How Robust is the Value-at-Risk of Credit Risk Portfolios?

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Abstract

In this paper, we assess the magnitude of model uncertainty of credit risk portfolio models, i.e., what is the maximum and minimum Value-at-Risk (VaR) that can be justified given a certain amount of available information. Puccetti and Rüschendorf (2012b) and Embrechts et al. (2013) propose the rearrangement algorithm (RA) as a general method to approximate VaR bounds when the default probabilities, exposures and recovery rates of the different loans are known but not their interdependence. Their numerical results show that the gap between worst-case and best-case VaR is typically very high, a feature that can only be explained by lack of dependence information.

Hence, sharpening the VaR bounds by considering the presence of dependence information is of great practical relevance. In this paper, we propose an efficient algorithm to approximate sharp VaR bounds for credit risk portfolios when besides the marginal distributions also higher order moments of the aggregate portfolio such as variance and skewness are available as sources of dependence information. We also give explicit sharp bounds for homogeneous credit risk portfolios. A numerical study shows that in all practical situations of interest, VaR assessments of credit portfolios that are performed at high confidence levels (as in Solvency II and Basel III) are subject to significant model uncertainty and thus not robust even with the additional moment information.

Keywords Rearrangement algorithm, Moment bounds, Value-at-Risk, Credit risk portfolio.

1 Introduction

The financial crisis that emerged in 2008 has shown that management of credit risk is of utmost importance for the stability of the worldwide financial system. Such stability is intimately connected to the amount of capital that is available as a cushion against adverse events and financial institutions and regulatory authorities use models to determine these capital buffers. In this regard, many industry participants as well as Basel III and Solvency II regulatory frameworks rely on the so-called "Merton's model of the firm" to estimate Value-at-Risk (VaR) of their credit risk portfolios and use this risk number as input to establish capital requirements. In the industry, this model is also essentially known as the KMV model (see also Gordy (2000)) and we refer to this name without further ado. However, like any other credit risk portfolio model, the KMV model requires several ad-hoc assumptions that are hard to justify and is thus inherently subject to model uncertainty. The basic reason for this feature is that large losses of a credit portfolio occur when several loans default together, but lack of default data implies that these joint probabilities are very hard to specify[1] (joint defaults are "rare events").

To illustrate that model uncertainty is a real concern in the context of credit risk portfolio modeling, Chernih et al. (2010) describe a portfolio model that is statistically indistinguishable of the MKMV model in the sense that it uses exactly the same basic parameters: These parameters are the probabilities of default (PDs), the exposures at default (EADs) and the loss given defaults (LGDs) of all individual loans as well as their default (asset) correlations used to describe the interactions among the loans. Yet, these authors show that, under their model, the VaR of a portfolio can be more than fifteen times larger than when using the MKMV model. At first, it may seem surprising that the VaRs of two models can be so different. However, (asset) credit correlations that are used to specify the dependence, in reality only reveal information on the likelihood that exactly *two* loans default together, but they do not make it possible to determine the likelihood that three or more loans default together. The MKMV model effectively deals with this issue by imposing that default of a loan occurs when the assets of the underlying debtor are insufficient to meet the liabilities and assumes also that the asset returns are multivariate normally distributed (with some asset correlation matrix).[2] By contrast, Chernih et al. (2010) make another than Gaussian dependence assumption (while preserving the same correlations) and this choice significantly impacts the VaRs.

Regulators are also increasingly concerned with model uncertainty and consistency of models. In a discussion paper, the Basel Committee 2010) explicitly states that a desired objective of a solvency framework concerns comparability: *"Two banks with portfolios having identical risk profiles apply the framework's rules and arrive at the same amount of risk-weighted assets and two banks with different risk profiles should produce risk numbers that are different proportionally to the differences in risk"*. However, there are a number of other reasons that explain why model uncertainty is important for business. Indeed, an important task of an Enterprise Risk Management (ERM) framework concerns capital (risk) alloca-

[1]Note also that the independence assumption cannot be invoked here as the credit quality of different credit loans depend on common factors such as the state of the global economic environment, industry, geography, monetary policy and so on.

[2]We remark that the use of multivariate normal models is often based on the (wrong) intuition that correlations are enough to model dependence. It is however clear that correlations only are not enough to model dependence, as a number (i.e., the correlation) can never be sufficient to describe the complex interaction between variables unless additional assumptions are made. (see e.g. Embrechts et al. (2013)). This fallacy may then also partially explain why the MKMV model has gained so much support in industry.

tion, i.e., the allocation of total capital held by the insurer across its various constituents (subgroups) such as business lines, risk types, geographical areas, among others. Doing so makes it possible to redistribute the cost of holding capital across the various constituents so that it can be transferred back to the clients in the form of charges (premiums). Risk allocation makes it also possible to assess the performance of the different business lines by determining the return on allocated capital for each line. Finally, the exercise of risk allocation may help to identify areas of risk consumption within a given organization and thus to support the decision making process concerning business expansions an reductions.

In this paper, we aim at assessing the magnitude of model uncertainty of credit risk portfolio models, i.e., what is the maximum (or minimum) value for a certain risk measure (typically the VaR) that can be justified given a certain set of information? In the unconstrained case (i.e., when all PDs, LGDs and EADs are assumed to be known but *not* the dependence), some explicit bounds were found by Rüschendorf (1982) for the two-dimensional case and by Puccetti and Rüschendorf (2012b) for homogeneous portfolios in higher dimensions. However, the problem is fairly more complicated when the portfolio is heterogeneous. In this regard, Puccetti and Rüschendorf (2012a) and Embrechts et al. (2013) propose the Rearrangement Algorithm (RA) to approximate the unconstrained VaR bounds of a heterogeneous portfolio. While their numerical examples provide evidence that the RA is indeed able to approximate the sharp bounds accurately, they also show that the gap between the minimum and the maximum possible VaR is typically very high. In particular, the upper bound on VaR is always larger than the VaR one would obtain in the case that all risks are assumed to be maximally correlated (comonotonic), a situation that is hard to accept by practitioners.

Hence, sharpening the VaR bounds by considering the presence of dependence information (constrained case) is of great practical relevance but also hard to do because, as pointed out before, knowledge of the joint default probabilities is not in reach practically. By contrast, the variance and perhaps also the skewness and the kurtosis of the aggregate portfolio can be estimated statistically and can potentially be used as a source of dependence information allowing to get improvements of the VaR bounds. This idea is actually inherent in Bernard, Rüschendorf and Vanduffel (2013) who propose a version of the RA that incorporates a variance constraint and who show that such constraint can have significant impact on the VaR bounds.

Our paper is a further development of theirs. We focus on credit risk and study portfolio of risky loans. In this context, marginal distributions are Bernoulli distribution that are characterized by a risk exposure (effective loss in case of default) and a default probability. We provide several contributions. We propose an efficient modification of the original RA to approximate sharp VaR bounds. When there are no dependence constraints our modified algorithm does not perform better than the original one. However, our algorithm is also directly applicable when there is dependence information available through knowledge of some higher order moment constraints (variance, skewness, kurtosis, etc.). An important feature of our approach is that it incorporates the statistical uncertainty on these additional moment constraints. Indeed, in practice the available information on the moments appears through statistical point estimates and the moments are never known with certainty. Hence, rather than imposing equality constraints for the moment information we work with inequality constraints as a robust (prudent) approach[3] to estimate VaR. Furthermore, we provide

[3]In presence of inequality constraints, the bounds on VaR will become wider as compared to a situation in which all moments are assumed to be known with certainty.

sharp VaR bounds for homogeneous portfolios. Finally, we provide a detailed numerical study showing that VaR assessments of credit risk portfolios that focus on "deep in the tail events" are not stable and prone to significant model uncertainty that we are able to quantify. We suggest that regulation should be based on VaR at lower confidence levels.

2 Problem description

We consider loan portfolios under the so-called default mode paradigm. Hence, a credit loss occurs if the loan (i.e., the underlying obligor) defaults during the considered time horizon and other value changes (e.g., due to a downgrade) are not recognized. Hence, let I_i be the indicator variable, which is equal to one if the i-th loan defaults and to zero otherwise. The default probability is denoted by p_i:

$$p_i := \mathrm{P}\left[I_i = 1\right].$$

Further, let EAD_i denote the "Exposure-At-Default" and LGD_i the "Loss-Given-Default" of risk i. The "Exposure-At-Default" is the maximum amount of loss on the i-th loan, provided that there is a default. The "Loss-Given-Default" is the percentage of the maximum amount that is effectively lost in the event of a default. We assume that all EAD_i and LGD_i are deterministic and known. The portfolio loss S during the reference period is then given by

$$S = \sum_{i=1}^{n} X_i,$$

in which $X_i = v_i I_i$ and $v_i = EAD_i LGD_i$. Hence, the credit losses X_i follow a scaled Bernoulli distribution (with known scaling factor v_i), i.e., $X_i \sim v_j \mathcal{B}(p_j)$. We denote its distribution by F_i. Without loss of generality, we assume that $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_n > 0$. We aim at computing the worst-case outcome, i.e., the Value-at-risk (VaR), of the portfolio loss S at a given confidence level q ($0 < q < 1$). Hence, we are interested in $\mathrm{VaR}_q^+[S]$ that is defined as

$$\mathrm{VaR}_q^+[S] = \sup\{x \in \mathbb{R} \mid F_S(x) \leqslant q\},$$

in which $F_S(x)$ is the distribution function of S.

It is then clear that a precise computation of VaRs of the portfolio loss S can only be obtained if and only if one knows the joint distribution of the default vector $(I_1, I_2, \ldots, I_n)$. However, this joint distribution is hard to get. In this regard, we point out that financial institutions typically use models that allow specification of default probabilities and default correlations. However, whilst default probabilities and correlations together reveal the level of all pairwise default probabilities (i.e., the specification of the distributions of the pairs (I_i, I_j)), they do not make it possible to completely specify the probability that three or more loans default together. In fact, lack of sufficient default statistics (joint defaults are rarely observed) make it hard, if not impossible, to specify the probabilities that several loans default together so that the joint distribution of $(I_1, I_2, \ldots, I_n)$ cannot readily be specified. In other words, all models that assess VaRs of credit risk portfolios strictly require additional ad-hoc (hard to justify) assumptions to describe the full dependence and all provide different VaR numbers. In this paper we aim at quantifying this inherent uncertainty on VaR estimates.

We first assume that (besides the information on the net exposures v_i) the only information that is available concerns the probabilites of default of each loan, i.e., the distributions of the different default events I_i $(i = 1, 2, \ldots, n)$ are known (but not their joint distribution). In this context, we solve for the maximum and minimum VaR of the portfolio of loans (Section 3). The bounds that we obtain are very wide confirming that using dependence information is crucial in improving the bounds.

However, it appears realistic to have a reasonable estimate for the variance and perhaps even the skewness of the portfolio loss S, providing indirect information on the dependence among credit loans. Hence, in this paper we are interested in the maximum possible VaR of a portfolio of loans in which the loss distributions F_i $(i = 1, 2, \ldots, n)$ of the constituent risky loans are known as well as some higher order moments of the portfolio loss (revealing information on dependence). Of course, these moments are typically not precisely known but have to estimated from available data. In order to capture the statistical uncertainty on these estimates, we propose a robust approach in the sense that we only assume a maximum value c_k for each unknown higher-order moments of S for $(k = 2, 3, \ldots, K)$.[4] Typically, c_k is the point estimate of the k-th moment but it can also be a higher value. In summary, we consider the following problem,

$$\begin{aligned} M \quad &= \quad \sup \mathrm{VaR}_q^+[S] \\ &\qquad \text{subject to } X_j \sim F_j \text{ and } E(S^k) \leqslant c_k \ (k = 2, 3, \ldots, K). \end{aligned} \tag{1}$$

As for the lower bound for VaR, we consider the problem

$$\begin{aligned} m \quad &= \quad \inf \mathrm{VaR}_q[S] \\ &\qquad \text{subject to } X_j \sim F_j \text{ and } E(S^k) \leqslant c_k \ (k = 2, 3, \ldots, K), \end{aligned} \tag{2}$$

in which $\mathrm{VaR}_q[S]$ is defined as

$$\mathrm{VaR}_q[S] = F_S^{-1}(q) = \inf\{x \in \mathbb{R} \mid F_S(x) \geqslant q\}.$$

In what follows, we always tacitly assume that the problems (1) and (2) are welll posed in the sense that there exist portfolios that satisfy the constraints. In particular, by denoting $E(S) := \mu$ and observing that (since $S \geqslant 0$),

$$\mu^k \leqslant E[S^k], \tag{3}$$

it follows that $c_k \geqslant \mu^k$ $(k = 2, 3, \ldots, K)$ will hold.

In our analysis, we make extensively use of two other risk measures, namely Tail Value-at-Risk (TVaR) and Left Tail Value-at-Risk (LTVaR), denoted by $\mathrm{TVaR}_q[S]$ and $\mathrm{LTVaR}_q[S]$, and defined as

$$\mathrm{TVaR}_q[S] = \frac{1}{1-q} \int_q^1 \mathrm{VaR}_u^+[S] du$$

and

$$\mathrm{LTVaR}_q[S] = \frac{1}{q} \int_0^q \mathrm{VaR}_u^+[S] du,$$

respectively. Loosely speaking, TVaR_q is the average of all upper VaRs and LTVaR_q is the average of all lower VaRs.

[4]Note that using inequality constraints is prudent in the sense that the VaR bounds will be wider as compared to a situation in which the moments are assumed to be known (and equal to c_k).

3 VaR Bounds when only the default probabilities are known

In this section, we first recall VaR bounds that were established in prior literature in the case that no dependence information is used at all. In other words, we consider the problems (1) and (2).in which all $c_k = \infty$ $(k = 2, 3, \ldots, K)$. In general, these bounds are not sharp. Therefore, we provide a rearrangement algorithm (RA) that makes it possible to determine (approximate) sharp VaR bounds when there is no information on the dependence available. The algorithm that we propose is inspired by the original RA of Puccetti and Rüschendorf (2012a) and Embrechts et al. (2013). It is as good as the original RA in the unconstrained case (only default probabilities are assumed to be known), but it is also directly applicable when the correlations among the risks are known or in the presence of higher order moment information on the portfolio (Sections 4 and 5). In the special case of credit risk portfolio with the same loss exposure ($\forall i, v_i = v$), we are able to derive explicit sharp bounds. We refer to this situation as an *homogeneous* portfolio of loans.

3.1 Analytical VaR Bounds

In what follows, we represent (without any loss of generality) the n credit risks X_i $(i = 1, 2, \ldots, n)$ as $X_i = f_i(U)$ for some random variable U that has a uniform distribution over $(0, 1)$, $U \sim U(0, 1)$. Each outcome u of U can be effectively interpreted as "a scenario" and translates into a particular loss $f_i(u)$ (that is either 0 or v_i). It follows that $f_i(U)$ and $F_i^{-1}(U)$ have the same distribution, namely F_i, and we say that f_i is a "rearrangement" of F_i^{-1} on $[0, 1]$. In fact, the rearrangements make it possible to describe dependence among the risks and one can show that that $(X_1, X_2, \ldots, X_n) =_d (f_1(U), f_2(U), \ldots, f_n(U))$ in which the f_i (i.e., the rearrangements) are suitably chosen and "$=_d$" reflects equality in distribution; (see Rüschendorf (Lemma 1, 1983) and Puccetti and Rüschendorf (2012)).[5]

Example 3.1 (perfectly dependent risks). Let $n = 2$ and take $f_1(U) = F_1^{-1}(U)$ and $f_2(U) = F_2^{-1}(U)$, then the risks X_1 and X_2 are both increasing in the same variable U. They are perfectly positively dependent (also called comonotonic). By contrast, taking $f_1(U) = F_1^{-1}(U)$ and $f_2(U) = F_2^{-1}(1 - U)$ results in risks X_1 and X_2 that are perfectly negatively dependent (also called antimonotonic).

When the individual risks X_i are comonotonic (i.e., when $f_i = F_i^{-1}$) we denote them by X_i^c and in this instance the portfolio loss is denoted by S^c. It seems intuitive that the highest possible VaR for the portfolio loss occurs when the risks are comonotonic. While this intuition turns out to be incorrect (as we show below), the comonotonic situation is still of great interest for finding VaR bounds. We explain this further as follows.

First, let us observe that for every dependence among the risks (thus also for the comonotonic dependence), the portfolio sum $S = X_1 + X_2 + \cdots + X_n$ satisfies the following inequalities

$$A := \mathrm{LTVaR}_q[S^c] \leqslant \mathrm{VaR}_q[S] \leqslant \mathrm{VaR}_q^+[S] \leqslant B := \mathrm{TVaR}_q[S^c]. \tag{4}$$

A proof for these inequalities can be found in Bernard, Rüschendorf and Vanduffel (2013) for instance. Note that A and B can be expressed as $A = \sum_{i=1}^n \mathrm{LTVAR}_q[X_i]$ and $B =$

[5]The traditional way to describe dependence is by copulas. Indeed, Sklar's theorem states that for a multivariate vector $(X_1, X_2, \ldots, X_n)$ it holds that $(X_1, X_2, \ldots, X_n) =_d (F_{X_1}^{-1}(U_1), F_{X_2}^{-1}(U_2), \ldots, F_{X_n}^{-1}(U_n))$ for some suitable chosen vector $(U_1, U_2, \ldots, U_n)$ in which the U_i are uniformly distributed. The joint distribution of $(U_1, U_2, \ldots, U_n)$ is called a copula.

$\sum_{i=1}^{n}\text{TVAR}_q[X_i]$, respectively. The inequalities (4) show that it is not possible to construct rearrangements f_i of F_i^{-1} with the property that $\text{VaR}_q(S)$ is larger than B.

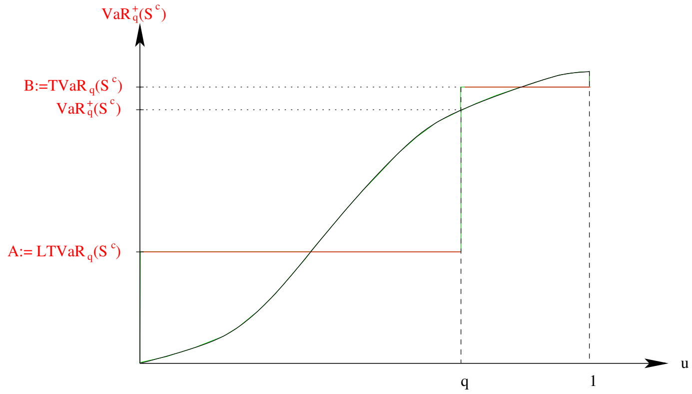

Figure 1: Representation of VaR_q^+ as a function of the level $q \in (0,1)$ for the comonotonic portfolio sum S^c.

In Figure 1, we depict the VaRs of the portfolio loss S^c constructed from comonotonic losses. For a given probability level q, B is the average of the upper VaRs (from level q onwards) in the comonotonic case. It is an upper bound for VaR_q^+ of the comonotonic sum. As the graph in Figure 1 indicates, in order to obtain the best case and worst case VaR one has to choose rearrangements f_i such that the quantile function of S assumes the value A on $[0,q]$ and the value B on $[q,1]$. Thus, we look for rearrangements f_i^* such that the portfolio sum $S^* = \sum_{i=1}^{n} f_i^*(U)$ takes two values only. Specifically, we aim for

$$S^* = \sum_{i=1}^{n} f_i^*(U) = \begin{cases} A & \text{if } U \in [0,q[\\ B & \text{if } U \in [q,1] \end{cases} \tag{5}$$

In general, the lower bound A and the upper bound B are not sharp (attainable), as it is often not possible to change the dependence among the risks such that the quantile function of the portfolio sum S becomes flat on $[0,q]$ and $]q,1]$, respectively.

3.2 Approximate sharp VaR bounds

As mentioned, an explicit dependence structure among the loss variables $X_1,\ldots,X_n$ that makes it possible to achieve the bounds A and B generally does not exist. We thus propose an algorithm that approximates sharp bounds by optimizing over all possible dependence among the $X_1,\ldots,X_n$. In this regard, it is useful to define the auxiliary (extra) variable X_{n+1},

$$X_{n+1} = \begin{cases} -B & \text{with probability } 1-q \\ -A & \text{with probability } q \end{cases} \tag{6}$$

and note that X_{n+1} can be represented as $X_{n+1} = f_{n+1}(U)$ in which f_{n+1} is denoting a rearrangement of $F_{X_{n+1}}^{-1}$. Hence, we can conclude that the bounds A and B are sharp if and only if we find rearrangements f_i^* $(i = 1, 2, \ldots, n+1)$ such that

$$\sum_{i=1}^{n+1} f_i^*(U) = 0 \tag{7}$$

or equivalently,

$$\operatorname{var}\left(\sum_{i=1}^{n+1} f_i^*(U)\right) = 0 \tag{8}$$

Without much loss of (practical) generality, we assume that the confidence level q and the default probabilities p_j are rational numbers so that we we can choose integer numbers d, d_j $(j = 1, 2, \dots, n)$ and k such that

$$\forall j \in \{1, 2, \dots, n\},\ p_j = \frac{d_j}{d} \tag{9}$$

and

$$1 - q = \frac{k}{d} \tag{10}$$

We sample each risk X_j $(j = 1, 2, \dots, n)$ into d equiprobable values. Hence, every X_j takes d values x_{ij} $(i = 1, 2, \dots, d)$ all occurring with probability $1/d$. We use these values to create a $d \times n$ matrix (x_{ij}). Specifically, in the j-th column $(j = 1, 2, \dots, n)$ the last d_j observations take the value v_j and the first $d - d_j$ observations take the value 0. Hence, the $d \times n$ matrix (x_{ij}) can be seen as a representation of a comonotonic loss vector $(X_1, X_2, \dots, X_n)$ and the j-th column is a representation of a variable X_j with loss distribution F_j.

To construct an approximation of the sharp bounds for the VaR, we make use of the following trick that consists in adding an extra column to the matrix. This added column reflects the possible outcomes of a variable X_{n+1} that can take the value $-B$ with probability $1 - q$ and the value $-A$ with probability q, where A and B have been defined in (4). Hence, we obtain the $d \times (n + 1)$ matrix $\mathbf{M}$,

$$\mathbf{M} := \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & x_{1,n+1} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} & x_{2,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{k,1} & x_{k,2} & \cdots & x_{k,n} & x_{k,n+1} \\ x_{k+1,1} & x_{k+2,2} & \cdots & x_{k+1,n} & x_{k+1,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{d,1} & x_{d,2} & \cdots & x_{d,n} & x_{d,n+1} \end{bmatrix},$$

in which

$$x_{1,n+1} = x_{2,n+1} = \cdots = x_{k,n+1} = -B \text{ and } x_{k+1,n+1} = x_{k+2,n+1} = \cdots = x_{d,n+1} = -A. \tag{11}$$

Note that if there exists a dependence structure between the risks X_i such that the bounds A and B are both sharp, then there exists a rearrangement of the matrix $\mathbf{M}$ such that the sum of each row i, $\sum_{j=1}^{n+1} x_{ij} = 0$ exactly. The algorithm that we describe below attempts to obtain this situation as much as possible.

The basic observation to be made is that rearranging values *within* each column of the matrix $\mathbf{M}$ has no impact on the marginal distributions involved, but rather only affects the allocation of the outcomes of the different risks to the different economic scenarios. Hence, consistent with (8), the algorithm effectively consists in rearranging the values within each column such that the rearranged matrix, denoted by $\mathbf{M}^*$, satisfies the condition that all columns are antimonotonic with the sum of all other columns; for this observation, see Puccetti and Rüschendorf (2012a, Theorem 2.1)).[6] Specifically,

Algorithm Consider $\mathbf{M} = (m_{ij})$.

1. Rearrange the values in each column such that the column becomes antimonotonic to the sum of all other columns and denote the matrix after rearrangement by $\mathbf{M}^*$.
2. for $i = 1, 2, \ldots, d$, consider the values $s_i := \sum_{j=1}^{j=n} m_{ij}$ and rank them in increasing order, $s_{[1]} \leqslant s_{[2]} \leqslant \ldots \leqslant s_{[d]}$.
3. The approximation for the lower bound m is then given by $s_{[d-k-1]}$ and the approximation for the upper bound M is given by $s_{[d-k]}$.

The output of the algorithm gives approximate values for m and M that correspond to a dependence between the loans that may happen and that is consistent with the only information available on the individual default probabilities.

Remark 3.2. As suggested by Bernard et al. (2013), it is possible to improve the algorithm by rearranging "blocks of columns" instead of one column at a time. The idea is simple. Split the number of columns into two disjoint sets. Then, sort the rows of each set according to their sums (in the first set, one arranges the rows in increasing order with their sum; in the other set, they are arranged in decreasing order with their sum).

3.3 Explicit VaR bounds for homogeneous portfolios

In this subsection, we assume that all exposures are equal, that is for all i, $v_i = EAD_i LGD_i = v$. In this case, we are able to give explicit sharp bounds. As each loss X_i takes value zero or v, it is clear that the portfolio sum S can only take values that are multiples of v and between zero (no loss) and nv (all loans default). Therefore the bounds A and B established in (4) cannot be sharp (attainable by a potential dependence between the loans) as soon as they are not a multiple of v.

For homogeneous credit risk portfolios the problem of finding a dependence structure that makes it possible to attain the lower and upper VaR of the portfolio of loans can be done without using the algorithm presented above. It is closely related to solving the problem of finding the dependence structure that minimizes the variance. First, we show that the problem

$$(\mathcal{P}) \qquad \begin{array}{l} \min \operatorname{var}(Y_1 + Y_2 + \cdots + Y_n) \\ \text{subject to } \; Y_j \sim v_j \mathcal{B}(p_j). \end{array} \tag{12}$$

can be solved exactly and an algorithm[7] is not needed. Armed with this result we provide sharp VaR bounds (Proposition 3.3). In this regard, it is convenient to introduce the notation $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) to reflect the smallest (resp. largest) integer number that is larger (resp. smaller) than x.

[6]Note indeed that minimizing the variance of a sum requires that each component has minimum correlation with the sum of all other components.

[7]The algorithm that we just described above can essentially be applied too (i.e., after removing the last column of M) but doing so is unnecessary and does not yield the sharp bounds in general. By contrast, when the portfolio is inhomogeneous it makes sense to apply this algorithm to obtain the approximate minimum variance portfolio.

Lemma 3.1 (Minimum variance portfolio)**.** *Consider the problem* $(\mathcal{P})$ *in* (12)*. Assume that the exposues are identical, i.e.* $v := v_j$ $(j = 1, 2, \dots, n)$*. Define for* $j = 1, \dots, n$,

$$a_j = \left(\sum_{i=1}^{j} p_i \right) \bmod 1,$$

and the sets

$$I_j = \begin{cases} [a_{j-1}, a_j] & \text{if } a_j > a_{j-1} \\ [0, a_j] \cup [a_{j-1}, 1] & \text{if } a_j < a_{j-1} \end{cases},$$

where we define $a_0 = 0$*. Then, the solution to* $(\mathcal{P})$ *in* (12) *is obtained for* Y_j^* *given as*

$$Y_j^* = v \mathbb{1}_{U \in I_j}, \tag{13}$$

where U *is a standard uniformly distributed random variable. Furthermore,*

$$\operatorname{var}(Y_1^* + Y_2^* + \cdots + Y_n^*) = v^2 p^* (1 - p^*),$$

where $p^* = \frac{\mu}{v} - \lfloor \frac{\mu}{v} \rfloor$.

Proof. See Appendix A.1. □

We make the following three observations.

(i) The minimum variance portfolio $(Y_1^*, Y_2^*, \dots, Y_n^*)$ has the property that it sum is concentrated on two values around the mean. Precisely,

$$Y_1^* + Y_2^* + \cdots + Y_n^* = \begin{cases} v \lfloor \frac{\mu}{v} \rfloor & \text{with probability } 1 - p^* \\ v \lceil \frac{\mu}{v} \rceil & \text{with probability } p^* \end{cases} \tag{14}$$

(ii) The variance is a traditional measure for comparing variability ("degree of riskiness") among risks. A more general concept to discuss and compare variability of risks is the so-called convex order. One says that a risk X is smaller than a risk Y in the sense of convex order if and only if $E(v(X)) \leqslant E(v(Y))$ for all convex functions v such that the expectations exist. Convex order is consistent with the preferences of all risk averse decision makers (who maximize the expected utility of wealth with a concave utility function). Consequently, it is often argued in the literature that when measuring risk one should use risk measures that are consistent with convex order such as the variance or the TVaR (but unlike VaR). From the proof of Lemma 3.1, one can see that the minimum variance portfolio $(Y_1^*, Y_1^*, \dots, Y_n^*)$ is also a convex minimum (among all portfolios with fixed marginal distributions). In other words, let ρ be a risk measure that is consistent with convex order, then the problem

$$\begin{array}{l} \min \rho(Y_1 + Y_2 + \cdots + Y_n) \\ \text{subject to } \; Y_j \sim v_j \mathcal{B}(p_j). \end{array}, \tag{15}$$

has the same solution as problem $(\mathcal{P})$.

(iii) A more specific algorithm to find the minimum variance portfolio (or more generally, the convex minimum) in the context of heterogeneous credit risk portfolios is available in Appendix A.5. An interesting feature of this specific algorithm is that it converges after $n - 1$ steps (n is the number of loans in the portfolio) to a (local) minimum, a feature that the original algorithm of Puccetti and Rüschendorf (2012a) nor our modification is having.

The next proposition gives exact sharp bounds for the VaR of the portfolio sum in (1) and (2) without moment constraints. It shows that there exists a dependence structure among the risks X_1, X_2,… X_n such that these bounds are attainable.

Proposition 3.3 (Unconstrained VaR bounds). *Consider the problems (1) and (2) in which $c_k = \infty$ $(k = 2, 3, \dots, K)$. Assuming that all exposures are identical $v_i = v$ for all $i = 1, \dots, n$, then*

$$v \left\lceil \frac{A}{v} \right\rceil \leqslant \mathrm{VaR}_q[S] \leqslant \mathrm{VaR}_q^+[S] \leqslant v \left\lfloor \frac{B}{v} \right\rfloor, \tag{16}$$

where $A := \mathrm{LTVaR}_q[S^c]$ and $B := \mathrm{TVaR}_q[S^c]$ (from (4)). Furthermore, these bounds are sharp.

Proof. See appendix A.2. □

The analytical bounds A and B, the approximate sharp bounds obtained by the RA or the exact bounds from Proposition 3.3 can be very wide. This feature will also be confirmed later in the examples and the distance between the largest possible value for VaR and its lowest value can only be reduced by considering possible available information on the dependence among the loans. Therefore, in order to reduce the uncertainty on the estimate of the VaR of a portfolio of loans, we consider different possibilities for incorporating dependence information.

4 VaR bounds when default probabilities and pairwise correlations are known

It is possible to improve the approximations for the VaR bounds when the correlations between the credit losses X_i $(i = 1, 2, \dots, n)$ are available. In this regard, we can assume that n is even (possibly by adding a risk with zero exposure, i.e., by taking $v_n = 0$). When the correlations are known, the distribution of each partial sum $S_{i,j} = X_i + X_j$ $(i \neq j = 1, 2, \dots, n)$ is also known.

4.1 Optimal reduction bounds

One way to use this information is to split all data into disjoint pairs. This split then leads to improved bounds. For every permutation π of $\{1, 2, \dots, n\}$, using a similar reasoning as in (4), we find that for every portfolio sum $S = X_1 + X_2 + \dots + X_n$,

$$\mathrm{VaR}_q^+[S] \leqslant \sum_{i=1}^{n/2} \mathrm{TVaR}_q[S_{\pi(i),\pi(i+1)}]. \tag{17}$$

Any permutation π leads to a reduction bound as in (17) using pairwise distributions.

Proposition 4.1 (Optimal reduction bounds). *Consider the permutation π^* of $\{1, 2, \dots, n\}$ such that*

$$\sum_{i=1}^{n/2} \mathrm{TVaR}_q \left(S_{\pi^*(i),\pi^*(i+1)}\right)$$

is minimized. Hence, for every portfolio sum $S = X_1 + X_2 + \dots + X_n$ holds:

$$C := \sum_{i=1}^{n/2} \mathrm{LTVaR}_q[S_{\pi^*(i),\pi^*(i+1)}] \leqslant \mathrm{VaR}_q[S] \leqslant \mathrm{VaR}_q^+[S] \leqslant D := \sum_{i=1}^{n/2} \mathrm{TVaR}_q[S_{\pi^*(i),\pi^*(i+1)}]. \tag{18}$$

This proposition gives explicit bounds but it can be very hard to compute these bounds exactly because of the too large number of possible permutations. However, it is possible to obtain some approximative bounds by using specific permutations. The idea of using information on pairwise distributions and some related examples on the magnitude of reduction can be also found in Puccetti and Rüschendorf (2012b) and Embrechts and Puccetti ((2009)).

4.2 Numerical evaluation of reduction bounds

As before, the bounds C and D from Proposition 4.1 are not sharp (attainable) as it is typically not possible to change the dependence among the risks such that the quantile function of the portfolio sum S becomes flat on $[0, q]$ and $[q, 1]$, respectively. However, we can apply the algorithm that we described above to approximate sharp bounds. In this regard, we make use of the auxiliary (extra) variable S_{n+1},

$$S_{n+1} = \begin{cases} -D & \text{with probability } 1-q \\ -C & \text{with probability } q \end{cases} \tag{19}$$

It is convenient to use the shorthand notation S_i to denote $S_{\pi^*(i),\pi^*(i+1)}$. Note that the different S_i can takes four values namely, 0, $v_{\pi^*(i)}$, $v_{\pi^*(i+1)}$ and $v_{\pi^*(i)} + v_{\pi^*(i+1)}$, occurring with the appropriate probabilities that are derived from the marginal PDs and the correlations. With no loss of practical generality we can assume that all probabilities are rational numbers. As before, we are going to sample each risk S_j $(j = 1, 2, \ldots, n/2)$ into d equiprobable values that are ordered from low to high (i.e., we start again with a portfolio that exhibits comonotonic dependence). Hence, every S_j takes d values s_{ij} $(i = 1, 2, \ldots, d)$, all occurring with probability $1/d$. The $d \times n$ matrix (s_{ij}) can then be seen as a representation of the multivariate vector $(S_1, S_2, \ldots, S_n)$. Next we add a column, reflecting the variable S_{n+1}, to this matrix to obtain the $d \times (n/2 + 1)$ matrix $\mathbf{S}$

$$\mathbf{S} := \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n/2} & s_{1,n/2+1} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n/2} & s_{2,n/2+1} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{k,1} & s_{k,2} & \cdots & s_{k,n/2} & s_{k,n/2+1} \\ s_{k+1,1} & s_{k+2,2} & \cdots & s_{k+1,n/2} & s_{k+1,n/2+1} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{d,1} & s_{d,2} & \cdots & s_{d,n/2} & s_{d,n/2+1} \end{bmatrix},$$

in which

$$s_{1,n/2+1} = s_{2,n/2+1} = \cdots = s_{k,n/2+1} = -D \text{ and } s_{k+1,n+1} = s_{k+2,n+1} = \cdots = s_{d,n+1} = -C \tag{20}$$

Hence, as before we rearrange the values in the columns of $\mathbf{S}$ such that the rearranged matrix $\mathbf{S}^*$ has the property that all columns are antimonotonic with the sum of all other columns; see algorithm in Section 3.2.

5 VaR bounds when default probabilities and bounds on higher order moments are known

Next, we consider the general constrained VaR maximization and minimization problems (1) and (2) in which $c_k < \infty$ for some $k = 2, 3, \ldots, K$.

5.1 Constrained Bounds

It is clear that the unconstrained bounds A and B are also bounds of the general problem if S^* (as in (5)) satisfies the moment constraints. Otherwise, it means that the two-point variable S^* exhibits too much spread. In this case, in order to obtain VaR bounds, the idea is to construct another two-point variable that is less dispersed (satisfies the constraints) and is "as close as possible" to S^*. To this end, let us define $A(\alpha)$ and $B(\alpha)$ $(0 \leqslant \alpha \leqslant q)$ as

$$B(\alpha) := \frac{1}{1-q} \int_{q-\alpha}^{1-\alpha} \mathrm{VaR}_u^+[S^c]du, \quad A(\alpha) := \frac{E(S) - B(\alpha)(1-q)}{q}, \tag{21}$$

and note that $B(0) = B$ and $A(0) = A$. Consider variables $X_{n+1}(\alpha)$ $(0 \leqslant \alpha \leqslant q)$,

$$X_{n+1}(\alpha) = \begin{cases} A(\alpha) & \text{with probability } 1-q \\ B(\alpha) & \text{with probability } q \end{cases} \tag{22}$$

and note that $X_{n+1}(0) = X_{n+1}$. For ease of exposition, we further denote $X_{n+1}(\alpha)$ by $X(\alpha)$. The moments of $X(\alpha)$ are given by

$$E[(X(\alpha))^k] = A^k(\alpha)q + B^k(\alpha)(1-q) \tag{23}$$

and note that $E[X(\alpha)] = E[S] = \mu$. For each k, $\alpha \to E[(X(\alpha))^k]$ is continuous on $[0, q]$. Precisely, the function first decreases, next increases and has minimum value μ^k (occurring when $A(\alpha) = B(\alpha)$). Note that $\mu^k \leqslant c_k$ (see (3)). Hence, there exists α^* given by

$$\alpha^* := \min\left\{\alpha \mid \mathrm{E}\left[(X(\alpha))^k\right] \leqslant c_k,\ k = 2, 3, \ldots, K\right\}. \tag{24}$$

The following theorem shows that the variable $X(\alpha^*)$ yields upper and lower VaR bounds, $B(\alpha^*)$ and $A(\alpha^*)$. Specifically, we obtain,

Proposition 5.1 (moment-constrained bounds)**.** *Consider the problems (1) and (2) and let α^* be defined by* (24)*. We have that*

$$A(\alpha^*) \leqslant \mathrm{VaR}_q[S] \leqslant \mathrm{VaR}_q^+[S] \leqslant B(\alpha^*).$$

Proof. See Appendix A.3. □

This proposition can be seen as a generalization of Theorem 3.3 in Bernard, Rüschendorf and Vanduffel (2013) who considered the case $K = 2$. The two-point distribution provides the best bounds as possible in all cases, which at first may seem counterintuitive. The reason is that we have inequality on moments and therefore all moment constraints are not binding. Note that Proposition 5.1 also covers the unconstrained case (in this case $\alpha^* = 0$ so that $A(\alpha^*) = A$ and $B(\alpha^*) = B$.

Proposition 5.1 shows that best-possible sharp upper and lower VaR bounds are obtained if one can construct a dependence among the risks X_i $(i = 1, 2, \dots, n)$ such that $S = X_1 + X_2 + \cdots + X_n$ takes values $A(\alpha^*)$ and $B(\alpha^*)$. Hence, one can use the same algorithm as described in Section 3.2 (in the context of the unconstrained problem) for approximating VaR bounds in the constrained situation. The only difference is that the last column in the $d \times (n+1)$ matrix $\mathbf{M} = (x_{ij})$ contains the realizations of the random variable $-X_{n+1}(\alpha^*)$ instead of $-X_{n+1}$, i.e.,

$$\begin{aligned} x_{1,n+1} = x_{2,n+1} = \cdots = x_{k,n+1} = -B(\alpha^*) \text{ and} \\ x_{k+1,n+1} = x_{k+2,n+1} = \cdots = x_{d,n+1} = -A(\alpha^*). \end{aligned} \tag{25}$$

Similarly, as in Section 3.3, the algorithm is not needed in the case of a homogeneous portfolio of risks since explicit sharp bounds can be computed (Proposition 5.2 below).

5.2 Homogeneous portfolios

In this section we assume that $v := v_j$ $(j = 1, 2, \dots, n)$. To discuss sharp bounds it is convenient to consider the following auxiliary variable Y taking three values and explicitly given as

$$Y = \begin{cases} kv & \text{with probability } qz, \\ (k+1)v & \text{with probability } q(1-z), \\ \left\lfloor \frac{B(\alpha^*)}{v} v \right\rfloor & \text{with probability } 1-q, \end{cases} \tag{26}$$

in which $0 \leqslant z \leqslant 1$ and $k \in \mathbb{N}$ are the (unique) values such that $E(Y) = E(S)$. The following proposition provides a sharp upper VaR bound for a homogeneous portfolio. The proof is completely similar to the proof in the unconstrained case (Proposition 3.3) and therefore omitted.

Proposition 5.2 (Sharp moment-constrained bounds for a homogeneous portfolio)**.** *Consider the problems (1) and (2) and define α^* by* (24)*. Assume that the variable Y as defined in(26) satisfies the moment constraints (i.e., $E(Y^k) \leqslant c_k$), then,*

$$\mathrm{VaR}_q^+[S] \leqslant v \left\lfloor \frac{B(\alpha^*)}{v} \right\rfloor. \tag{27}$$

Furthermore, these bounds are sharp.

Similarly, one gets the lower bound

$$\mathrm{VaR}_q(S) \geqslant v \left\lceil \frac{A(\alpha^*)}{v} \right\rceil, \tag{28}$$

which is attained by a corresponding three point distribution assuming that the moment constraints are satisfied.

The variable Y satisfies the moment constraints in particular in the case where $\frac{A(\alpha^*)}{v}$ and $\frac{B(\alpha^*)}{v} \in \mathbb{N}$ (see also the analysis in Section 5.1). In general the moment constraints are satisfied approximatively and (27) gives approximative best bounds. In the case when $K = 2$, it is possible to prove the sharpness of the bounds in (27).

The maximum portfolio VaR is typically strictly larger than the VaR that is obtained when assuming the risks are fully dependent (comonotonic). It has a VaR larger than the sum of the individual VaRs.

In practice, however, it is often believed that comonotonic dependence among the risks should yield the maximum possible VaR in the sense that portfolios that give rise to higher than comonotonic VaRs are then considered as being unrealistic. However, it is not so clear whether comonotonic scenarios are more realistic (i.e., occur more often) than other extreme scenarios. In addition, this feature of having risk bounds that go beyond the comonotonic bounds does not occur when using a measure that is consistent with the convex order (unlike VaR). Nevertheless, if one goes "deep enough in the tail", we still have that the worst case VaR occurs in the case of full dependence.

Proposition 5.3 (Maximum VaR = comonotonic VaR)**.** *Consider the problem (1) and assume there exists a portfolio $(X_1, X_2, \dots X_n)$ that strictly satisfies the moment constraints. Then, there exists $1 > q^* > 0$ and a portfolio $(X_1^*, X_2^*, \dots X_n^*)$ satisfying the constraints such that for $q \in [q^*, 1]$, $\mathrm{VaR}_q^+(X_1^* + X_2^* + \cdots + X_n^*) = \sum_{i=1}^n v_i$.*

Proof. For the proof see Appendix A.4. □

6 Model risk of industry models for credit risk

In this section, we discuss two main models that are used in the financial industry to assess VaRs of credit risk portfolios, namely the KMV model and CrediRisk$^+$ (see Gordy (2000), Vanderdorpe et al. (2008)). Next, we will analyze to which extent these industry standards are robust with respect to model misspecification.

6.1 KMV model (Merton's model of the firm).

Description

Many financial institutions as well as Basel III and Solvency II regulation rely on "Merton's model of the firm" when computing the VaR of a portfolio of loans (see also the survey of McKinsey (2009)). The basic idea is very simple: a default is an event in which *the asset value drops below a threshold value (a liability that is due).* Formally, after normalization, default of the i-th risk occurs when $\{N_i < c_i\}$ where N_i is the normalized asset return and c_i is the threshold value such that $p_i = P(N_i < c_i)$. Merton's model further assumes that the joint asset (log-)returns are multivariate normally distributed. Hence, for a loan portfolio we find that the loss S writes as

$$S = \sum_{i=1}^{n} v_i \mathbb{1}_{N_i < c_i}, \tag{29}$$

in which $(N_1, N_2, \dots, N_n)$ is a multivariate normally distributed with correlation matrix $\boldsymbol{\rho}$. Each N_i can be expressed as a linear combination of independent factors that are standard normally distributed. Specifically,

$$N_i = \sum_{j=1}^{k} \sqrt{\rho_{ij}} M_j + \varepsilon_i \sqrt{1 - \sum_{j=1}^{k} \rho_{ij}}, \tag{30}$$

in which M_j is the explaining factor of the asset return N_j, and in which ε_i represents the idiosyncratic (individual) risk. The weights $\sqrt{\rho_{ij}}$ can be interpreted as the correlation between the i-th return N_i and the factor M_j. It is natural to assume that there is always a portion of idiosyncratic risk that remains inherent in N_i, i.e., $1 - \sum_{j=1}^{k} \rho_{ij} > 0$ and $k < n$. Hence, we can also write the portfolio loss S as

$$S = \sum_{i=1}^{n} v_i I_i, \tag{31}$$

in which I_i is a Bernoulli random variable with (stochastic) probability $p_i(M_1, M_2, \ldots, M_k)$ given as

$$p_i(M_1, M_2, \ldots, M_k) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sum\limits_{j=1}^{k} \sqrt{\rho_{ij}} M_j}{\sqrt{1 - \sum\limits_{j=1}^{k} \rho_{ij}}}\right) \tag{32}$$

and where Φ is the distribution of the standard normal random variable. It is then clear that $\mathrm{VaR}_q\,[S]$ can be obtained for instance using Monte Carlo simulations.

In summary, KMV merely chooses one particular multivariate structure for the default events, while, in fact, there are many models that are consistent with the available information and the different models will differ even when using the same marginal distributions and the same set of default correlations. In other words, for a given set of marginals and correlations, several copulas that preserve the correlations will exist and each of these copulas will give rise to one particular probability distribution function for the total credit portfolio loss; Then, as shown by Frey *et al.* (2001) and Frey and McNeil ((2003)) it is not difficult to build credit risk models that are consistent with this maximum available information while providing very different results.

Single factor model

Assuming a homogeneous portfolio ($v_i = v, \rho_{ij} = \rho$) and asset returns that are driven by one single factor M only, then when the number of loans $n \to \infty$, we find that

$$\lim_{n\to\infty} \mathrm{VaR}_q\left[\frac{S}{nv}\right] = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho}\cdot\Phi^{-1}(q)}{\sqrt{1-\rho}}\right), \tag{33}$$

see also Vasicek (2002). This model is then an example of a one-factor mixture model in which the default event of the obligor is assumed to be driven by a common economic factor M. It can also be seen as the one-factor version of the KMV model that is highly used in the industry and also appears in regulatory frameworks. For example, the Basel III standard framework relies on formula (33) to determine the required capital that banks need to hold for their credit portfolios; see the Basel Committee on Banking Supervision 2010). The Solvency II framework also uses this formula to decide the amount of capital that insurers need to hold as a buffer if reinsurance or derivative counterparts fail.

Assessing model risk in KMV Model

We consider a corporate portfolio of a major European Bank. The portfolio contains 4495 loans mainly to large corporate clients but there are also some loans that were granted to (semi-)public entities. The total exposure (EAD) is 18642.7 and the top 10% of the portfolio (in terms of EAD) accounts for 70.1% of it. In Table 1 we provide some further summary statistics for the portfolio, which confirm that the portfolio exhibits some heterogeneity. The bank also has models in place for getting estimates for the PDs, LGDs, EADS and these will be used for the further analysis. We are first going to compute 99.5%-VaRs assuming the single factor KMV model. Specifically, we make various assumptions on the degree of asset correlation (which we assume constant among the loans) and we report the results of our calculations in Table 1. As the KMV model is fully parameterized once the asset correlation ρ^A and all PDS, LGDs and EADs are fixed, we are also able to compute the moments of the portfolio.

Summary statistics of a corporate portfolio			
	Minimum	Maximum.	Average
Default probability	0.0001	0.15	0.0119
EAD	0	750.2	116.7
LGD	0	0.90	0.41

Table 1: Some summary statistics of a corporate portfolio containing 4495 loans of a major European bank. EADS are reported in mio Euros.

VaR assessment of a corporate portfolio							
		KMV	Comon.	No constraints	$K=2$	$K=3$	$K=4$
	$q=0.95$	297.0	393.5	2153.6	509.7	457.7	439.2
$\rho^A=0.05$	$q=0.99$	430.4	2429.3	7278.4	1001.9	757.9	651.8
	$q=0.995$	487.9	5170.4	10648.0	1366.7	946.9	772.3
	$q=0.999$	625.5	13558.3	15769.3	2843.3	1548.9	1152.0
	$q=0.95$	355.6	393.5	2153.6	641.5	589.6	589.6
$\rho^A=0.10$	$q=0.99$	568.1	2429.3	7278.4	1303.2	987.6	878.9
	$q=0.995$	667.1.	5170.4	10647.9	1791.0	1237.6	1042.4
	$q=0.999$	916.0	13558.3	15769.3	3813.8	2067.8	1548.9
	$q=0.95$	404.7	393.5	2153.6	769.2	733.0	733.0
$\rho^A=0.15$	$q=0.99$	708.8	2429.3	7278.4	1590.5	1233.8	1146.1
	$q=0.995$	858.3	5170.4	10648.0	2198.2	1547.7	1360.8
	$q=0.999$	1251.7	13558.3	15769.3	4767.1	2623.8	1970.9

Table 2: We report the maximum VaRs of a corporate portfolio under the KMV framework for various levels of the asset correlation ρ^A. The VaRs computed under KMV assumptions can be compared with comonotonic VaRs and the VaR upper bounds in the unconstrained and the constrained case (K reflects the number of moments of the portfolio sum that are known). All numbers are in mio Euros.

We make the following observations. First, if we only trust the marginal distributions (no constraints), then the VaR upper bound is very large and very different from the KMV VaR. By adding dependence information, the VaR upper bound sharpens considerably. Furthermore, the higher the probability level the more information one needs to obtain reductions. Finally, from the table, we observe that even if one completely trusts all PDs, LGDs, EADs as well as the first four moments of the portfolio there is still significant model risk left.

6.2 Credit Risk$^+$

Description

Starting from the expression of the loss of a portfolio of loans in (31), it is clear that other assumptions for the default probabilities can be made (reflecting other choices for dependence among the risks), and (32) merely reflects only one such possibility. Note that we can rewrite the portfolio loss (31) as

$$S = \sum_{i=1}^{n} \sum_{j=1}^{I_i} v_i \tag{34}$$

in which I_i is a Bernoulli random variable with (stochastic) probability $p_i(M_1, M_2, \ldots, M_k)$ given by (32). Since the dependent Bernoulli r.v's I_i are too difficult to work with, one substitutes them by other dependent r.v's N_i that are "close" to the I_i but that are more tractable. Hence, we consider

$$S_* = \sum_{i=1}^{n} \sum_{j=1}^{N_i} v_i \tag{35}$$

in which N_i is a Poisson random variable with (stochastic) intensity $p_i(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ given as

$$p_i(\Gamma_1, \Gamma_2, \ldots, \Gamma_k) = p_i \left(w_i + \sum_{j=1}^{K} w_{ij} \Gamma_j \right) \tag{36}$$

The coefficient $w_i \geqslant 0$ reflects the portion of idiosyncratic risk that can be attributed to the i-th risk whereas $w_{ij} \geqslant 0$ reflects its affiliation to the j-th common factor. The random variables Γ_i are assumed to be independent Gamma distributed variables with respective variances σ_i^2. Since for any k and $a > 0$, the r.v. $a\Gamma_k$ will be distributed like a Gamma r.v. we can assume without loss of generality that $E[\Gamma_i] = 1$. Assuming that conditionally on $(\Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2, \ldots, \Gamma_k = \gamma_k)$, the random variables N_i are mutually independent, we find after some computations for the moment generating function S_*,

$$m_{S_*}(t) = \exp\left(\sum_{i=1}^{n} w_i p_i (\exp(tv_i) - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln \left[1 - \sigma_k^2 \sum_{i=1}^{n} w_{ik} p_i (\exp(tv_i) - 1) \right] \right) \tag{37}$$

Using the Fast Fourier Transform, one can easily derive an algorithm that can be used to find the probability distribution function of S_*; see e.g. Haaf et al. (2003).

Single factor model

Similarly as for the KMV model, let us consider the single factor model and assume that there exists a single random variable $\Gamma = \gamma$ representing the "global state of the economy" with variance σ^2 $(= 1/\beta)$ such that, conditionally given $\Lambda = \lambda$, the random variables N_i are Poisson distributed with parameters $p_i\ \lambda$. Then, the expression of the moment generating function in (37) can be simplified to

$$m_{S_*}(t) = \left[\frac{\beta}{\beta - \sum_{i=1}^{n} p_i (\exp(tv_i) - 1)} \right]^{\beta}, \tag{38}$$

which is the moment generating function of a Compound Negative Binomially distributed variable, i.e.,

$$S_* = \sum_{i=1}^{N} Y_i$$

with

$$N \sim \mathcal{NB}\left(\beta, \frac{\beta}{\beta + \sum_{i=1}^{n} p_i}\right)$$

and where the $Y_i \stackrel{d}{=} Y$ are i.i.d. and independent of N, with moment generating function given as

$$m_Y(t) = \frac{\sum_{i=1}^{n} p_i \exp(tv_i)}{\sum_{i=1}^{n} p_i}.$$

The Compound Negative Binomial distribution can be computed using (37) or by using the recursion of Panjer (1981). Observe that this model formally allows that a credit loan defaults more than once. However, a realistic model calibration based on the given default probabilities and the default correlations generally ensures that the probability that this occurs is very small. For more details; see also Credit Suisse (1997).

Assessing model risk in Credit Risk$^+$

We analyze a small concentrated portfolio of 25 exposures as described in Appendix B of Credit Suisse (1997). The exposures range from 0.4 to 20.4 and the total exposure is 130.5 (all numbers mentioned are in mio Euros). As for the default rates, they range between 1.5% and 30% (Credit Suisse (1997), Page 61, Table 9). The portfolio expected loss is then equal to 14.2. Also the portfolio standard deviation is assumed to be known (it is derived from default statistics) and is equal to 12.7; see Credit Suisse (1997, Page 62). Assuming the single factor version of the CreditRisk+ model, it is then straightforward to compute the value of the remaining parameter β in (38) (by simple moment matching). Next, the VaRs can be computed using Panjer's recursion for instance. The results of the CreditRisk$^+$ model are given in the second column of Table 3; see also page 63 in Credit Suisse (1997). For example, the 99.5%-VaR is equal to 62 mio Euros.

VaR assessment of a small portfolio (Appendix B of Credit Suisse (1997))						
	CreditRisk$^+$	Comon.	No constraints	$K = 2$	$K = 3$	$K = 4$
$q = 0.750$	20.5	21.9	(1.4; 52.5)	(15.4; 31.6)	(15.4; 31.6)	(15.4; 31.6)
$q = 0.950$	38.9	85.2	(8.8; 117.4)	(20.2; 64.6)	(20.2; 55.1)	(20.2; 51.6)
$q = 0.990$	55.3	130.5	(13.0; 130.5)	(20.2; 130.5)	(20.2; 94.9)	(20.2; 80.7)
$q = 0.995$	62.0	130.5	(13.6; 130.5)	(20.2; 130.5)	(20.2; 115.1)	(20.2; 94.8)
$q = 0.999$	77.1	130.5	(14.1; 130.5)	(20.2; 130.5)	(20.2; 130.5)	(20.2; 130.5)

Table 3: Column 2 contains VaRs under the Credit Risk$^+$. They can be compared with comonotonic VaRs and the VaR bounds (displayed in brackets) in the unconstrained and the constrained case (K reflects the number of moments of the portfolio sum that are known). All numbers are in mio Euros.

However, other modeling assumptions could be made. For instance, if we only trust the marginal distributions and the portfolio variance (moments up to $K = 2$), we observe from Table 3 that the true 99.5%-VaR can actually be any value between 13.6 and 130.5

showing that the 99.5%-VaR can easily be underestimated by a factor 2. In Table 3, we also provide VaR bounds assuming that more information on the higher order moments[8] is available. This table makes clear that adding higher order information reduces the gap between the upper and lower bound for VaR significantly, thus reduces model uncertainty on VaR assessment. When the probability level q that is used to assess the VaR is not too big (e.g., $q = 0.95$) then this has a positive impact on model uncertainty in the sense that the range of possible VaRs becomes "reasonable" as soon as second and/or third moment information is added. At high probability levels (e.g., $q = 0.999$), significant model risk is always present.

6.3 Assessing model risk when using a Beta distribution

The Beta distribution has always been a benchmark model for credit portfolio risk calculations. To discuss the model uncertainty let us consider the portfolio presented on page 365 in McNeil et al. (2005). Using the same parameters as in Table 8.6 page 365 from McNeil et al. (2005), we set the default probability of all loans equal to $p = 0.049$ and for the correlation we take $\rho^D = 0.0157$. The variance of the portfolio sum of n correlated loans (all with exposure that is equal to $1/n$) can thus be easily calculated. Next, the two parameters of the beta distribution can be inferred (by moment matching) and one can then compute higher order moments. We use the associated higher moments as the moment constraints. Specifically, we consider the Beta distribution for the sum of $n = 10,000$ loans. We are interested in the VaR of the portfolio of loans at confidence levels 95%, 99% and 99.5%. The discretization parameter is set to $d = 10,000$.

$n = 10,000$				
	$\text{VaR}_{95\%}$	$\text{VaR}_{99\%}$	$\text{VaR}_{99.5\%}$	$\text{VaR}_{99.9\%}$
Beta	10%	13.1%	14.4%	17.1%
Comon.	0%	100%	100%	100%
K=2	16.72%	31.89%	43.17%	90.65%
K=3	14.95%	24.29%	30.24%	50.95%
K=4	14.00%	20.55%	24.34%	36.23%
K=5	13.52%	18.53%	21.26%	29.28%

Table 4: We report VaRs for a homogeneous portfolio assuming the portfolio loss follows a Beta distribution. We provide then exact upper and lower VaR bounds.

According to the numerical results, we observe that taking additional moments constraints into account can improve the VaR bounds significantly, especially at the high percentage (e.g. 99.5% in Table 4). We observe that including information on the skewness and kurtosis (the 3rd and 4th moment) reduces the model risk on the VaR assessment. But bounds on VaR are still very wide even when we include information on the first 5 moments of the portfolio sum.

[8] They have been computed under the specification of the Credit Risk$^+$ model.

7 Conclusions

In order to assess credit portfolio risk one needs to model the marginal risks as well as the way they interact, i.e. the dependencies. Dependence modeling in a credit context usually focuses on finding the economic dimensions that influence the default behavior for the different loans. Apart from factors describing the global state of the economy such as, for example, interest rates the default drivers that are typically considered are asset size, industry sector and geographical situation; see Lopez (2002), Duellmann and Scheule (2003), Duellmann et al. (2008) or Dietsch and Petey (2004). As a result companies of similar size, industry activity and geographical situation will be grouped together meaning that they behave similarly, which is akin to saying that they are positively dependent. However, all these dimensions together still do not fully capture all sources of dependence and in this paper we assess the potential impact for some of these sources of uncaptured dependence across portfolios. Specifically, we study model risk on the estimation of VaR of a portfolio of loans when one knows the individual default probabilities and possibly the pairwise correlation or bounds on some moments of the portfolio. We provide some explicit results for homogeneous portfolios and also develop an efficient algorithm to deal with the general case.

We show that the VaR computed in typical credit models that the financial institutions report do not necessarily reflect the true risk and are hard to confirm. In this regard we note that under the internal model approach of Basel III and Solvency II, the financial institutions are allowed to use their own model for setting their capital requirements. However, it is hard, if not impossible, to show which model is better than the others, as they are all prone to model error, in particular when high confidence levels are used. In practice, one typically has information on the variance, the skewness and may be on further moments of the portfolio and it seems that in this case VaR assessments at lower confidence levels (e.g., $q = 0.95$) are more "stable" and in reasonable range while for higher levels as $q \geqslant 0.99$ the range of values typically remains wide. In summary, we do not recommend as currently used in Basel III or Solvency II to compute the VaR of portfolios at high confidence levels with specialized models when no further information is available to support the use of these models.

A Appendix

A.1 Proof of Lemma 3.1

Proof. Let us first observe that $Y_j^* \sim v\mathcal{B}(p_j)$. Furthermore, one can easily verify that $S_n^* = Y_1^* + Y_2^* + \cdots + Y_n^*$ only takes values ℓv with probability $(1-p^*)$ or $(\ell+1)v$ with probability [9] p^*. It is straightforward to show that

$$\mathrm{var}(S_n^*) = v^2 p^*(1-p^*),$$

and we only need to show that any other sum $S_n = Y_1 + Y_2 + \cdots + Y_n$ with $Y_j \sim v\mathcal{B}(p_j)$ has a larger variance. Consider any sum S_n. In particular, S_n takes values in $\{0, v, 2v, \ldots, nv\}$ with respective probabilities $q_0, q_1, \ldots, q_n$.

It is clear that $\forall x \in]0, \ell v[$, $F_{S_n}(x) \geqslant F_{S_n^*}(x) = 0$ and $\forall x \in [(\ell+1)v, +\infty[$, $F_{S_n}(x) \leqslant F_{S_n^*}(x) = 1$. Since $F_{S_n}(x)$ and $F_{S_n^*}(x)$ are constant on the interval $[\ell v, (\ell+1)v[$ one has,

$$\exists c \geqslant 0, \quad \begin{cases} \forall x \in (0, c), & F_{S_n}(x) \geqslant F_{S_n^*}(x) \\ \forall x \in (c, +\infty), & F_{S_n}(x) \leqslant F_{S_n^*}(x) \end{cases} \tag{39}$$

namely, $c = (\ell+1)v$ if $F_{S_n}(\ell v) > F_{S_n^*}(x)$ and $c = \ell v$ if $F_{S_n}(\ell) \leqslant F_{S_n^*}(x)$. In other words, the distribution function F_{S_n} crosses $F_{S_n^*}$ exactly once from above. Since $\mathbb{E}(S_n) = \mathbb{E}(S_n^*)$ this implies the well-known fact that $\mathbb{E}(h(S_n^*)) \leqslant \mathbb{E}(h(S_n))$ for all convex functions $h(x)$ (see, for example, Müller and Stoyan (2002)). Taking $h(x) = x^2$ ends the proof. □

A.2 Proof of Proposition 3.3

Proof. The proof follows from Lemma 3.1 essentially. Consider variables Y_i given as $Y_i = v\mathbb{1}_{U \leqslant q} V_i + v\mathbb{1}_{U > q} W_i$ in which V_i and W_i are Bernoulli distributed random variables that are independent of the uniform random variable U and such that $Y_i \sim vB(p_i)$ $(i = 1, 2, \ldots, n)$, $\mathrm{E}(\sum_{i=1}^n V_i) = \frac{A}{v}$, $\mathrm{E}(\sum_{i=1}^n W_i) = \frac{B}{v}$. Applying Lemma 3.1, they can be chosen and such that the portfolio sum $\sum_{i=1}^n Y_i$ takes four values, namely $v\left\lfloor \frac{A}{v} \right\rfloor$, $v\left\lceil \frac{A}{v} \right\rceil$, $v\left\lfloor \frac{B}{v} \right\rfloor$ and $v\left\lceil \frac{B}{v} \right\rceil$. One observes that for this dependence among V_i and W_i, $\mathrm{VaR}\, q^+[Y_1 + Y_2 + \cdots + Y_n] = v\left\lfloor \frac{B}{v} \right\rfloor$ and $\mathrm{VaR}_q[Y_1 + Y_2 + \cdots + Y_n] = v\left\lceil \frac{A}{v} \right\rceil$, which ends the proof. □

A.3 Proof of Proposition 5.1

Proof. We show that $B(\alpha^*)$ is an upper bound of M. To this end, assume that there exists $T = \sum_{i=1}^n X_i$ such that satisfies all moment constraints, i.e. $\mathrm{E}[T^k] \leqslant c_k$, $k = 2, 3, \ldots, K$ and that $\mathrm{VaR}_q^+[T] > B(\alpha^*)$. Denote the distribution function of $X(\alpha^*)$ as G. Then $\forall a \leqslant x < b$, $F_T(x) \leqslant G(x) = q$. When $b \leqslant x$, $F_T(x) \leqslant G(x) = 1$. Since $G(x) = 0$ for $x < a$, this implies that,

$$\begin{cases} \forall x < a, & F_T(x) \geqslant G(x) \\ \forall x \geqslant a, & F_T(x) \leqslant G(x) \end{cases} \tag{40}$$

In other words, the distribution function F_T crosses G once from above. Since $\mathbb{E}[T] = \mathbb{E}(X[\alpha^*)]$, this implies that $X_{\alpha^*}^* \leqslant_{cx} T$ (cut criterion of Karlin and Novikoff (1963)). Moreover, since T and $X^k(\alpha^*)$ are positive, it holds that $\mathrm{E}[X^k(\alpha^*)] \leqslant \mathrm{E}[T^k]$ for all $k =$

[9] Note that $p = 0$ may hold.

$2, 3, \ldots, K$. On the other hand, there exists $\widehat{k} \geqslant 2$ such that $\mathrm{E}[X^{\widehat{k}}(\alpha^*)] = c_{\widehat{k}}$ (by definition of α^*). This implies that $\mathrm{E}[T^{\widehat{k}}] = \mathrm{E}[X^{\widehat{k}}(\alpha^*)]$ (because $\mathrm{E}[T^{\widehat{k}}] \leqslant c(\widehat{k})$ because of the moment constraint). As $\phi(x) = x^{\widehat{k}}$ ($\widehat{k} \geqslant 2$) is strictly convex, it follows that $T \stackrel{d}{=} X(\alpha^*)$ must hold (Shaked and Shanthikumar (2007, Theorem 3.A.43)). This ends the proof. The proof that $A(\alpha^*)$ is an absolute lower bound can be done in a similar way. □

A.4 Proof of Proposition 5.3

Proof. Without loss of generality we can express the X_i as $X_i = F_i^{-1}(V_i)$ for uniformly distributed $V_i.(i = 1, 2, \ldots, n)$. Next, we consider variables $X_i^* = \mathbb{1}_{U \leqslant u} F_i^{-1}(uV_i) + \mathbb{1}_{U > u} F_i^{-1}(U)$ in which $0 < u < 1$ is chosen such that $(X_1^*, X_2^*, \ldots, X_n^*)$ also satisfies the moment constraints. One observes then that for $q > q^* := \max(1 - \min(p_1, p_2, \ldots, p_n), u)$, $\mathrm{VaR}_q^+(X_1^* + X_2^* + \cdots + X_n^*) = \sum_{i=1}^n v_i$. □

A.5 Specific Algorithm for getting minimum variance portfolio of risky loans

Algorithm

Recall that $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_d > 0$.

1. Put in the first column of the matrix $\mathbf{M}^*$ the vector $[0, \ldots, 0, v_1, \ldots, v_1]^T$ where v_1 appears d_1 times.
2. For $j = 2, 3, \ldots, n$, add in the j-th column, the vector $[0, \ldots, 0, v_1, \ldots, v_1]^T$ where v_j appears d_j times, such that it is antimonotonic to the sum of the $j - 1$ first vectors of the matrix.

The matrix $\mathbf{M}^*$ that we obtain as an output of this algorithm is a representation of a random vector $(X_1^*, X_2^*, \ldots, X_n^*)$ satisfying $X_i^* \sim F_i$ and we only need to show it is a possible solution to the minimum variance problem $\mathcal{P}$ given by (12).

Proposition A.1 (convergence of algorithm)**.** *The random vector $(X_1, X_2, \ldots, X_n)^T$ as constructed above gives rise to a a candidate solution for Problem ($\mathcal{P}$) in* (12)*. That is, for all $l = 1, 2, \ldots, n$, X_l is antimonotonic with $\sum_{k=1, k \neq l}^n X_k$.*

Proof. Fix some elements l_{kj} and $l_{\ell j}$. By construction,

$$(l_{kj} - l_{\ell j}) \sum_{r=1}^{j-1} (l_{kr} - l_{\ell r}) \leqslant 0.$$

We first want to prove that

$$(l_{kj} - l_{\ell j}) \sum_{r=1, r \neq j}^{n} (l_{kr} - l_{\ell r}) \leqslant 0. \tag{41}$$

Clearly, the difference $|l_{kr} - l_{\ell r}|$ is either equal to 0 or v_r $(r = 1, 2, \ldots, n)$. If $l_{kj} - l_{\ell j} = 0$, then property (41) is obvious. Without loss of generality, assume then that $l_{kj} - l_{\ell j} > 0$, i.e. $l_{kj} - l_{\ell j} = v_j$. We want to prove that

$$\sum_{r=1, r\neq j}^{n} (l_{kr} - l_{\ell r}) \leqslant 0. \tag{42}$$

We know already that $\sum_{r=1}^{j-1}(l_{kr} - l_{\ell r}) \leqslant 0$ because by construction the j-th vector $[v_j, \ldots, v_j, \ldots, 0]^T$ is antimonotonic with the sum of $j-1$ first vectors. Hence, if for all $s > j$, $l_{ks} - l_{\ell s} \leqslant 0$ then it is clear that (42) holds true. Let thus $s > j$ be the smallest element such that $l_{ks} - l_{\ell s} > 0$ and observe that $l_{ks} - l_{\ell s} = v_s$. We then have that

$$\sum_{r=1}^{s-1}(l_{kr} - l_{\ell r}) = \sum_{r=1}^{j-1}(l_{kr} - l_{\ell r}) + v_j + \sum_{r=j+1}^{s-1} (l_{kr} - l_{\ell r}) \leqslant 0.$$

Since $v_s \leqslant v_j$, we obtain,

$$\sum_{r=1, r\neq j}^{s} (l_{kr} - l_{\ell r}) = \sum_{r=1}^{s-1}(l_{kr} - l_{\ell r}) - v_j + v_s \leqslant 0.$$

Next, we consider the first element $s' > s$ such that $(l_{ks'} - l_{\ell s'}) = v_{s'} > 0$. Then,

$$\sum_{r=1}^{s'-1}(l_{kr} - l_{\ell r}) \leqslant 0.$$

Therefore,

$$\sum_{r=1}^{s'-1}(l_{\ell j} - l_{\ell i}) = \sum_{r=1}^{j-1}(l_{kr} - l_{\ell r}) + v_{s'} + \sum_{r=s+1}^{s'-1} (l_{kr} - l_{\ell r}) \leqslant 0,$$

and thus, as $v_{s'} \leqslant v_j$,

$$\sum_{r=1, r\neq j}^{s'} (l_{kr} - l_{\ell r}) \leqslant 0.$$

We repeat this until we reach the last element with the property that $(l_{kh} - l_{\ell h}) > 0$. We obtain,

$$\sum_{r=1, r\neq j}^{h} (l_{kr} - l_{\ell r}) \leqslant 0.$$

All remaining elements are non-positive, therefore

$$\sum_{r=1, r\neq j}^{n} (l_{kr} - l_{\ell r}) \leqslant 0.$$

We thus have proven that (42) holds and since this holds for all j, k and ℓ thus also that every X'_j is antimonotonic with $\sum_{r=1, r\neq j}^{n} X'_r$. This ends the proof. $\square$

References

Basel Committee on Banking Supervision. (2006). Basel II: International Convergence of Capital Measurement and Capital Standards: A Revised Framework – Comprehensive Version.

Basel Committee on Banking Supervision. (2010). Basel III: A global regulatory framework for more resilient banks and banking systems.

Bernard. C., Rüschendorf. L., Vanduffel. S (2013). Value-at-Risk bounds with variance constraints, *Working Paper available at SSRN.*

Bernard., C., Denuit, M., Vanduffel. S (2014). Measuring Portfolio Risk Under Partial Dependence Information. Available at SSRN: `http://dx.doi.org/10.2139/ssrn.2406377`.

Chernih, A., Henrard, L., Vanduffel, S. (2010). Reconciling asset correlations. *The Journal of Risk Model Validation,* **4**(2), 47–64.

Credit Suisse Financial Products (1997). CreditRisk$^+$, A Credit Risk Management Framework, *Technical document*, available online at *www.csfb.com/creditrisk*

de Servigny, A., Renault, O. (2002). Default correlation: Empirical evidence. *Standard & Poor's.*

Denuit, M., Genest, C., Marceau, É. (1999). Stochastic bounds on sums of dependent risks. *Insurance: Mathematics and Economics,* **25**, 85–104.

Dietsch, M., Petey, J. (2004). Should SME exposures be treated as retail or corporate exposures? A comparative analysis of default probabilities and asset correlations in French and German SMEs. *Journal of Banking and Finance,* **28**, 773–788.

Duellmann, K., Scheicher, M., Schmieder, C. (2008). Asset correlations and credit portfolio risk: an empirical analysis. *Journal of Credit Risk*, **4**(2), 37–62.

Duellmann, K., Scheule, H. (2003), Determinants of the asset correlations of German corporations and implications for regulatory capital. *Unpublished Working Paper.*

Embrechts, P., Puccetti, G., (2009). Bounds for the sum of dependent risks having overlapping marginals. *Journal of Multivariate Analysis,* **101**, 177–190.

Embrechts, P., Puccetti, G., Rüschendorf, L., (2013). Model uncertainty and VaR aggregation, *Journal of Banking and Finance,* **37**, 2750–2764.

Frey, R., McNeil, A., Nyfeler, M.A. (2001). Modelling Dependent Defaults: Asset Correlations Are Not Enough!, http://e-collection.ethbib.ethz.ch/show?type=bericht&nr=273, Department of Mathematics, ETHZ, Zurich.

Frey, R., McNeil, A. (2003). Dependent defaults in models of portfolio credit risk. *Journal of Risk,* **6**(1), 59–92.

Frye, J. (2008). Correlation and Asset Correlation in the Structural Portfolio Modd. *Journal of Credit Risk,* **4**(2), 75–96.

Genest, C., Marceau, É. and Mesfioui, M. (2002). Upper stop-loss bounds for sums of possibly dependent risks with given means and variances. *Statistics & Probability Letters,* **57**, 33–34.

Gordy, M. (2000). A comparative anatomy of credit risk models. *Journal of Banking and Finance,* **24**(1–2), 119–149.

Haaf, H., Reiss. O., Schoenmakers. J. (2003). Numerically stable computation of CreditRisk$^+$. *Technical report, Weierstrass-Institut.*

Hamerle, A., Rösch D. (2006). Parameterizing credit risk models. *Journal of Credit Risk* **3**, 101–122,

Karlin, S., Novikoff, A. (1963). Generalized convex inequalities. *Pacific Journal of Mathematics,* **13**, 1251–1279.

Lopez, J. A. (2002), The empirical relationship between average asset correlation, firm probability of default and asset size. *Working Paper*, BIS Workshop: "Basel II: An Economic Assessment".

McKinsey (2009). Best Practices for estimating Economic Capital. *Mc Kinsey's Working Papers on Risk*, 11, April 2009.

McNeil, A. J., Frey, R., Embrechts, P. (2005). *Quantitative Risk Management: Concepts, Techniques, Tools.* Princeton, NJ: Princeton University Press.

Meilijson, I., Nadas, A. (1979). Convex majorization with an application to the length of critical paths. *Journal of Applied Probability,* **16**, 671–677.

Müller, A., Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks.* John Wiley & Sons, Inc.

Panjer, H. H. (1981). Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* **12**(1), 22–26.

Pitts, A. (2004). Correlated defaults: Let's go back to the data. *Risk Magazine*, **17**(6), 75–79.

Puccetti, G., Rüschendorf, L. (2012a). Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Maths*, **236**, 1833–1840.

Puccetti, G., Rüschendorf, L. (2012b). Bounds for joint portfolios of dependent risks. *Statistics and Risk Modeling*, **29**, 107–132.

Rösch, D., Scheule. H. (2007) Stress-Testing Credit Risk Parameters: An Application to Retail Loan Portfolios. *Journal of Risk Model Validation* **1**(1), 55–75.

Rüschendorf, L. (1982). Random variables with maximum sums. *Advances in Applied Probability*, **14**, 623–632.

Rüschendorf, L. (1983). Solution of a statistical optimization problem by rearrangement methods, *Metrika*, **30**, 55–61.

Rüschendorf, L. (2013). *Mathematical Risk Analysis*, Springer.

Shaked, M., Shanthikumar, J.G. (2007). *Stochastic Orders.* Springer.

Vandendorpe, A., Ho, N., Vanduffel, S., Van Dooren, P. (2008). On the parameterization of the CreditRisk+ model for estimating credit portfolio risk. *Insurance: Mathematics and Economics*, **42**, 736–745.

Vanduffel, S., Shang, Z., Henrard, L., Dhaene, J., Valdez, E. (2008). Analytic bounds and approximations for annuities and Asian options. *Insurance: Mathematics and Economics*, **42**, 1109–1117.

Vasicek, O. (2002). Loan portfolio value, *Risk Magazine*, December 2002, 160–162.