,m_o)$ denotes the corresponding class of processes. For $m_o \in M^1(C_T)$ and $m \in \tilde{M}_p(C_T,m_o)$ consider the Liouville equation

$$X_t = B_t + \operatorname*{ess\,sup}_{s\leq t} \left| \int_{C_T} b(X_s,y_s)dm(y)\right|. \tag{4.21}$$

Lemma 4.6 *Assume (4.17) and let*

$$\operatorname*{ess\,sup}_{s\leq T} \left| \int_{C_T} b(0,y_s)m(dy)\right| < \infty.$$

Then

(a) *Equation (4.21) has a unique solution X.*

(b) *If $\Phi(m)$ is the law of X, then $\Phi(m) \in M_p(C_T,m_o)$, that is* $\Phi : \tilde{M}_p(C_T,m_o) \to \tilde{M}_p(C_T,m_o)$.

Proof: Let $X \in \tilde{\mathcal{X}}_p(C_T,m_o)$ and define

$$\begin{aligned}
(SX)_t &:= B_t + \operatorname*{ess\,sup}_{s\leq t} \left| \int_{C_T} b(X_s,y_s)m(dy)\right|. \\
|(SX)_t - (SY)_t|^p &= \Big| \operatorname*{ess\,sup}_{0\leq s<t} \left| \int_{C_T} b(X_s,y_s)m(dy)\right| \\
&\quad - \operatorname*{ess\,sup}_{0\leq s\leq t} \left| \int_{C_T} b(Y_s,y_s)m(dy)\right|\Big|^p \\
&\leq \operatorname*{ess\,sup}_{0\leq s\leq t} c^p |X_s - Y_s|^p,
\end{aligned}$$

by the Lipschitz condition (4.17).

This implies

$$\operatorname*{ess\,sup}_{s\leq t} |(SX)_s - (SY)_s|^p \leq c^p \operatorname*{ess\,sup}_{0\leq s\leq t} |X_s - Y_s|^p,$$

and

$$\tilde{L}^*_{p,t}(SX,SY) \leq c\ \tilde{L}^*_{p,t}(X,Y).$$

Define, inductively, $X^0 = B,\ X^n = SX^{n-1}$, then

$$\tilde{L}^*_{p,t}(X^n,X^{n-1}) \leq c^n\ \tilde{L}^*_{p,t}(X^1,X^0).$$

Furthermore,

$$\begin{aligned}\tilde{L}^*_{p,T}(X^1,X^0) &= (E\operatorname{ess\,sup}_{s\le T}|\int b(B_s,y_s)m(dy)|^p)^{1/p}\\ &\le (E\operatorname{ess\,sup}_{s\le T}[|\int b(0,y_s)m(dy)|+c\int|B_s|m(dy)]^p)^{1/p}\\ &\le c'(\operatorname{ess\,sup}_{s\le T}|\int b(0,y_s)m(dy)|+(E\operatorname{ess\,sup}_{s\le t}|B_s|^p)^{1/p}<\infty.\end{aligned}$$

This implies

$$\sum_{n\ge1}\tilde{L}^*_{p,T}(X^n,X^{n-1})\le\sum_{n\ge1}c^n\ \tilde{L}^{*p}_{p,T}(X^1,X^0)<\infty.$$

Therefore, $X^n\xrightarrow{a.s.}X$, uniformly on bounded intervals, and
$E\operatorname{ess\,sup}_{s\le t}|X_s|^p<\infty$. □

In addition, suppose that b is Lipschitz in both arguments

$$|b(x_1,y_1)-b(x_2,y_2)|\le c[|x_1-x_2|+|y_1-y_2|] \tag{4.22}$$

with $0<c<\frac{1}{2}$, and consider the map

$$\Phi:\tilde{M}_p(C_T,m_o)\to\tilde{M}_p(C_T,m_o).$$

Lemma 4.7 *(Contraction of ϕ w.r.t. $\tilde{\ell}^*_{p,t}$)*
Under (4.22) and the assumptions of Lemma 4.1, for $t<T$ and $m_1,m_2\in\tilde{M}_p(C_T,m_o)$, it holds:

$$\tilde{\ell}^*_{p,t}(\Phi(m_1),\Phi(m_2))\le\frac{c}{1-c}\tilde{\ell}^*_{p,t}(m_1,m_2). \tag{4.23}$$

Proof: Let for $i=1,2$, and $t\le T$,

$$X^{(i)}_t=B_t+\operatorname{ess\,sup}_{0<s<t}|\int_{C_T}b(X^{(i)}_s,y_s)dm_i(y)|,$$

and let $m\in M^1(m_1,m_2)$. Then

$$\begin{aligned}&E\operatorname{ess\,sup}_{s\le t}|X^{(1)}_s-X^{(2)}_s|^p\\ =\ &E|\operatorname{ess\,sup}_{s\le t}|\int_{C_T}b(X^{(1)}_s,y^{(1)}_s)dm_1(y^{(1)})|-\operatorname{ess\,sup}_{s\le t}|\int_{C_T}b(X^{(1)}_s,y^{(2)}_s)dm_2(y^{(2)})||^p\\ \le\ &E|\operatorname{ess\,sup}_{s\le t}c[|X^{(1)}_s-X^{(2)}_s|+\int_{C_T\times C_T}|y^{(1)}_s-y^{(2)}_s|dm(y^{(1)},y^{(2)})]|^p.\end{aligned}$$

Therefore, passing to minimal metrics on the RHS,

$$\begin{aligned}&(E\operatorname{ess\,sup}_{s\le t}|X^{(1)}_s-X^{(2)}_s|^p)^{1/p}\le c(E\operatorname{ess\,sup}_{s\le t}|X^{(1)}_s-X^{(2)}_s|^p)^{1/p}\\ &+c[(\inf_{m\in M^1(m_1,m_2)}\int_{C_T\times C_T}\operatorname{ess\,sup}_{s\le t}|y^{(1)}_s-y^{(2)}_s|dm(y_1,y_2))^p]^{1/p},\quad \text{i.e.}\\ &(1-c)(E\operatorname{ess\,sup}_{s\le t}|X^{(1)}_s-X^{(2)}_s|^p)^{1/p}\le c\,\tilde{\ell}^*_{1,s}(m_1,m_2)\le c\,\tilde{\ell}^*_{p,s}(m_1,m_2).\end{aligned}$$

Passing to the minimal metrics in the LHS, we obtain

$$\tilde{\ell}^{*p}_{p,T}(\Phi(m_1), \Phi(m_2)) \leq \frac{c}{1-c} \tilde{\ell}^{*}_{p,T}(m_1, m_2).$$

$\square$

Next we conclude the existence of a unique solution of the McKean-Vlasov type equation

$$X_t = B_t + \operatorname*{ess\,sup}_{s \leq t} |\int b(X_s, y_s) u_s(dy_s)|, \; X_{t=0} = X_0. \tag{4.24}$$

Theorem 4.8 *Under (4.22) and*

$$\operatorname*{ess\,sup}_{s \leq T} |\int_{C_T} b(0, y_s) dm_o(y)| < \infty,$$

equation (4.24) has for $m \in \tilde{M}_p(C_T, m_o)$ a unique weak and strong solution in $\tilde{\mathcal{X}}_p(C_T, m_o)$.

Proof: From Lemma 4.7 with $C := \frac{c}{1-c}$, $m \in \tilde{M}_p(C_T, m_o)$

$$\tilde{\ell}^{*p}_{p,T}(\Phi^{k+1}(m), \Phi^k(m)) \leq C^k \, \tilde{\ell}^{*p}_{p,T}(\Phi(m), m) < \infty,$$

which implies the theorem. $\square$

Consider the system of N interacting particles driven by equation (4.3) i.e.

$$\begin{aligned} X^{i,N}_t &= W^i_t + \operatorname*{ess\,sup}_{s \leq t} |\frac{1}{N} \sum_{j=1}^N b(X^{i,N}_s, X^{j,N}_s)| \\ X^{i,N}_o &= X^i_o, \qquad 1 \leq i \leq N. \end{aligned} \tag{4.25}$$

We show that $X^{i,N}$ has a natural limit $\bar{X}^i$, where $\bar{X}^i$ are iid copies of the solution of (4.25).

Theorem 4.9 *Suppose that (4.22) holds and that the r.v. $Y_{s,j} := b(\bar{X}^1_s, \bar{X}^j_s)$ on $C[0,T]$ are in the domain of normal attraction (dna) or satisfy the bounded law of the iterated logarithm (BLIL) and satisfy $E\|b(\bar{X}^1_s, \bar{X}^j_s\|^2_\infty < \infty$, then for any $i \geq 1$*

$$\sup_N a_N \, E \, \|X^{i,N}_t - \bar{X}^i_t\|_\infty < \infty, \tag{4.26}$$

where $a_N = \sqrt{N}$ or $a_N = \sqrt{N \log\log N}$.

Proof: Similar to the proof of Theorem 4.4, we obtain from the condition $2c < 1$, for $\alpha \geq 1$

$$\tilde{L}^*_{\alpha,T}(X^i, \bar{X}^i) \leq \frac{1}{1-2c} \frac{1}{N} \sum_{i=1}^N E \operatorname*{ess\,sup}_{s \leq T} |\frac{1}{N} \sum_{j=1}^N b(\bar{X}^i_s, \bar{X}^j_s) - \int b(\bar{X}^i_s, y) u_s(dy)|^\alpha.$$

If $(Y_{s,j}$ are in dna (cf. Hoffmann-Jörgensen (1977)) then (4.26) follows with $a_N = \sqrt{N}$. If $(Y_{s,j})$ satisfy the BLIL, then for the corresponding centered sum S_N $\overline{\lim} E\|\frac{S_N}{a_N}\|_\infty \le \overline{\lim}\frac{\|S_N\|_\infty}{a_N} < \infty$ a.s. (cf. Kuelbs (1977)) and (4.26) is a consequence. $\square$

We remark that by Corollary 5.7 of Hoffmann-Jörgensen (1977) a sufficient condition for the dna of $S_N = \sum_{j=1}^N X_j$ is given bx

$$E\|X_1\|_{bL}^2 < \infty, \tag{4.27}$$

$\| \ \|_{bL}$ the bounded Lipschitz-norm w.r.t. any Gaussian metric ρ.

Corollary 4.10 *Under the assumptions of Theorems 4.8 and 4.9 let m denote the law of $\bar{X}$, m_N the law of $(X^{1,N}, \dots, X^{N,N})$, then m_N is m-chaotic.*

Remark 3. In the case $\underline{0 < p < 1}$ we see by similar methods that there exists no unique solution of the Liouville equation and also there is no propagation of chaos.

5 Minimal Mean Interaction in Time

Consider the analogue of equation (4.3) with minimal mean interaction in time

$$\begin{aligned} X_t^{i,N} &= W_t^i + \operatorname*{ess\,inf}_{s\le t} \Big|\frac{1}{N}\sum_{j=1}^N b(X_s^{i,N}, X_s^{j,N})\Big| \\ X_o^{i,N} &= X_o^i, \qquad 1 \le i \le N. \end{aligned} \tag{5.1}$$

The corresponding Boltzmann type equation is

$$\begin{aligned} X_t &= B_t + \operatorname*{ess\,inf}_{s\le t} \Big| \int b(X_s, y) u_s(dy)\Big| \\ X_{t=0} &= X_o. \end{aligned} \tag{5.2}$$

We obtain the following results. As the proofs are similar to those in Section 4, we omit them.

Theorem 5.1 *If $m_o \in M^1(C_T)$ and*

$$|b(x_1, y_1) - b(x_2, y_2)| \le c[|x_1 - x_2| + |y_1 - y_2|], \ 0 < c < \frac{1}{2} \tag{5.3}$$

and

$$\operatorname*{ess\,sup}_{s\le T} \Big| \int_{C_T} b(0, y_s) dm_o(y)\Big| < \infty, \tag{5.4}$$

then (5.2) has a unique strong solution in $\mathcal{X}_p(C_T, m_o)$.

The system $(X^{i,N})$ in (5.1) has a natural limiting process $(\bar{X}^i)$, which are iid copies of the solution X of (5.2).

Theorem 5.2 *Under the assumptions of Theorem 5.1 and Theorem 4.9 holds for any $i \geq 1$*

$$\sup_N a_N \, E \sup_{t \leq T} |X_t^{i,N} - \bar{X}_t^{i,N}| < \infty. \tag{5.5}$$

As corollary we obtain:

Corollary 5.3 *Under the conditions of Theorems 5.1 and 5.2, the system (5.1) has the propagation of chaos property.*

6 Interaction with the Normalized Variation of the Neighbours; Relaxed Lipschitz-Conditions

Consider the following system

$$\begin{aligned} X_t^{i,N} &= W_t^i + \int_0^t \Big(\frac{1}{N} \sum_{j=1}^N b(X_s^{i,N}, \overset{\circ}{X}{}_s^{j,N})\Big) ds \\ X_o^{i,N} &= X_o^i, \qquad 1 \leq i \leq N, \end{aligned} \tag{6.1}$$

where

$$\overset{\circ}{X}{}_s^{i} := \frac{X_s^i - EX_s^i}{E|X_s^i - EX_s^i|} \tag{6.2}$$

is the normalized variation of particle i, $((W_t^i), X_o^i)$ are independent identically distributed processes on $C_T \times \mathbb{R}$. The drift is given by the mean of the interactions with the normalized variation of all particles. We assume that

$$b(x, 0) = 0, \quad \forall \, x, \tag{6.3}$$

i.e. the interaction is zero, if the relative variation is zero.

The McKean-Vlasov type equation corresponding to (6.1) is given by

$$\begin{aligned} X_t &= B_t + \int_0^t \Big(\int b(X_s, y) dP^{\overset{\circ}{X}_s}(y)\Big) ds \\ X_{t=0} &= X_o, \end{aligned} \tag{6.4}$$

where $B \overset{d}{=} W^i$. Note that B in this section is not necessarily a Brownian motion. We study these equations under a relaxed Lipschitz condition. Assume that b has a partial derivative

$$b_2' := \frac{\partial b}{\partial y} \tag{6.5}$$

w.r.t. the second coordinate and consider
(L1) $\quad |b_2'(x_1, y) - b_2'(x_2, y)| \leq c|x_1 - x_2|$
and
(L2) $\quad |b_2'(x_1, y_1) - b_2'(x_2, y_2)| \leq c[|x_1 - x_2| + |y_1 - y_2|]$.

(L2) allows a quadratic growth of b w.r.t. the second component. To obtain contraction properties in this case, we have to switch to a suitable probability metric with regularity conditions of higher order. This makes necessary an essential change in the method of the proofs given so far.

For $m \in M_1(C_T)$ the distribution of a process (ξ_s) let $\overset{\circ}{m}$ denote the distribution of the normalized process $(\overset{\circ}{\xi}_s)$ assuming an absolute first moment of m_s. Define

$$N_s \quad := \quad \overset{\circ}{m}_s - \delta_o = N_s^m \quad \text{and} \tag{6.6}$$

$$F_{N_s}^{(-1)}(y) \quad := \quad \int_{-\infty}^{y} F_{N_s}(u)du. \tag{6.7}$$

In concordance with the usual notation of derivates of a function f by $f^{(s)}$, $s \geq 1$, we define the s-fold integrated function by $f^{(-s)}$ since this is the inverse operation and $(f^{(-1)})^{(1)} = f$, etc.

Note that by (6.3) we can replace the integration w.r.t. $\overset{\circ}{m}_s$ in (6.4) by integration w.r.t. N_s. Consider the Liouville equation

$$X_t = B_t + \int_0^t (\int b(X_s, y)dN_s(y_s))ds. \tag{6.8}$$

By integration by parts (6.7) is equivalent to

$$X_t = B_t + \int_0^t (\int b_2'(X_s, y)dF_{N_s}^{(-1)}(y))ds. \tag{6.9}$$

Theorem 6.1 *Suppose that $m \in M^1(C_T)$ has a finite first moment and $E\sup_{s\leq T}|B_s| < \infty$. Furthermore, let (L1) be satisfied and suppose that*

$$\int_0^T \int |b_2'(0,y)|\,|F_{N_s}(y)|dy = \int_0^T E\int_0^{\overset{\circ}{\xi}_s} |b_2'(0,t)|dt < \infty. \tag{6.10}$$

Then (6.8) has a unique strong solution X and, moreover, $E\sup_{s\leq T}|X_s| < \infty$.

Proof: Let

$$\begin{aligned}(SX)_t : \quad &= \quad B_t + \int_0^t ds(\int_{\mathbb{R}} b(X_s, y_s)dN_s(y_s)),\\ &= \quad B_t + \int_0^t ds(\int_{\mathbb{R}} b_2'(X_s, y_s)dF_{N_s}^{(-1)}(y_s)).\end{aligned}$$

Then by the Lipschitz condition (L1),

$$\begin{aligned}|(SX)_t - (SY)_t| \quad &\leq \quad \int_0^t ds|\int_{\mathbb{R}}(b_2'(X_s, y_s) - b_2'(Y_s, y_s))dF_{N_s}^{(-1)}(y_s)|\\ &\leq \quad \int_0^t ds\int_{\mathbb{R}} c|X_s - Y_s|\,|F_{N_s}(y_s)|dy_s.\end{aligned}$$

Observe that the total variation norm of the measure $F_{N_s}^{(-1)}(dy)$ is 1;

$$\mathrm{Var}\,(F_{N_s}^{(-1)}) = \int_{\mathbb{R}} |F_{N_s}(y)|dy = \int_{-\infty}^0 F_{\overset{\circ}{\xi}_s}(y)dy + \int_0^\infty (1 - F_{\overset{\circ}{\xi}_s}(y))dy = E|\overset{\circ}{\xi}_s| = 1.$$

Therefore,

$$|(SX)_t - (SY)_t| \leq c \int_0^t |X_s - Y_s| ds \tag{6.11}$$

implying

$$L^*_{1,t}(SX, SY) \leq c \int_0^t L^*_{1,s}(X, Y) ds.$$

Define inductively, $X^\circ = B,\ X^n = SX^{n-1}$, then

$$L^*_{1,T}(X^n, X^{n-1}) \leq c^n \frac{T^n}{n!} L^*_{1,T}(X^1, X^0). \tag{6.12}$$

Note that

$$\begin{aligned}
L^*_{1,T}(X^1, X^0) &= E \sup_{s \leq T} | \int_0^s ds(\int_{\mathbb{R}} b_2'(B_s, y_s) dF^{(-1)}_{N_s}(y_s)| \qquad (6.13) \\
&\leq E \int_0^T ds \int_{\mathbb{R}} |b_2'(B_s, y_s)| \, |F_{N_s}(y_s)| dy_s \\
&\leq E \int_0^T ds \int_{\mathbb{R}} (c|B_s| + |b_2'(0, y_s)|) |F_{N_s}(y_s)| dy_s \\
&\leq E \int_0^T ds \, c|B_s| + \int_0^T ds \int_{\mathbb{R}} |b_2'(0, y_s)| \, |F_{N_s}(y_s)| dy_s < \infty.
\end{aligned}$$

The equality in (6.10) results from

$$\begin{aligned}
& \int_{\mathbb{R}} |b_2'(0, y)| |F_{N_s}(y)| dy \\
= & \int_{\mathbb{R}} |b_2'(0, y)| |F_{\overset{\circ}{\xi}_s}(y) - F_0(y)| dy \\
= & \int_{-\infty}^0 |b_2'(0, y)| F_{\overset{\circ}{\xi}_s}(y) dy + \int_0^\infty |b_2'(0, y)| (1 - F_{\overset{\circ}{\xi}_s}(y)) dy \\
= & \int_{-\infty}^{+\infty} (\int_0^y |b_2'(0, t)| dt) dF_{\overset{\circ}{\xi}_s}(y) = E \int_0^{\overset{\circ}{\xi}_s} |b_2'(0, t)| dt < \infty.
\end{aligned}$$

Consequently, $L^*_{1,T}(X^1, X^0) < \infty$. (6.12), (6.13) imply the existence and the uniqueness of a strong solution X. Moreover,

$$L^*_{1,T}(X, B) \leq \sum_{n \geq 1}^{\infty} L^*_{1,T}(X^n, X^{n-1}) \leq e^{cT} L^*_{1,T}(X_1, B) < \infty;$$

that is, $E \sup_{s \leq T} |B_s| < \infty$ implies $E \sup_{s \leq T} |X_s| < \infty$. □

We next extend the result of Theorem 6.1 to the case where p-th moments exist, $p \geq 1$. Denote $\|X\|^*_{T,p} = (E \sup_{t \leq T} |X(t)|^p)^{1/p},\ 1 \leq p < \infty$, and $\|X\|^*_{T,\infty} = E \operatorname{ess\,sup}_{0 < t \leq T} |X(t)|$.

Theorem 6.2 *Suppose that $\|B\|^*_{T,p} < \infty$ for some $1 \le p \le \infty$. Suppose that (L1) holds and suppose that*

$$\int_0^T ds(\int_{\mathbb{R}} |b_2'(0,y_s)F_{N_s}(y_s)|^p dy_s)^{1/p} < \infty \quad (1 \le p < \infty) \quad resp. \tag{6.14}$$

$$\int_0^T ds(\operatorname*{ess\,sup}_{y_s} |b_2'(0,y_s)||F_{N_s}(y_s)|) < \infty \quad (p = \infty). \tag{6.15}$$

Then (6.8) has a unique solution X and $\|X\|^*_{T,p} < \infty$. In particular, if $\Phi(m)$ is the distribution of the solution of (6.8), then $\Phi(m)$ maps $M_p(C_T, \delta_o)$ into $M_p(C_T, \delta_o)$.

Proof: As in Theorem 6.1

$$|(SX)_t - (SX)_t| \le c\int_0^t |X_s - Y_s|ds,$$

implying for any $1 < p \le \infty$

$$L^*_{p,T}(SX, SY) \le \int_0^t L^*_{p,T}(X,Y)ds.$$

Further, for $1 \le p < \infty$ (the case $p = \infty$ is similar)

$$\begin{aligned}
L^*_{p,T}(X,B) &= (E\sup_{s\le T}|\int_0^s ds(\int_{\mathbb{R}} b_2'(B_s,y_s)dF^{(-1)}_{N_s}(y_s)|^p)^{1/p} \\
&\le (E(\int_0^T ds\int_{\mathbb{R}} |b_2'(B_s,y_s)||F_{N_s}(y_s)|dy_s)^p)^{1/p} \\
&\le \int_0^T ds[E(\int_{\mathbb{R}} |b_2'(B_s,y_s)||F_{N_s}(y_s)|dy_s)^p]^{1/p} \\
&\le \int_0^T ds[E(\int_{\mathbb{R}} (c|B_s| + |b_2'(0,y_s)|)|F_{N_s}(y_s)|dy_s)^p]^{1/p} \\
&\le c\int_0^T ds(E|B_s|^p)^{1/p} + \int_0^T ds(\int_{\mathbb{R}} |b_2'(0,y_s)F_{N_s}(y_s)|^p\, dy_s)^{1/p} < \infty.
\end{aligned}$$

Then continue as in Theorem 6.1, to complete the argument. □

Denote by $M_2^*(C_T, \delta_o)$ the space of all $m \in M_2(C_T, \delta_o)$ such that

$$\inf_{0<s\le T} E|\xi_s - E\xi_s| =: A_T^* > 0, \ \xi \stackrel{d}{=} m. \tag{6.16}$$

Condition (6.16) postulates that the L^1-variation does not converges to zero for $0 < s < T$. For a Brownian motion this means that we do not start deterministically at one fixed point at $s = 0$.

Let $\Phi(m)$ be the solution of (6.8)

$$X_t = B_t + \int_0^t ds(\int_{\mathbb{R}} b(X_s,y_s)d\stackrel{\circ}{m}_s(y_s))$$

under the assumptions of Theorem 6.2 with $p = 2$. Then by Theorem 6.2, Φ maps $M_2(C_T, \delta_o)$ into $M_o(C_T, \delta_o)$.

Theorem 6.3 *(Contraction to Φ)*
Suppose that the Lipschitz condition (L2) holds, and $m_1, m_2 \in M_2^*(C_T, \delta_o)$. Then a contraction of Φ w.r.t. $\ell_{2,t}^*$ holds:

$$\ell_{2,t}^*(\Phi(m_1), \Phi(m_2)) \leq c_t \int_0^t \ell_{2,u}^*(m_1, m_2) du. \tag{6.17}$$

Proof: For $m_1, m_2 \in M_2^*(C_T, \delta_o)$ let

$$\begin{aligned} X_t^{(i)} &= B_t + \int_0^t \Big(\int_{\mathbb{R}} b(X_s^{(i)}, y_s^{(i)}) dF_{N_s^{(i)}}(y_s^{(i)})\Big) ds \\ &= B_t + \int_0^t \Big(\int_{\mathbb{R}} b_2'(X_s^{(i)}, y_s^{(i)}) dF_{N_s^{(i)}}^{(-1)}(y_s^{(i)})\Big) ds. \end{aligned}$$

Then

$$\begin{aligned} X_t^{(1)} - X_t^{(2)} &= \int_0^t \Big[\int_{\mathbb{R}} b_2'(X_s^{(1)}, y_s^{(1)}) dF_{N_s^{(1)}}^{(-1)}(y_s^{(1)}) \\ &\quad - \int_{\mathbb{R}} b_2'(X_s^{(2)}, y_s^{(2)}) dF_{N_s^{(2)}}^{(-1)}(y_s^{(2)})\Big] ds. \end{aligned}$$

Since the total variation norm of $F_{N_s}^{(-1)}$ is 1 and the total mass is 0, by the Jordan decomposition

$$F_{N_s}^{(-1)}(dx) = \mu_s^+(dx) - \mu_s^-(dx),$$

where $\mu_s^+(\mathbb{R}) + \mu_s^-(\mathbb{R}) = 1,\ \mu_s^+(\mathbb{R}) - \mu_s^-(\mathbb{R}) = 0$, in other words,

$$\mu_s^+(\mathbb{R}) = \mu_s^-(\mathbb{R}) = \frac{1}{2}.$$

We write

$$F_{N_s^{(i)}}^{(-1)}(ds) = \mu_s^{(i)+}(dx) - \mu_s^{(i)-}(dx)$$

and so,

$$\begin{aligned} X_t^{(1)} - X_t^{(2)} &= \int_0^t ds\Big[\int_{\mathbb{R}} b_2'(X_s^{(1)}, y_s^{(1)})(\mu_s^{(1)+} - \mu_s^{(1)-})(dy_s^{(1)}) \\ &\quad - \int_{\mathbb{R}} b_2'(X_s^{(2)}, y_s^{(2)})(\mu_s^{(2)+} - \mu_s^{(2)-})(dy_s^{(2)})\Big). \end{aligned}$$

Let $dm_s^+(y_s^{(1)}, y_s^{(2)})$ be a coupling for $\mu_s^{(1)+}$ and $\mu_s^{(2)+}$, that is, m_s^+ is a positive measure with total mass $\frac{1}{2}$ and such that $\pi_i m_s^+ = \mu_s^{(i)+},\ i = 1, 2,\ \pi_i$ the i-th component. Similarly, let $dm_s^-(y_s^{(1)}, y_s^{(2)})$ be a coupling for $\mu_s^{(1)-}$ and $\mu_s^{(2)-}$. Then

$$\begin{aligned} X_t^{(1)} - X_t^{(2)} &= \int_0^t ds\Big[\int_{\mathbb{R}} (b_2'(X_s^{(1)}, y_s^{(1)}) - b_2'(X_s^{(2)}, y_s^{(2)})) dm_s^+(y_s^{(1)}, y_s^{(2)}) \\ &\quad - \int_{\mathbb{R}} (b_2'(X_s^{(1)}, y_s^{(1)}) - b_2'(X_s^{(2)}, y_s^{(2)})) dm_s^-(y_s^{(1)}, y_s^{(2)})\Big]. \end{aligned}$$

Consequently, by the Lipschitz condition

$$\begin{aligned}
&|X_t^{(1)} - X_t^{(2)}| \qquad (6.18)\\
\le\ & \int_0^t ds\Big(\int_{\mathbb{R}^2} |b_2'(X_s^{(1)}, y_s^{(1)}) - b_2'(X_s^{(2)}, y_s^{(2)})| d(m_s^+ + m_s^-)(y_s^{(1)}, y_s^{(2)})\Big)\\
\le\ & \int_0^t ds \int_{\mathbb{R}^2} (c|X_s^{(1)} - X_s^{(2)}| + c|y_s^{(1)} - y_s^{(2)}|) d(m_s^+ + m_s^-)(y_s^{(1)}, y_s^{(2)})).
\end{aligned}$$

Observe that the total mass of $m_s^+ + m_s^-$ is 1, and for $i = 1,2$, $\pi_i m_s^+ + \pi_i m_s^- = \mu_s^{(i)+} + \mu_s^{(i)-}$ is the variation of $F^{(-1)}_{N_s^{(i)}}$. Minimizing w.r.t. all couplings $m_s^+ + m_s^-$ with marginals $\mu_s^{(i)+} + \mu_s^{(i)-}$, $i = 1,2$, we get

$$|X_t^{(1)} - X_t^{(2)}| \le \int_0^t c\, ds |X_s^{(1)} - X_s^{(2)}| + \int_{\mathbb{R}} |F_{\mu_s^{(1)+} + \mu_s^{(1)-}}(x) - F_{\mu_s^{(2)+} + \mu_s^{(2)-}}(x)| dx.$$

As $F_{\mu_s^{(1)+} + \mu_s^{(1)-}}(x) = F_{\mathrm{Var}(F^{(-1)}_{N_s^{(1)}})}(x)$, we have that the integral on the right hand side equals, using $\int |F_{\mu_1}(x) - F_{\mu_2}(x)| dx \le \int |x| \mathrm{Var}(\mu_1 - \mu_2)(dx)$

$$\begin{aligned}
& \int_{\mathbb{R}} |F_{\mathrm{Var}(F^{(-1)}_{N_s^{(1)}})}(x) - F_{\mathrm{Var}(F^{(-1)}_{N_s^{(2)}})}(x)| dx \qquad (6.19)\\
\le\ & \int_{\mathbb{R}} |x| \mathrm{Var}(\mathrm{Var}(F^{(-1)}_{N_s^{(1)}}) - \mathrm{Var}(F^{(-1)}_{N_s^{(2)}}))(dx)\\
\le\ & \int_{\mathbb{R}} |x| \mathrm{Var}(F^{(-1)}_{N_s^{(1)}} - F^{(-1)}_{N_s^{(2)}})(dx)\\
=\ & \int_{\mathbb{R}} |x| \Big|\frac{d}{dx}(F^{(-1)}_{N_s^{(1)}}(x) - F^{(-1)}_{N_s^{(2)}}(x))\Big| dx\\
=\ & \int_{\mathbb{R}} |x| |F_{N_s^{(1)}}(x) - F_{N_s^{(2)}}(x)| dx \quad (\text{as} \quad N_s := P^{\overset{\circ}{\xi}_s} - \delta_o)\\
=\ & \int_{\mathbb{R}} |x| |F_{\overset{\circ}{\xi}{}_s^{(1)}}(x) - F_{\overset{\circ}{\xi}{}_s^{(2)}}(x)| dx\\
=\ & : \kappa_2(\overset{\circ}{\xi}{}_s^{(1)}, \overset{\circ}{\xi}{}_s^{(2)}),
\end{aligned}$$

where $\overset{\circ}{\xi}{}_s^{(i)}$ are r.v.'s with laws $P^{\overset{\circ}{\xi}{}_s^{(i)}} = P^{(\xi_s^{(i)} - E\xi_s^{(i)})/E|\xi_s^{(i)} - E\xi_s^{(i)}|}$, and $P^{\xi_s^{(i)}} = m_s^{(i)}$.

By the minimality of κ_2,

$$\kappa_2(\xi_s^{(1)}, \xi_s^{(2)}) = \inf\{E||\eta_s^{(1)}|\eta_s^{(1)} - |\eta_s^{(2)}|\eta_s^2|; \eta_s^{(i)} \overset{d}{=} \xi_s^{(i)}\}. \qquad (6.20)$$

By means of (6.20) we estimate $\kappa_2(\overset{\circ}{\xi}{}_s^{(1)}, \overset{\circ}{\xi}{}_s^{(2)})$ by $\kappa_2(\xi_s^{(1)}, \xi_s^{(2)})$ making use of the assumption that

$$\sup_{s \le T} E|\xi_s^{(i)}|^2 \le A_T$$

$$\inf_{s \le T} E|\xi_s^{(i)} - E\xi_s^{(i)}| =: A_T^* > 0.$$

Then

$$
\begin{aligned}
& \kappa_2(\overset{\circ(1)}{\xi_s}, \overset{\circ(2)}{\xi_s}) && (6.21)\\
\leq\ & \ell_2(\overset{\circ(1)}{\xi_s}, \overset{\circ(2)}{\xi_s})(E|\overset{\circ(1)}{\xi_s}|^2 + E|\overset{\circ(2)}{\xi_s}|^2)\\
\leq\ & 2A_T\ell_2(\frac{\xi_s^{(1)} - E\xi_s^{(1)}}{E|\xi_s^{(1)} - E\xi_s^{(1)}|}, \frac{\xi_s^{(2)} - E\xi_s^{(2)}}{E|\xi_s^{(2)} - E\xi_s^{(2)}|})\\
\leq\ & 2A_T\ell_2(\frac{\xi_s^{(1)} - E\xi_s^{(1)}}{E|\xi_s^{(1)} - E\xi_s^{(1)}|}, \frac{\xi_s^{(2)} - E\xi_s^{(2)}}{E|\xi_s^{(1)} - E\xi_s^{(1)}|})\\
+\ & 2A_T\ell_2(\frac{\xi_s^{(2)} - E\xi_s^{(2)}}{E|\xi_s^{(1)} - E\xi_s^{(1)}|}, \frac{\xi_s^{(2)} - E\xi_s^{(2)}}{E|\xi_s^{(2)} - E\xi_s^{(2)}|})\\
\leq\ & \frac{2A_T}{E|\xi_s^{(1)} - E\xi_s^{(1)}|} \cdot \ell_2(\xi_s^{(1)} - E\xi_s^{(1)}, \xi_s^{(2)} - E\xi_s^{(2)})\\
+\ & 2A_T(E|\xi_s^{(2)} - E\xi_s^{(2)}|^2)^{1/2}\frac{|E|\xi_s^{(1)} - E\xi_s^{(1)}| - E|\xi_s^{(2)} - E\xi_s^{(2)}||}{(E|\xi_s^{(1)} - E\xi_s^{(1)}|)(E|\xi_s^{(2)} - E\xi_s^{(2)}|)}\\
\leq\ & c_T\ell_2(\xi_2^{(1)}, \xi_s^{(2)}),
\end{aligned}
$$

using the fact that

$$|E\xi_s^{(1)} - E\xi_s^{(2)}| \leq \ell_1(\xi_s^{(1)}, \xi_s^{(2)}) \leq \ell_2(\xi_s^{(1)}, \xi_s^{(2)}),$$

and $\frac{1}{|E\xi_s^{(i)} - E\xi_s^{(i)}|} \leq \frac{1}{A_T^*}$. Combining our estimates we write

$$|X_t^{(1)} - X_t^{(2)}| \leq \int_0^t c\, ds |X_s^{(1)} - X_s^{(2)}| + c_T\ell_2(m_s^{(1)}, m_s^{(2)}), \tag{6.22}$$

using the assumptions $E(\xi_s^{(i)})^2 < \infty,\ i = 1, 2$, and $E|\xi_s^{(i)} - E\xi_s^{(i)}| \geq A_T^* > 0$ uniformly on $s \in (0, T]$. Then, by the Gronwall inequality, with $c_T^* = c \vee c_t$,

$$\sup_{s\leq t} |X_t^{(1)} - X_t^{(2)}| \leq c_T^* e^{C_T^* T} \int_0^t \ell_{2,s}^*(m_s^{(1)}, m_s^{(2)}) ds.$$

The above implies by passing to minimal metrics that
$\ell_{2,t}^*(\Phi(m_1), \Phi(m_2)) \leq c\, e^{cT} \int_0^t \ell_{2,s}^*(m_1, m_2) ds.$ □

Theorem 6.4 *Suppose that $\|B\|_{T,2}^* < \infty$, (L2) and for some $m_o \in M_2(C_T, \delta_o)$ with $N_s = N_s^{m_o}$ holds*

$$\int_0^T ds (\int |b_2'(0, y)F_{N_s}(y)|^2\, dy)^{1/2} < \infty \tag{6.23}$$

and

$$\phi^n(m_o) \in M_2^*(C_T, \delta_o),\ \forall\, n \in \mathbb{N}, \tag{6.24}$$

then the Boltzmann type equation (6.4) has a unique weak and strong solution in $M_2(C_T, \delta_o)$.

Proof: From Theorem 6.3, for $m \in M_2(C_T, \delta_o)$

$$\ell^*_{2,T}(\Phi^{(k+1)}(m), \Phi^{(k)}(m)) \le C_T^k \frac{T^k}{k!}(\ell^*_{2,T}(\Phi(m), \delta_o) + \ell^*_{2,T}(m, \delta_o)) < \infty.$$

Therefore, $(\Phi^k(m))$ is a Cauchy sequence in $(C_T, \ell^*_{2,T})$ and converges to a fixed point. If $X^{(k+1)}, X^{(k)}$ are the optimal couplings of $\Phi^{(k+1)}(m), \phi^{(k)}(m)$ we get that $(X^{(k)})$ is a $L^2_{*,T}$-Cauchy sequence, leading to a (unique) $L^*_{2,T}$-fix point X. $\square$

Remark 4. Condition (6.24) postulates that the solutions of the Liouville equations corresponding to $\Phi^m(m_o)$ have strict positive variation. A simple sufficient condition for this to hold in the case that b is bounded, $|b| \le M$ is that $\inf_{s\le T} |B_s - EB_s| \ge TM + \varepsilon$. This condition is useful only for fixed T but not for $T \to \infty$. But it might be possible in examples (as in the construction of solutions of SDE's is typically done) to construct a solution piecewise on small time intervalls and to join the pieces to a solution on the whole real line. For special choices of b it is possible to obtain weaker sufficient conditions for (6.24). Condition (6.16) is needed in order to reconstruct the process. Without this condition we only can reconstruct the normalized process (cf. (6.21)).

We now turn to equation (6.1). The next theorem asserts that as $N \to \infty$ each $X^{i,N}$ has a limit $\bar{X}^i$. $(\bar{X}^i)$ are independent copies of the solution of (6.4) considered in Theorem 6.4.

Theorem 6.5 *Suppose that (L2) holds and moreover,* $\|b\|_\infty = \sup_{x,y} |b(x,y)| < \infty$. *Suppose also that uniformly on* i,

$$|W^i|_{T,\infty} := \operatorname{ess\,sup} \sup_{0<s<T} |W^i_s| \le X < \infty.$$

Then for any $i \ge 1,\ T > 0$,

$$\sup_N \sqrt{N} E \sup_{0<t\le T} |X^{i,N}_t - \bar{X}^i_t| < \infty.$$

Corollary 6.6 *(Propagation of Chaos)*
Let m *denote the law of* $\bar{X}^i$ *satisfying (6.4) and let* W_N *denote the law of* $(X^{1,N}, \dots, X^{N,N})$. *Then under the assumptions of Theorems 6.4 and 6.5* W_N *is* m*-chaotic.*

Proof of Theorem 6.5. Omitting the index N, we get

$$\begin{aligned} X^i_t - \bar{X}^i_t &= \int_0^t ds \frac{1}{N}\sum_{j=1}^N b(X^i_s, \overset{\circ}{X}{}^j_s) - \int_0^t ds \int_{C_T} b(\bar{X}_s, y_s) P^{\overset{\circ}{X}_s}(dy) \\ &= : I_1(t) + I_2(t) + I_3(t), \quad \text{where} \\ I_1(t) : &= [\int_0^t ds \frac{1}{N}\sum_{j=1}^N b(X^i_s, \overset{\circ}{X}{}^j_s) - \int_0^t ds \frac{1}{N}\sum_{j=1}^N b(\bar{X}^i_s, \overset{\circ}{X}{}^j_s)] \end{aligned}$$

$$\begin{aligned}
I_2(t) : &= [\int_0^t ds \frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \overset{\circ}{X}{}_s^j) - \int_0^t ds \frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \overset{\circ}{\bar{X}}{}_s^j)] \\
I_3(t) : &= [\int_0^t ds \frac{1}{N}\sum_{j=1}^N b(\bar{X}_s^i, \overset{\circ}{\bar{X}}{}_s^j) - \int_0^t ds \int_{C_T} b(\bar{X}_s, y_s) P^{\overset{\circ}{\bar{X}}_s}(dy)] \\
E|I_1|_T : &= E \sup_{0<t<T} E|I_1(t)| \\
&= E\int_0^T ds |\frac{1}{N}\sum_{j=1}^N [b(X_s^i, \overset{\circ}{X}{}_s^j) - b(\bar{X}_s^i, \overset{\circ}{X}{}_s^j)]|.
\end{aligned}$$

From (L2)

$$\begin{aligned}
|b(x,y) - b(\bar{x},y)| &= |b(x,y) - b(x,0) - (b(\bar{x},y) - b(\bar{x},0))| \\
&= |\int_0^y b_2'(x,t)dt - \int_0^t b_2'(\bar{x},t)dt| \\
&\le \int_0^{|y|} |b_2'(x,t) - b_2'(\bar{x},t)|dt \\
&\le c|x-\bar{x}||y|.
\end{aligned}$$

Therefore,

$$E|I_1|_T \le c\, E\int_0^T \frac{1}{N}\sum_{j=1}^N |X_s^i - \bar{X}_s^i||\overset{\circ}{X}{}_s^j|.$$

Assuming that $\|b\|_\infty = \sup_{x,y}|b(x,y)| < \infty$ and

$$|W^{i,N}|_{T,\infty} := \operatorname{ess\,sup} \sup_{0<s\le T} |W_s^{i,N}| \le K,$$

then

$$\sup_{i,N} |X_t^{i,N}| \le K + T\cdot\|b\|_\infty.$$

Therefore,

$$\begin{aligned}
E|I_1|_{T,1} &\le C\int_0^T ds\, E|X_s^i - \bar{X}_s^i| \\
E|I_2|_{T,1} &\le \int_0^T ds \frac{1}{N}\sum_{j=1}^N |b(\bar{X}_s^i, \overset{\circ}{X}{}_s^j) - b(\bar{X}_s^i, \overset{\circ}{\bar{X}}{}_s^i)|.
\end{aligned}$$

For $0 < y < \bar{y}$,

$$\begin{aligned}
|b(x,y) - b(x,\bar{y})| &= \int_y^{\bar{y}} |b_2'(x,t)|dt \\
&\le c\int_y^{\bar{y}} |b_2'(x,t) - b_2'(0,t)|dt \\
&+ c\int_y^{\bar{y}} |b_2'(0,t) - b_2'(0,0)|dt
\end{aligned}$$

$$
\begin{aligned}
&\le \quad c|x||\bar{y}-y| + \frac{1}{2}|\bar{y}^2 - y^2| \\
&\le \quad c|\bar{y}-y|(|x| + \frac{\bar{y}+y}{2}).
\end{aligned}
$$

In general,

$$
|b(x,y) - b(x,\bar{y})| \le c|y-\bar{y}|(|x| + \frac{|y|+|\bar{y}|}{2}).
$$

Assuming that $X_s^{j,N}$ are bounded a.s., $|X^{j,N}|_{T,\infty} := \operatorname{ess\,sup} \sup_{0<s<T} |X_s^{j,N}| < \infty$, we obtain

$$
\begin{aligned}
\|I_2\|_{T,1} := E|I_2|_T \quad &\le \quad \int_0^T ds \frac{1}{N} \sum_{j=1}^N |b(\bar{X}_s^i, \mathring{X}_s^j) - b(\bar{X}_s^i, \mathring{\bar{X}}_s^j)| \\
&\le \quad c \int_0^T ds \frac{1}{N} \sum_{j=1}^N E| \mathring{X}_s^j - \mathring{\bar{X}}_s^j | \times (|\bar{X}_s^i| + \frac{1}{2}(| \mathring{X}_s^j | + | \mathring{\bar{X}}_s^j |)) \\
&\le \quad c_{abs} \int_0^T ds \frac{1}{N} \sum_{j=1}^N E| \mathring{X}_s^j - \mathring{\bar{X}}_s^j |.
\end{aligned}
$$

Using the estimates for $|I_1|_T$ and $|I_2|_T$, we have

$$
\begin{aligned}
N\|X^i - \bar{X}^i\|_{T,1} &= \sum_{i=1}^N \|X^i - \bar{X}^i\|_{T,1} \\
\le \quad c_{abs} \int_0^T ds \{ &\sum_{i=1}^N E|X^i - \bar{X}^i|_{s,1} + \sum_{j=1}^N E|X^j - \bar{X}^j|_{s,1} \} \\
+ \quad \sum_{i=1}^N \int_0^T ds | &\frac{1}{N} \sum_{j=1}^N b(\bar{X}_s^i, \mathring{\bar{X}}_s^j) - \int_{C_T} b(\bar{X}_s, y_s) P^{\mathring{\bar{X}}_s}(dy)|.
\end{aligned}
$$

By the Gronwall lemma
$\|X^i - \bar{X}^i\|_{T,1} \le c_{abs} \int_0^T ds [\frac{1}{N} \sum_{i=1}^N \int_0^T ds | \frac{1}{N} \sum_{j=1}^N b(\bar{X}_s^i, \mathring{\bar{X}}_s^j) - \int_{C_T} b(\bar{X}_s, y_s) P^{\mathring{\bar{X}}_s}(dy_s)| \le c_{abs} \cdot (O(\frac{1}{\sqrt{N}}))$ by the Pyke and Root (1968) inequality. □

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