RISK BOUNDS, WORST CASE DEPENDENCE, AND OPTIMAL CLAIMS AND CONTRACTS

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Abstract

Some classical results on risk bounds as the Fréchet bounds, the Hoeffding–Fréchet bounds and the extremal risk property of the comonotonicity dependence structure are used to describe worst case dependence structures for portfolios of real risks. An extension of the worst case dependence structure to portfolios of risk vectors is given. The bounds are used to (re-)derive and extend some results on optimal contingent claims and an optimal (re-)insurance contracts.

1. Risk bounds and comonotonicity

For a risk vector $X = (X_1, \dots, X_n)$ of risks X_i with distributions P_i resp. distribution functions F_i it is a classical problem to determine (sharp) bounds for a risk functional of the form $E\Psi(X)$ induced by dependence between the components X_i of X. The class of all possible dependence structures is given by the Fréchet class $\mathcal{M}(P_1, \dots, P_n)$ of joint distributions with marginals P_i. For the case of real risks one can consider equivalently the class $\mathcal{F}(F_1, \dots, F_n)$ of joint distribution functions with marginal distribution functions $F_i \sim P_i$.

The sharp upper and lower dependence bounds for the risk function Ψ are given by

$$\begin{aligned} M(\Psi) &= \sup\left\{\int \Psi dP; P \in M(P_1, \dots, P_n)\right\} \\ \text{and} \quad m(\Psi) &= \inf\left\{\int \Psi dP; P \in M(P_1, \dots, P_n)\right\}. \end{aligned} \tag{1}$$

They are called *(generalized) upper resp. lower Fréchet bounds*. Typical risk functionals of interest are in the case of real risks X_i risk functionals of the joint portfolio like $(\sum_{i=1}^n X_i - K)_+$, $1_{[t,\infty)}(\sum_{i=1}^n X_i)$ or $\max_i X_i$ leading to bounds for the excess of loss, for the value at risk and for the maximal risk of the joint portfolio.

The following three examples of sharp risk bounds are classical examples of generalized Fréchet bounds.

B1) Sharpness of Fréchet bounds
If X_i are real random variables with distribution functions F_i, $1 \le i \le n$, then for the distribution function $F = F_X$ of $X = (X_1, \dots, X_n)$ the following bounds are sharp:

$$F_c(x) := \Big(\sum_{i=1}^n F_i(x_i) - (n-1)\Big)_+ \le F(x) \le F^c(x) := \min F_i(x_i). \tag{2}$$

The upper bound $F^c(x)$ is the distribution function of the comonotonic vector $X^c := (F_1^{-1}(U), \dots, F_n^{-1}(U)) = (X_1^c, \dots, X_n^c)$ where $U \sim U(0,1)$. In consequence the upper bound in (2) is sharp. The lower bound $F_c(x)$ is a distribution function if $n = 2$ and then corresponds to the antithetic (countermonotonic) vector

$$X^{cm} := (F_1^{-1}(U), F_2^{-1}(1-U)) = (X_1^{cm}, X_2^{cm}).$$

The bounds in (2) go back to Fréchet (1951) and Hoeffding (1940) for $n = 2$. The upper and lower bounds were described in Dall'Aglio (1972). Sharpness of the lower bound in (2) was first given in Rü[1] (1981).

B2) Hoeffding–Fréchet bounds
For real random variables X_1, X_2 Hoeffding (1940) found the following representation of the covariance:

$$\mathrm{Cov}(X_1, X_2) = \int\int (F(x,y) - F_1(x)F_2(y))dxdy. \tag{3}$$

Together with the Fréchet bounds in (2) this representation implies the sharp upper and lower Hoeffding–Fréchet bounds:

$$\mathrm{Cov}(F_1^{-1}(U), F_2^{-1}(1-U)) \le \mathrm{Cov}(X_1, X_2) \le \mathrm{Cov}(F_1^{-1}(U), F_2^{-1}(U)), \tag{4}$$

or, equivalently,

$$EF_1^{-1}(U)F_2^{-1}(1-U)) \le EX_1X_2 \le EF_1^{-1}(U), F_2^{-1}(U)). \tag{5}$$

The comonotonic resp. countermonotonic vectors are the unique (in distribution) vectors which attain the upper resp. lower risk bounds in (4), (5).

If $V \sim U(0,1)$ is a random variable uniformly distributed on $(0,1)$ and independent of X_1, X_2 then defining the distributional transform

$$U_i := F_i(X_i, V) = \tau_{X_i}, \quad i = 1, 2, \tag{6}$$

where $F_i(x, \lambda) := P(X_i < x) + \lambda P(X_i = x)$ are the modified distribution functions. Then

$$U_i \sim U(0,1) \quad \text{and} \quad X_i = F_i^{-1}(U_i) \ \text{a.s.} \tag{7}$$

[1] Rüschendorf is abbreviated with Rü in this paper.

In fact the pair (U_1, U_2) is a copula vector of X (see Rü (1981, 2009)). Further the pairs

$$\begin{aligned} &(X_1, F_2^{-1}(F_1(X_1, V))) = (X_1, F_2^{-1}(\tau_{X_1})) \\ \text{and } &(X_1, F_2^{-1}(1 - F_1(X_1, V))) = (X_1, F_2^{-1}(1 - \tau_{X_1})) \end{aligned} \tag{8}$$

are comonotonic resp. countermonotonic pairs with marginal distribution functions F_1, F_2 and thus attain the upper resp. lower Hoeffding–Fréchet bounds in (5). The interesting point in (8) is that the solution can be written as pair $(X_1, F_2^{-1}(\tau_{X_1}))$ resp. $(X_1, F_2^{-1}(1-\tau_{X_1}))$ with distributional transform $\tau_{X_1} = F_1(X_1, V)$ which is increasing in (X_1, V).

B3) Comonotonic vector as worst case dependence structure

The third classical result concerns sharp upper bounds on the excess of loss. It states that the comonotonic vector $X^c = (F_1^{-1}(U), \dots, F_n^{-1}(U))$ is the *worst case dependence structure* w.r.t. excess of loss. Formulated in terms of convex ordering $\leq_{\text{cx}}$ it says:

$$\text{If} \quad X_i \sim F_i, 1 \leq i \leq n, \text{ then} \quad \sum_{i=1}^n X_i \leq_{\text{cx}} \sum_{i=1}^n F_i^{-1}(U). \tag{9}$$

This result was first established in Meilijson and Nadas (1979) together with the following equivalent representation: For all $d^* \in \mathbb{R}^1$ holds

$$\begin{aligned} \sup_{X_i \sim F_i} E\Big(\sum_{i=1}^n X_i - d^*\Big)_+ &= E\Big(\sum_{i=1}^n F_i^{-1}(U) - d^*\Big)_+ \\ &= \Psi_+(d) := \inf_{\sum_{i=1}^n d_i = d^*} \sum_{i=1}^n E(X_i - d_i)_+. \end{aligned} \tag{10}$$

For continuous distribution functions one can choose a solution (d_i^*) of (10) as

$$d_i^* = F_i^{-1}\Big(F_{\sum_{i=1}^n X_i^c}(d^*)\Big). \tag{11}$$

In general, if d^* is a u_0-quantile of $\mathcal{L}(\sum_{i=1}^n X_i^c)$, then d_i^* can be chosen as u_0-quantiles of F_i.

As consequence of (9) one obtains that

$$\Psi\Big(\sum_{i=1}^n X_i\Big) \leq \Psi\Big(\sum_{i=1}^n X_i^c\Big) \tag{12}$$

for all law invariant convex risk measures Ψ (see Föllmer and Schied (2004), Burgert and Rü (2006)). Thus the comonotonic risk vector is in this sense a *universal* worst case dependence structure for the joint portfolio.

2. Worst case dependence for risk vectors

In the case that the components X_i of the risk vector X are d-dimensional, $1 \leq d$, there does not exist a universal worst case dependence structure corresponding to the comonotonic vector in

$d = 1$. Several aspects of this problem have been described in Rü (2004) and Puccetti and Scarsini (2010). Corresponding to each law invariant risk measure Ψ there corresponds one worst case dependence structure which is described in Rü (2006, 2010).

Let for a density vector $Y = (Y_1, \dots, Y_d)$ with $Y_i \geq 0$, $EY_i = 1$, $1 \leq i \leq d$, with distribution μ

$$\Psi_\mu(X) := \sup\{E\tilde{X} \cdot Y; \tilde{X} \stackrel{d}{=} X\} \tag{13}$$

denote the *max-correlation risk measure* in direction Y (resp. μ) as introduced in Rü (2006). Then $\Psi_\mu(X)$ defines a law invariant convex risk measure defined for risk vectors $X \in \mathbb{R}^d$. Any lsc convex law invariant risk measure Ψ on $L_d^p(P)$, the class of risk vectors with components $X_i \in L^p(P)$, has a representation as

$$\Psi(X) = \sup_{\mu \in A}(\Psi_\mu(X) - \alpha(\mu)), \tag{14}$$

where A is a weakly closed class of scenario measures and $\alpha(\mu)$ is a law invariant penalty function. Thus the max-correlation risk measures play in the multivariate case a similar role as the spectral risk measures in $d = 1$ and are the building blocks of the class of convex, law invariant risk measures.

The worst case dependence structure of a joint portfolio $\sum_{i=1}^n X_i$ with $X_i \in \mathbb{R}^d$, $X_i \sim F_i$ w.r.t. a law invariant convex risk measure Ψ as in (14) is defined as $X_i^* \sim F_i$, $1 \leq i \leq n$, such that

$$\Psi\Big(\sum_{i=1}^n X_i^*\Big) = \sup_{X_i \sim F_i} \Psi\Big(\sum_{i=1}^n X_i\Big). \tag{15}$$

Its determination involves two steps:

Step 1) Determine a worst case scenario measure $\mu^* \in A$ solving an optimization problem of the form

$$F_a(\mu^*) = \sup_{\mu \in A} F_a(\mu), \tag{16}$$

where $F_a(\mu) = \sum_{i=1}^n \Psi(X_i) - \alpha(\mu)$ is the sum of the marginal risks. $F_a(\mu)$ depends only on the marginals F_i.

Step 2) Let $X_i^* \sim F_i$, $1 \leq i \leq n$ be μ^*-comonotone, i.e. for some $Y^* \sim \mu^*$

$$X_i^* \sim_{\text{oc}} Y^*, \quad 1 \leq i \leq n. \tag{17}$$

All X_i^* are optimally coupled to the same vector Y^*, $1 \leq i \leq n$, in the L^2-sense, i.e. they solve the classical mass transportation problem

$$E\|X_i^* - Y^*\|^2 = \inf\{E\|X_i - Y\|^2; X_i \sim F_i, Y \sim \mu\}. \tag{18}$$

Step 1) and Step 2) imply that

$$(X_1^*, \dots, X_n^*) \text{ is a worst case dependence structure w.r.t. } \Psi. \tag{19}$$

One could call the vector $X^* = (X_1^*, \dots, X_n^*)$ in analogy to the case $d = 1$ a Ψ-*comonotonic vector*. Some examples like elliptical distributions, Archimedian copulas and location-scale families are discussed in Rü (2006, 2010). In general both steps needed to determine worst case dependent vectors can be done only numerically.

3. Applications of dependence bounds

Our aim in this section is to use the classical dependence bounds for risk functionals to derive in a simple and unified way some results on the optimization of financial products and of (re-)insurance contracts.

3.1. Optimal contingent claims

As a step to derive optimal portfolio results as in the classical paper of Merton (1971), He and Pearson (1991a,b) formulated the static problem of optimal claims. This also fitted with economic theory on optimal investments following the Markowitz theory. As reference we mention Merton (1971) and for more recent formulation Dybvig (1988), Dana (2005), Schied (2004), and Föllmer and Schied (2004). The problem of cost efficient options was formulated in Dybvig (1988) and discussed in detail in Bernard and Boyle (2010) and Bernard et al. (2011a,b).

3.1.1. Optimal investment problem

Given an investment (claim) X and a price measure $Q = \varphi \cdot P$ with price density φ w.r.t. P the *optimal investment problem* is formulated as follows:

Find an optimal investment C^* such that

$$E_Q C^* = \int \varphi C^* dP = \inf_{C \leq_{\mathrm{cx}} X} . \tag{20}$$

C^* has the lowest price under all investments C, which are less risky than X in the sense of convex order $\leq_{\mathrm{cx}}$. The minimal price

$$e(X, \varphi) := E_Q C^* \tag{21}$$

is called *reservation price* in Jouini and Kallal (2001). The following result is stated in Dybvig (1988), Dana (2005), and Föllmer and Schied (2004) in various generality.

Theorem 3.1 (Optimal investment) *Let X be an investment with $F_X = F$ and let φ be a price density. Then the reservation price is given by*

$$e(X, \varphi) = \int_0^1 F_\varphi^{-1}(1-t) F^{-1}(t) dt. \tag{22}$$

An optimal investment is given by

$$C^* = F^{-1}(1 - \tau_\varphi(\varphi; V)), \tag{23}$$

where τ_φ is the distributional transform of φ (see (8)*).*

Proof. By the Hoeffding–Fréchet bounds in (5) for any investment C

$$A_\varphi(C) := \inf_{\tilde{C} \sim C} \int \varphi \tilde{C} dP = \int_0^1 F_\varphi^{-1}(1-t) F_C^{-1}(t) dt. \tag{24}$$

Also by a well-known stochastic ordering result

$$C_1 \le_{\mathrm{cx}} C_2 \text{ implies } \int_0^1 h(t) F_{C_1}^{-1}(t)dt \ge \int_0^1 h(t) F_{C_2}^{-1}(t)dt$$

for decreasing functions h. This implies that

$$\inf_{C \le_{\mathrm{cx}} X} A_\varphi(C) = A_\varphi(X) = e(X,\varphi) = \int_0^1 F_\varphi^{-1}(1-t)F^{-1}(t)dt.$$

The representation of the optimal claim in (23) by the distributional transform follows from the fact that the pair (φ, C^*) attains the lower Fréchet bound (see (8)). ■

Remark 3.1 *a)* (C^*,φ) *is a pair of antithetic variables. The distribution of the optimal pair is unique and is given by the anticomonotone distribution. Defining*

$$\tilde{C} := E(C^* \mid \varphi) = \int_0^1 F^{-1}(1-\tau_\varphi(\varphi, v))dv, \tag{25}$$

then $\tilde{C} = g(\varphi)$ *where* $g \downarrow$ *is a decreasing function of the price density* φ *alone. Further,* $\tilde{C} \le_{\mathrm{cx}} C$ *and* $E_Q\tilde{C} = E_Q C^*$*. Thus there exists an optimal investment* $C^* = g^*(\varphi)$, $g^* \downarrow$ *which is a decreasing function of the price density* φ.

b) ***Transformed measure.*** *Defining the transformed measure*

$$Q^* := \varphi^* P \text{ with } \varphi^* := F^{-1}(1-\tau_F(X,V)), \tag{26}$$

then φ^* *is decreasing in* X *and*

$$e(X,\varphi) = E_Q C^* = E_{Q^*} X. \tag{27}$$

Thus the reservation price is identical to the expectation of X *w.r.t. the transformed price measure* Q^*. Q^* *describes a worst case price density for the claim* X.

c) ***Path dependent options.*** *Let* $S = (S_t)_{0\le t\le T}$ *be a price process and assume that the price density* φ *is a function of* S_T, $\varphi = \varphi(S_T)$*, then*

$$C^* = g(S_T). \tag{28}$$

Thus any path dependent option $C = f(S)$ *can be improved by a European option*

$$C^* = g(S_T).$$

If φ *is increasing (decreasing), then* g *can be chosen decreasing (increasing). For this observation see Bernard et al. (2011b).*

d) ***Cost efficient options.*** *Given an option* X *with distribution function* F *we consider the class* $\mathcal{C} = \mathcal{C}(F)$ *of all options which have the same payoff distribution as* X,

$$\mathcal{C} = \{C; F_C = F\} = \mathcal{C}(X). \tag{29}$$

As corollary Theorem 3.1 implies

Theorem 3.2 (Cost efficient claims) *For a given claim X and price density φ the claim*

$$C^* := F^{-1}(1-\tau_\varphi(\varphi, V)) \in \mathcal{C}(X) \tag{30}$$

is a cost efficient claim, i.e.

$$E_Q C^* = \inf_{C \in \mathcal{C}(X)} E_Q C.$$

Proof. For the proof note that any $C \in \mathcal{C}(X)$ satisfies that $C \leq_{\text{cx}} X$. Thus Theorem 3.2 follows from Theorem 3.1. ∎

The notion of cost efficient claims was introduced in Dybvig (1988) and studied in the discrete case. It was extended in recent papers in Bernard and Boyle (2010) and Bernard et al. (2011b) to the case of continuous distributions. Several explicit results on lookback options, Asian options or related path dependent options in Black–Scholes type models are given in these papers.

3.1.2. Minimal demand problem

Closely related to the optimal investment problem is the minimal demand problem. Given a law invariant convex risk measure Ψ, a price measure $Q = \varphi P$ and a budget set

$$B = \{C; C \text{ claim}, E_Q C \leq c\}. \tag{31}$$

The *minimal demand problem* aims to find a claim C^* in the budget set with minimal risk

$$C^* \in B; \quad \Psi(C^*) = \inf\{\Psi(C); C \in B\}. \tag{32}$$

This problem has been discussed in Dana (2005), Schied (2004), and Föllmer and Schied (2004). An existence result is obtained in these papers for lsc convex risk measures. For law invariant convex risk measures the Hoeffding–Fréchet bounds imply similarly as in Theorem 3.1.

Theorem 3.3 (Minimal demand problem) *There exists a solution C^* of the minimal demand problem* (32) *such that*

$$C^* = g(\varphi) \text{ for some } g \downarrow.$$

Remark 3.2 *For the corresponding utility maximization problem w.r.t. an expected utility function U*

$$U(C^*) = Eu(C^*) = \sup_{C \in B} U(C),$$

where u is a utility function explicit solutions are derived in He and Pearson (1991a,b) and many related papers for the standard utility functions. The solutions are obtained in the form

$$C^* = I(\lambda_Q(c))\varphi, \quad I(x) := (u')^{-1}(x), \tag{33}$$

where $\lambda_Q(x)$ is a constant chosen such that

$$E_Q C^* = c.$$

The main methods applied to solve this problem are a duality approach closely connected to a martingale approach (see Merton (1971), He and Pearson (1991a,b), and Kramkov and Schachermayer (1999) and a projection approach based on φ-divergence distances (see Goll and Rü (2001) and Biagini and Frittelli (2005, 2008)).

3.2. Optimal (re-)insurance contracts

Optimal (re-)insurance contracts can be seen as particular instances of the optimal risk allocation problem. In this section we discuss some variations on the optimality of the classical stop-loss contracts which are obtained from the risk bound results for the comonotonic risk vector in Section 1, B1)–B3).

A (re-)insurance contract $I(X)$ for a risk $X \geq 0$ is defined by a function $I = \mathbb{R}_+ \to \mathbb{R}_+$, $0 \leq I(x) \leq x$, $I(0) = 0$. Let I denote the class of all reinsurance contracts (see Kaas et al. (2001)). The premium to be paid for the contract $I(X)$ is given by

$$\pi_I(X) = (1 + \vartheta)EI(X). \tag{34}$$

The stop-loss contract $I_d(X)$ – with retention limit d – is defined by

$$I_d(X) = (X - d)_+. \tag{35}$$

By a classical result going back to Arrow (1963, 1974) the stop-loss contract minimizes the retained risk $X - I(X)$ given a fixed premium π_0. The strongest version of this result is given in (Kaas et al. 2001, Example 10.4.4, p. 238).

Theorem 3.4 (Optimality of stop-loss contracts) *For any $I \in \mathcal{I}$ with $EI(X) = EI_d(X) = \frac{\pi_0}{1+\vartheta}$ with retained risks $R_I(X) := X - I(X)$ and $R_d(X) := X - I_d(X)$ holds*

$$R_d(X) \leq_{\mathrm{cx}} R_I(X). \tag{36}$$

Remark 3.3 *The proof in Kaas et al. (2001) is based on stochastic ordering and in particular on the Karlin–Novikov criterion for convex ordering. As consequence of* (36) *it holds for any law invariant convex risk measure Ψ that*

$$\Psi(R_d(X)) \leq \Psi(R_I(X)).$$

This result is reproved in Cheung et al. (2010a).

As in classical Markowitz theory we can formulate a corresponding efficient boundary result. Let Ψ be a law invariant convex risk measure and define for $I \in \mathcal{I}$

$$\begin{aligned} \mu_I &:= E(X - I(X)), & \sigma^2_\Psi(I) &:= \Psi(X - I(X)),\\ \mu(d) &:= \mu_{I_d}, & \sigma^2_\Psi(d) &:= \sigma^2_\Psi(I_d). \end{aligned}$$

Corollary 3.5 *Consider the risk sets $\mathcal{R}_\Psi := \{(\mu_I, \sigma^2_\Psi(I)); I \in \mathcal{I}\}$ of all reinsurance contracts and the risk set $T_\Psi := \{(\mu(d)), \sigma^2_\Psi(d)); d \geq 0\}$ of all stop-loss contracts. Then the risk set T_Ψ of the stop-loss contracts is the lower boundary of the risk set $\mathcal{R}_\Psi$.*

Proof. The proof is similar as in the classical variance case. ■

An interesting risk minimizing insurance protection problem for risks of joint portfolios was introduced in a recent paper of Cheung et al. (2010b). For a portfolio $X = \sum_{i=1}^n X_i$ with $X_i \sim F_i$ the risk is strongly influenced by the dependence of the components X_i of the joint portfolio. Let

$\mathcal{I}_n = \{I = (I_1, \dots, I_n); I_j$ reinsurance contracts$\}$ denote the set of reinsurance contracts of the joint portfolio. Let $\pi(I) = (1-\vartheta)\sum_{k=1}^n EI_k(X_k)$ denote the premium of contract I and π_0 be a given level of premium. $I^* \in \mathcal{I}_n$ is called *optimal worst case reinsurance contract* if it solves the following problem:

$$R_\Psi(\pi_0) := \inf_{\substack{I\in\mathcal{I}_n \\ \pi(I)=\pi_0}} \sup_{X_i\sim F_i} \Psi\Big(\sum_{k=1}^n (X_k - I_k(X_k))\Big), \tag{37}$$

where Ψ is a law invariant convex risk measure. Thus with problem (37) one aims to find robust versions of reinsurance contracts which take into account the possible worst case dependence structure in the portfolio.

Cheung et al. (2010b) show in a recent paper that certain stop-loss contracts solve problem (37). This result can be obtained in a simplified way from the risk bound results in Section 1, which also allow to extend the result to general distributions not assuming continuity and strictly increasing distribution functions.

Theorem 3.6 (Optimal worst case reinsurance contracts) *The stop-loss contracts*

$$I_k^*(x) = I_{d_k^*}(x) = (x - d_k^*)_+, \quad 1 \le k \le n$$

as defined in (40) *are optimal worst case reinsurance contracts at premium* $\pi(I) = \pi_0$ *for any choice of law invariant convex risk measure* Ψ.

Proof. The proof follows from the risk bounds in Section 1 by the following two steps.

1) Since for $I \in \mathcal{I}_n : X_k - I_k(X_k) = (id - I_k)(X_n)$ is an increasing function of X_k. It follows from B1) and B3) that for any $X_k \sim F_k$ and $I \in \mathcal{I}_n$

$$\sum_{k=1}^n (X_k - I_k(X_k)) \le_{\mathrm{cx}} \sum_{n=2}^n (X_k^c - I_k(X_k^c)), \tag{38}$$

where $X^c = (X_k^c)$ is the comonotonic vector.

The comonotonic vector X^c is by (9) the worst case dependence structure. As consequence of (38) we obtain

$$\Psi\Big(\sum_{k=1}^n (I_k - I_k(X_k))\Big) \le \Psi\Big(\sum_{k=1}^n (X_k^c - I_k(X_k^c))\Big) \tag{39}$$

for any law invariant convex risk measure Ψ (see (12)).

2) Let $d^* \ge 0$ satisfy $E(\sum_{k=1}^n X_k^c - d^*)_+ = \frac{\pi_0}{1+\vartheta}$, then there exist $d_k^* \ge 0$ such that $\sum_{k=1}^n d_k^* = d^*$ and

$$\Big(\sum_{k=1}^n X_k^c - d^*\Big)_+ = \sum_{k=1}^n (X_k^c - d_k^*)_+ \tag{40}$$

(see (11) for the choice of d_k^*.).

As a result these two points together with the classical optimality result for stop-loss contracts in Theorem 3.4 imply optimality of (I_k^*). ∎

Remark 3.4 *Theorem 3.6 can be extended to the worst case risk problem with upper bounds on the premiums* $\pi(I) \le \pi_0$. *This follows from the fact, that* $d_i^* = d_i^*(\pi_0)$ *are increasing in* π_0 *(see* (11) *for the continuous case). For this result and examples see Cheung et al. (2010b).*

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