Optimal risk allocation for convex risk functionals in general domains

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Abstract

In this paper we extend the classical optimal risk allocation problem to the case of general convex risk functionals defined on real Banach spaces. In particular we characterize optimal allocations and give existence and uniqueness results. The second part of the paper is concerned with an application to expected risk functionals. This case can be dealt with by the Banach space approach applied to Orlicz hearts associated to the risk functionals. We give a detailed discussion of the necessary continuity and differentiability properties and also establish an ordering result for Orlicz hearts which allows extensions of this frame to different Orlicz hearts as domain of risk functionals. In some numerical results optimal redistributions are determined for the expected risk case and the approximation errors are evaluated.

Key words: Optimal risk allocations, optimal reinsurance problem, Orlicz space, Orlicz heart, expected utility

JEL classification: C61, G11

1 Introduction

The optimal allocation problem of risks is a classical problem in mathematical economics and insurance and is of considerable practical and theoretical interest. It has been studied in the case of real risks in the classical papers of Borch (1962), Gerber (1979), Bühlmann and Jewell (1979), Deprez and Gerber (1985) and others in the context of risk sharing in insurance and reinsurance contracts with respect to expected utility. In more recent years this problem has also been studied in the context of the risk measure theory with focus on financial risks as in risk exchange, assignment of liabilities to daughter companies and individual hedging problems (see the papers of Heath and Ku (2004), Barrieu and El Karoui (2005a,b), Burgert and Rüschendorf (2006); Burgert and Rüschendorf (2008), Jouini et al. (2007), Acciaio (2007), Filipović and Svindland (2008), Balbás et al. (2009), [KR] (2008; 2010)[1], Grechuk and Zabarankin (2011), and others).

In particular we give in this paper an extension of some results in [KR] (2008; 2010). We consider the problem of optimal risk allocation or risk exchange problem where the main focus lies on multivariate convex risk functionals which not necessarily are monotone or cash invariant defined on the product space E^d of a Banach space E. For $E = L^p(P)$ this generality enables us to take into consideration some risk functionals of practical interest like mean variance or standard deviation or related one-sided risk functionals in the one dimensional case and general expected risk functionals in the multivariate case. We provide characterization and existence results for optimal risk allocations minimizing the total risk as well as for Pareto optimal allocations and we give a uniqueness result for optimal allocations. This implies in particular in the case of cash invariant, strictly convex risk functionals on E^d the uniqueness of Pareto optimal allocations up to additive constants.

A main motivation for considering the multivariate risk framework is to include the effects of the dependence between the single risky positions on the risk of a portfolio. In recent papers several of the aspects of multivariate risks like worst case portfolios, diversification effects or strong coherence have been studied (see e.g. Carlier et al. (2012), Ekeland et al. (2012), Rüschendorf (2006, 2012)). It is shown in [KR] (2010) that optimal risk allocations in the case $E = L^p_d(P)$ are described by worst case scenario measures μ_0 and comonotone allocations w.r.t. μ_0.

In Section 2 we give characterizations of allocations with minimal total risk as well as of Pareto optimal allocations. Section 3 is concerned with existence and Section 4 with uniqueness of optimal allocations. These results are mostly extensions of [KR] (2010). We give detailed arguments only for those which need new methodology. In Section 5 we demonstrate the usefulness of the general frame of Banach spaces used in this paper. Dealing with expected risk functionals lacking polynomial growth leads to the consideration of related Orlicz hearts where this allocation problem can be solved in our framework. Some extensions to the case of non-identical domains of the risk functionals are also detailed. Finally in Section 6 we give some numerical results for optimal redistributions for the expected risk case.

2 Optimal risk allocations

As our basic space we consider a real Banach space E containing the constants with dual pairing $(E, E^*, \langle \cdot \mid \cdot \rangle_E)$. We consider convex proper normed lower semicontinuous functions, called in the following *risk functionals* $\varrho_i : E^d \longrightarrow (-\infty, \infty]$, $1 \leqslant i \leqslant n$, defined on risk vectors $x = (x^1, \dots, x^d)$ with $x^j \in E$, where E^d is the

[1]Kiesel and Rüschendorf is abbreviated within this paper to [KR].

d-fold product of E. As typical examples we can think of $E = L^p(P)$ the space of risks in an L^p-space. The risk functionals ϱ_i describe the risk evaluation of the n traders in the market.

For a given portfolio of d risky positions described by a risk vector $x \in E^d$ we define the set $\mathcal{A}(x)$ of *allocations* of the portfolio x by

$$\mathcal{A}(x) := \left\{ (\xi_1, \dots, \xi_n) \mid \xi_i \in E^d, \sum_{i=1}^n \xi_i = x \right\}. \tag{2.1}$$

For an allocation $(\xi_1, \dots, \xi_n) \in \mathcal{A}(x)$ trader i is exposed to the risk vector $\xi_i \in E^d$ which is evaluated by the risk functional ϱ_i. Note that ξ_i may contain some zero components and thus trader i may only be exposed to some of the d components of risk in our formulation. Let

$$\mathfrak{R} := \{ (\varrho_1(\xi_1), \dots, \varrho_n(\xi_n)) \mid (\xi_1, \dots, \xi_n) \in \mathcal{A}(x) \} \tag{2.2}$$

denote the corresponding risk set. One aim is to characterize Pareto optimal allocations $(\xi_i) \in \mathcal{A}(x)$ which are allocations such that the corresponding risk vectors are minimal elements of the risk set $\mathfrak{R}$ in the componentwise ordering. A related optimization problem is to characterize allocations $(\eta_i) \in \mathcal{A}(x)$ which minimize the total risk, i.e.

$$\begin{aligned} \sum_{i=1}^n \varrho_i(\eta_i) &= \inf \left\{ \sum_{i=1}^n \varrho_i(\xi_i) \mid (\xi_i) \in \mathcal{A}(x) \right\} \\ &=: \bigwedge \varrho_i(x). \end{aligned} \tag{2.3}$$

In the literature the latter minimization problem (2.3) is also referred to as the *infimal convolution* of $\varrho_1, \dots, \varrho_n$.

The following lemma is a classical result of convex optimization theory (see Rockafellar (1970); Ioffe and Tikhomirov (1979); Barbu and Precupanu (1986)). It collects several useful properties of the infimal convolution.

Lemma 2.1 *For proper convex functions $\varrho_i : E^d \longrightarrow \overline{\mathbb{R}}$, $i \in \{1, \dots, n\}$ the following statements are true.*

1) $\bigwedge \varrho_i : E^d \longrightarrow \overline{\mathbb{R}}$ *is convex.*

2) $\operatorname{dom}(\bigwedge \varrho_i) = \sum_{i=1}^n \operatorname{dom}(\varrho_i)$.

3) $(\bigwedge \varrho_i)^* = \sum_{i=1}^n \varrho_i^*$.

4) $\operatorname{dom}(\bigwedge \varrho_i)^* = \bigcap_{i=1}^n \operatorname{dom}(\varrho_i^*)$

5) $\bigcap_{i=1}^n \partial \varrho_i(x_i) \subseteq \partial (\bigwedge \varrho_i)(x)$ *for all* $(x_1, \dots, x_n)$ *with* $\sum_{i=1}^n x_i = x$.

6) *If all ϱ_i are additionally lower semicontinuous and fulfill the property $\bigcap_{i=1}^n \operatorname{dom}(\varrho_i^*) \neq \emptyset$, then $\bigwedge \varrho_i$ is proper.*

We recast the optimization problem in (2.3) into a global minimization problem by defining the function $\bar{\varrho} : (E^d)^n \longrightarrow \overline{\mathbb{R}}$ with $\bar{x} := (x_1, \dots, x_n) \in (E^d)^n$ by

$$\bar{\varrho}(\bar{x}) := \sum_{i=1}^n \varrho_i(x_i).$$

With the convex indicator function

$$\mathbb{1}_A(x) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

we obtain the representation

$$\bigwedge \varrho_i(x) := \inf \left\{ \bar{\varrho}(\bar{z}) + \mathbb{1}_{\mathcal{A}(x)}(\bar{z}) \mid \bar{z} \in (E^d)^n \right\}, \tag{2.4}$$

which is of the form of an unrestricted minimization problem. The infimal convolution as a restricted minimization problem is called *well posed* for a given $x \in E^d$ if

$$\operatorname{domc}(\bar{\varrho}) \cap \mathcal{A}(x) \neq \emptyset, \tag{2.5}$$

where $\operatorname{domc}(\bar{\varrho})$ denotes the *domain of continuity* of $\bar{\varrho}$. For $x \in E^d$ and $z \in (E^*)^d$ the scalar product of the dual pairing $(E^d, (E^*)^d, \langle \cdot \mid \cdot \rangle_d)$ is given by

$$\langle x \mid z \rangle_d := \sum_{j=1}^d \langle x^j \mid z^j \rangle_E. \tag{2.6}$$

For $\bar{x} \in \left(E^d\right)^n$ and $\bar{z} \in \left((E^*)^d\right)^n$ the scalar product of the dual pairing $\left(\left(E^d\right)^n, \left((E^*)^d\right)^n, \langle \cdot \mid \cdot \rangle_d^n\right)$ is given by

$$\langle \bar{x} \mid \bar{z} \rangle_d^n := \sum_{i=1}^n \langle x_i \mid z_i \rangle_d = \sum_{i=1}^n \sum_{j=1}^d \langle x_i^j \mid z_i^j \rangle_E. \tag{2.7}$$

The following result is the basic characterization of minimal total risk allocations which extends the developments for real risks to the portfolio case in Banach spaces. For the ample literature to this theorem see the references mentioned in the introduction to this section.

Theorem 2.2 (Characterization of minimal total risk) *Let ϱ_i be risk functionals on E^d. Let $x \in E^d$ be a risk portfolio such that the minimal total risk problem is well posed and let $(\eta_i) \in \mathcal{A}(x)$ be a risk allocation. Then the following statements are equivalent:*

1) (η_i) *has minimal total risk (w.r.t.* $\varrho_1, \dots, \varrho_n$ *and* X*),*

2) $\exists V \in (E^*)^d : V \in \partial\varrho_i(\eta_i), \quad 1 \leqslant i \leqslant n,$ (2.8)

3) $\exists V \in (E^*)^d : \eta_i \in \partial\varrho_i^*(V), \quad 1 \leqslant i \leqslant n.$ (2.8′)

The proof of Theorem 2.2 is a slight extension of the proof of Theorem 3.1 in [KR] (2010). It is based on Fermat's rule, on the subdifferential sum formula in Banach spaces, and uses the following representation of the subgradient of the convex indicator function $\mathbb{1}_A$. For details see [K] (2013)[2].

Lemma 2.3 *For all* $x \in E^d$ *and* $\bar\xi \in \mathcal{A}(x)$ *holds*

$$\partial\mathbb{1}_{\mathcal{A}(x)}(\bar\xi) = \Big\{\bar z \in \big((E^*)^d\big)^n \mid \bar z = \sum_{i=1}^n z\mathbf{e}_i, z \in (E^*)^d\Big\}, \tag{2.9}$$

where $\mathbf{e}_i$ *is the* i*-th unit vector of the* n*-fold product space* $\big((E^*)^d\big)^n$*. Thus the element* $z\mathbf{e}_i$ *is understood as that element of* $\big((E^*)^d\big)^n$ *which has* $z \in (E^*)^d$ *as its* i*-th component and* $0 \in (E^*)^d$ *otherwise.*

To obtain a connection between minimization of the total risk and of Pareto optimality we introduce as in [KR] (2008) a condition called non–saturation property. We say that ϱ has the *non–saturation property* if

$$\text{(NS)} \qquad \inf_{x \in E^d} \varrho(x) \text{ is not attained.} \tag{2.10}$$

The non–saturation property is a weak property of risk functionals. It is implied in particular by the cash invariance property. This is easily deduced for $1 \leqslant i \leqslant d$ by $\lim_{c\to-\infty} \varrho(x + \mathbf{e}_i c) = \lim_{c\to-\infty} \varrho(x) + c = -\infty$. Under the (NS) condition Pareto optimality is related to the problem of minimizing the total weighted risk. This is described by the *weighted minimal total risk* $(\bigwedge \varrho_i)_\gamma(x)$ defined for weight vectors $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ by

$$(\bigwedge \varrho_i)_\gamma(x) := \inf\Big\{\sum_{i=1}^n \gamma_i\varrho_i(\xi_i) \;\Big|\; (\xi_1, \dots, \xi_n) \in \mathcal{A}(x)\Big\}. \tag{2.11}$$

The connection between Pareto optimality and minimizing weighted total risk dates back to early papers in insurance in mathematical economics (see e.g. Gerber (1979) or Aubin (1993, Proposition 12.3)).

Theorem 2.4 (Characterization of Pareto optimal allocations) *For* $1 \leqslant i \leqslant n$ *let* ϱ_i *be risk functionals on* E^d *satisfying the non–saturation conditions (NS). Then for* $x \in E^d$ *and* $(\xi_1, \dots, \xi_n) \in \mathcal{A}(x)$ *the following statements are equivalent*

[2] Kiesel is abbreviated within this paper to [K].

1) $(\xi_1, \dots, \xi_n)$ *is a Pareto optimal allocation of* x *w.r.t.* $\varrho_1, \dots, \varrho_n$. (2.12)

2) $\exists \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n_{>0}$ *such that* $(\bigwedge \varrho_i)_\gamma(x) = \sum_{i=1}^n \gamma_i \varrho_i(\xi_i)$. (2.13)

3) $\exists \gamma = (\gamma_i) \in \mathbb{R}^n_{>0}$ *and* $\exists V \in (E^*)^d$ *such that* $V \in \gamma_i \partial \varrho_i(\xi_i),\ \forall i$. (2.14)

4) $\exists \gamma = (\gamma_i) \in \mathbb{R}^n_{>0}$ *and* $\exists V \in (E^*)^d$ *such that* $\xi_i \in \partial(\gamma_i \varrho_i)^*(V),\ \forall i$. (2.15)

The proof follows in a similar way as in [KR] (2008, Proof of Theorem 3.3) by a Hahn–Banach separation argument.

For cash invariant risk functionals this characterization result implies by a rebalancing argument (see Jouini et al. (2007), Burgert and Rüschendorf (2006; 2008), Acciaio (2007), [KR] (2008)) that it is enough to consider the optimal risk allocation problem for the weight vector $\gamma = (1, \dots, 1)$.

Corollary 2.5 *If* ϱ_i *are cash invariant risk functionals on* E^d*, then for* $x \in E^d$*,* $(\xi_i) \in A(x)$ *the following are equivalent:*

1) (ξ_i) *is Pareto optimal.*

2) (ξ_i) *has minimal total risk (w.r.t.* $\varrho_1, \dots, \varrho_n$ *and* X*)*

3) $\bigcap \partial \varrho_i(\xi_i) \neq \emptyset$

4) $\exists V \in (E^*)^d : \xi_i \in \partial \varrho_i^*(V), \quad \forall i$

Further for any V *as above holds*

$$(\bigwedge \varrho_i)(X) = \langle V \mid X \rangle_d - \sum_{i=1}^n \varrho_i^*(V). \tag{2.16}$$

The intersection condition (2.14) can also be described by saying that

$$V \in \partial(\bigwedge \varrho_i)_\gamma(x). \tag{2.17}$$

This is a consequence of the following proposition (see [KR] (2010, Proposition 3.4)).

Proposition 2.6 *If* $(\xi_i) \in \mathcal{A}(x)$ *minimizes the total risk w.r.t.* $\varrho_1, \dots, \varrho_n$*, then*

$$\partial(\bigwedge \varrho_i)(x) = \bigcap_{i=1}^n \partial \varrho_i(\xi_i). \tag{2.18}$$

As a consequence we get a further characterization for the optimality of allocations.

Proposition 2.7 *The allocation $(\xi_i) \in \mathcal{A}(x)$ minimizes the total risk w.r.t. $\varrho_1, \dots, \varrho_n$ if and only if* (2.18) *holds and is non empty, i.e.*

$$\partial(\wedge \varrho_i)(x) = \bigcap_{i=1}^{n} \partial\varrho_i(\xi_i) \neq \emptyset. \tag{2.19}$$

Proof: See [K] (2013). □

Example 2.1 (Expected risk) *Let $r : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a strictly convex, continuously differentiable risk function satisfying the polynomial growth condition*

$$|r(x)| \leqslant c(1 + \|x\|_p^p) \text{ for some } c \in \mathbb{R}, \quad p \geqslant 1. \tag{2.20}$$

It is shown in [KR] (2010) that under the growth condition ϱ_r is Gâteaux differentiable with Gâteaux gradient

$$\nabla\varrho_r(X) = \nabla r(X) \in L_d^q, \quad X \in L_d^p. \tag{2.21}$$

For a family $r_1, \dots, r_n$ of convex functions as above with corresponding expected risk functionals $\varrho_{r_1}, \dots, \varrho_{r_n}$ on L_d^p we consider the optimal risk allocation problem for $X \in L_d^p$. By Theorem 2.2 an optimal allocation $(\xi_i) \in \mathcal{A}(X)$ with minimal total risk is characterized by the optimality equation

$$\nabla r_i(\xi_i) = \nabla r_j(\xi_j), \quad 1 \leqslant i \leqslant n. \tag{2.22}$$

This is a multivariate extension of the classical Borch theorem to $d \geqslant 1$. For strictly convex r_i, ∇r_i is one-to-one and as a consequence we obtain

$$\xi_i = ((\nabla r_i)^{-1} \circ \nabla r_1)(\xi_1), \quad 2 \leqslant i \leqslant n, \tag{2.23}$$

where the critical allocation condition for ξ_1 is determined by

$$\xi_1 + \sum_{i=2}^{n} ((\nabla r_i)^{-1} \circ \nabla r_1)(\xi_1) = X. \tag{2.24}$$

For identical strictly convex risk functions r_i fulfilling the growth condition (2.20) *for a $p \geqslant 1$ an optimal allocation of $X \in L_d^p$ is of the form*

$$\xi_i = \frac{1}{n} X \text{ for } i = 1, \dots, n. \tag{2.25}$$

This follows directly from (2.24) *and* (2.23)*. For univariate risk functions of the form*

$$r_i(x) := c_i x^{\alpha_i}, \alpha_1 \leqslant \dots \leqslant \alpha_n,\ \alpha_i \in 2\mathbb{N},\ c_i > 0, i \in \{1, \dots, n\}$$

and $X \in L^{\alpha_n}$ the critical allocation condition for ξ_1 is given by the implicit equation

$$\xi_1 + \sum_{i=2}^{n} \left(\frac{c_1\alpha_1}{c_i\alpha_i}\right)^{\frac{1}{\alpha_i - 1}} \xi_1^{\frac{\alpha_1 - 1}{\alpha_i - 1}} = X \quad P\text{-a.s.} \tag{2.26}$$

The expected risk case under polynomial growth thus can be dealt with in the frame of the Banach space $E = L^p$ which is the frame used in the paper of [KR] (2010). We will consider the expected risk functional without polynomial constraint in Section 5.

3 Existence of optimal allocations

In this section we extend some of the existence results for optimal allocations known from the literature. Existence results for optimal allocations have been based in the one-dimensional case on comonotone improvement theorems (see Jouini et al. (2007), Acciaio (2007), and Filipović and Svindland (2008)) which allow to restrict to allocations $\xi_i = f_i(X)$ with some monotone functions f_i with $\sum_{i=1}^n f_i = \mathrm{Id}$ and thus allows to apply Dini's theorem. Alternatively a strong intersection condition (SIS) from convex analysis (see Barbu and Precupanu (1986)) has been used in [KR] (2008). Several *interior point* conditions for existence have been given in [KR] (2010). In this section we review and extend some of the results in this paper. We first state results on the existence of risk minimizing allocations for the case $n = 2$ and show subsequently how they carry over to the general case of arbitrary $n \in \mathbb{N}$.

In the following we shall make use of the *subdifferential sum formula* for functions f, g at a point x (see Barbu and Precupanu (1986)):

$$\text{(SD}(x)) \qquad \partial(f+g)(x) = \partial f(x) + \partial g(x), \tag{3.1}$$

which has a close link with the *epigraph condition* for the conjugates f^*, g^*:

$$\text{(EC)} \qquad \operatorname{epi}((f+g)^*) = \operatorname{epi}(f^*) + \operatorname{epi}(g^*) \tag{3.2}$$

where $\operatorname{epi}(f) := \{(x,\alpha) \in E \times \mathbb{R} \mid f(x) \leqslant \alpha\}$.

A classical result of Burachik and Jeyakumar (2005, Theorem 3.1) states that the epigraph condition (EC) implies the subdifferential sum formula (SD(x)) for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$. The following theorems extend this result and give a link to existence of optimal allocations.

Theorem 3.1 (Local existence) *Let $f, g : E \longrightarrow (-\infty, \infty]$ be proper, lsc convex functions on a Banach space E such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. *Then the following holds:*

$f \wedge g$ is subdifferentiable at x and the subdifferential sum formula w.r.t. f^* and g^* holds for some $x^* \in \partial(f \wedge g)(x)$, i.e.

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*), \tag{3.3}$$

if and only if there exists an allocation $(\xi_1, \xi_2) \in \mathcal{A}(x)$ which minimizes the total risk, namely:

$$(f \wedge g)(x) = f(\xi_1) + g(\xi_2). \tag{3.4}$$

Proof:

a) Let $(\xi_1, \xi_2) \in \mathcal{A}(x)$ be an allocation with $(f \wedge g)(x) = f(\xi_1) + g(\xi_2)$. From Proposition 2.7 we get that $f \wedge g$ is subdifferentiable at x. Thus for $x^* \in \partial(f \wedge g)(x)$. Then

$$x \in \partial(f \wedge g)^*(x^*) = \partial(f^* + g^*)(x^*). \tag{3.5}$$

(This inclusion does not need the lsc.) On the other hand for any solution (ξ_1, ξ_2) of $(f \wedge g)(x)$ we have

$$x^* \in \partial f(\xi_1) \cap \partial f(\xi_2) \tag{3.6}$$

by Theorem 2.2.This again implies

$$\xi_1 \in \partial f^*(x^*) \text{ and } \xi_2 \in \partial g^*(x^*).$$

Thus we obtain

$$x = \xi_1 + \xi_2 \in \partial f^*(x^*) + \partial g^*(x^*),$$

and we get the inclusion

$$\partial(f^* + g^*)(x^*) \subseteq \partial f^*(x^*) + \partial g^*(x^*).$$

Therefore, equality holds since the opposite inclusion holds generally.

b) For the converse direction the assumptions imply, together with Lemma 2.1, the existence of an $x^* \in E^*$ such that

$$x \in \partial(f \wedge g)^*(x^*) = \partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

Thus there exists an allocation $(\xi_1, \xi_2) \in \mathcal{A}(x)$ with $\xi_1 \in \partial f^*(x^*)$ and $\xi_2 \in \partial g^*(x^*)$ which implies that (ξ_1, ξ_2) minimizes the total risk (see Theorem 2.2).

□

The infimal convolution $f \wedge g$ is called *exact in* x if the infimum is attained at x as in (3.4); it is called *exact* if this holds for all $x \in E$. Let f, g be functions as in Theorem 3.1 then we get as corollary:

Corollary 3.2 *The following statements are equivalent*

1) $f \wedge g$ *is exact.*

2) $f \wedge g$ *is subdifferentiable at all* $x \in E$ *and* $\forall x^* \in \partial(f \wedge g)(x)$ *the subdifferential sum formula holds for* f^*, g^*, *i.e.*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

By a result of Boţ and Wanka (2006) for functions f, g with $\mathrm{dom}(f^*) \cap \mathrm{dom}(g^*) \neq \emptyset$ holds:

The epigraph condition (EC) for f^*, g^*

$$\text{i.e. } \mathrm{epi}((f^* + g^*)^*) = \mathrm{epi}(f) + \mathrm{epi}(g) \tag{3.7}$$

is equivalent to

$$(f^* + g^*)^* = f \wedge g \text{ and } f \wedge g \text{ is exact.} \tag{3.8}$$

Based on this result global existence was characterized in [KR] (2010).

Theorem 3.3 (Existence of optimal allocations) *Let f, g be proper lower semicontinuous convex functions from a Banach space E to $[-\infty, \infty]$ such that* $\mathrm{dom}(f^*) \cap \mathrm{dom}(g^*) \neq \emptyset$.

Then the following statements are equivalent:

1) *$f \wedge g$ is exact.*

2) *The epigraph condition (EC) holds for f^*, g^*, i.e.*

$$\mathrm{epi}((f^* + g^*)^*) = \mathrm{epi}\, f + \mathrm{epi}\, g. \tag{3.9}$$

3) *$f \wedge g$ is subdifferentiable at all $x \in E$ and for all $x^* \in \partial(f \wedge g)(x)$ the subdifferential sum formula holds*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

The local exactness in a point $x \in E$ can also be geometrically related to a suitable epigraph condition as is explained in the following. First we give a rather simple sufficient condition on the subdifferential sum formula (3.3) which concerns the right directional derivatives and we discuss some geometric interpretations of the (local) exactness in a point $x \in E$ related to the epigraph condition.

We remind that the directional derivative $y \mapsto \mathcal{D}(f, x)(y)$ is convex and positively homogeneous and therefore sublinear. Further the subdifferentials of a proper convex function $f : E \longrightarrow \overline{\mathbb{R}}$ and its right directional derivative are connected by

$$\partial f(x) = \partial(\mathcal{D}(f, x))(0).$$

A special case of statement 1.4.12 (1) in Kusraev and Kutateladze (1995) provides the subdifferential sum formula at zero for sublinear finite mappings $p_i : E \longrightarrow \mathbb{R}$, $i \in \{1, \ldots, n\}$.

Lemma 3.4 (Kusraev and Kutateladze (1995, 1.4.12 (1))) *Let E be a Banach space and $p_i : E \longrightarrow \mathbb{R}$ be proper finite sublinear functions. Then*

$$\partial\Big(\sum_{i=1}^{n} p_i\Big)(0) = \sum_{i=1}^{n} \partial p_i(0). \tag{3.10}$$

Thus we get for the subdifferential of the sum of convex functions.

Theorem 3.5 *Let $f_i : E \longrightarrow \overline{\mathbb{R}}$, $i \in \{1, \dots, n\}$ be proper convex functions on a Banach space E. For $x_0 \in E$ such that mappings $y \mapsto \mathcal{D}(f_i, x_0)(y)$ are finite, i.e.* $\operatorname{dom}(\mathcal{D}(f_i, x_0)(\cdot)) = E$ *for all $i \in \{1, \dots, n\}$, it holds*

$$\partial\Big(\sum_{i=1}^{n} f_i\Big)(x_0) = \sum_{i=1}^{n} \partial f_i(x_0). \tag{3.11}$$

Proof: Define $g_i(y) := \mathcal{D}(f_i, x_0)(y)$. Since $y \mapsto g_i(y)$ are finite for all $i \in \{1, \dots, n\}$, it holds $\mathcal{D}\big(\sum_{i=1}^{n} f_i, x_0\big)(y) = \sum_{i=1}^{n} g_i(y)$ and we get from Lemma 3.4 and using that $\partial(\mathcal{D}(f, x))(0) = \{x^* \in E^* \mid \forall y \in E : \langle x^* \mid x\rangle \leqslant \mathcal{D}(f, x)(y)\}$ the following sequence of equations:

$$\begin{aligned}\partial\Big(\sum_{i=1}^{n} f_i\Big)(x_0) &= \partial(\mathcal{D}\Big(\sum_{i=1}^{n} f_i, x_0\Big))(0) = \partial\Big(\sum_{i=1}^{n} g_i\Big)(0)\\ &= \sum_{i=1}^{n} \partial g_i(0) = \sum_{i=1}^{n} \partial f_i(x_0).\end{aligned}$$

□

In consequence, $f \wedge g$ is exact at x if the hypotheses of Theorem 3.1 and of Theorem 3.5 hold for the convex conjugates f^*, g^* and $x_0^* \in \partial(f \wedge g)(x)$.

The *strict epigraph* of a function f is defined by

$$\operatorname{epi}_s(f) := \{(x, r) \in E \times \mathbb{R} \mid f(x) < r\}. \tag{3.12}$$

Then using Rockafellar and Wets (1998, Chapter 1 H.) we obtain for functions $f, g : E \longrightarrow \overline{\mathbb{R}}$

$$\operatorname{epi}_s(f \wedge g) = \operatorname{epi}_s(f) + \operatorname{epi}_s(g). \tag{3.13}$$

For a detailed proof of this statement we refer to [K] (2013).

We need the following lemma.

Lemma 3.6 *For functions $f, g : E \longrightarrow \overline{\mathbb{R}}$, the following statements hold*

1) *For $(x, r) \in \operatorname{epi}_s(f)$ there exists $\varepsilon > 0$ such that* $\{(x, s) \mid s \in [r - \varepsilon, r]\} \subset \operatorname{epi}_s(f)$.

2) *For* $(x, r) \in \mathrm{epi}_s(f) + \mathrm{epi}_s(g)$ *there exists* $\varepsilon > 0$ *such that* $\{(x, s) \mid s \in [r - \varepsilon, r]\} \subset \mathrm{epi}_s(f) + \mathrm{epi}_s(g)$.

Proof: See [K] (2013). □

As consequence we obtain a characterization of local exactness by a geometric epigraph condition.

Theorem 3.7 (Local existence and epigraph condition) *For functions* $f, g : E \longrightarrow \bar{\mathbb{R}}$ *the following statements are equivalent*

1) $f \wedge g$ *is exact at* $x \in E$.

2) $(x, (f \wedge g)(x)) \in \mathrm{epi}(f) + \mathrm{epi}(g)$.

3) $(f \wedge g)(x) = \min\{\alpha \mid (x, \alpha) \in \mathrm{epi}(f) + \mathrm{epi}(g)\} \in \mathbb{R}$.

Proof:

1) $\Rightarrow$ 2) There exist x_1, x_2 such that $f(x_1) + g(x_2) = (f \wedge g)(x)$ thus it follows

$$(x, (f \wedge g)(x)) = (x_1, f(x_1)) + (x_2, g(x_2)) \in \mathrm{epi}(f) + \mathrm{epi}(g).$$

2) $\Rightarrow$ 3) a) First we show that it generally holds

$$(f \wedge g)(x) = \inf\{\alpha \mid (x, \alpha) \in \mathrm{epi}(f) + \mathrm{epi}(g)\}. \tag{3.14}$$

Since $\mathrm{epi}(f) + \mathrm{epi}(g) \subseteq \mathrm{epi}(f \wedge g)$ and obviously

$$(f \wedge g)(x) = \inf\{\alpha \mid (x, \alpha) \in \mathrm{epi}(f \wedge g)\} \tag{3.15}$$

we get $(f \wedge g)(x) \leqslant \inf\{\alpha \mid (x, \alpha) \in \mathrm{epi}(f) + \mathrm{epi}(g)\}$. Assume now that $(f \wedge g)(x) < \inf\{\alpha \mid (x, \alpha) \in \mathrm{epi}(f) + \mathrm{epi}(g)\} := r^*$. For the strict epigraph this implies together with (3.13)

$$(x, r^*) \in \mathrm{epi}_s(f \wedge g) = \mathrm{epi}_s(f) + \mathrm{epi}_s(g). \tag{3.16}$$

Lemma 3.6 then grants the existence of an $s^* < r^*$ such that $(x, s^*) \in \mathrm{epi}_s(f) + \mathrm{epi}_s(g)$. Then the inclusion

$$(x, s^*) \in \mathrm{epi}_s(f) + \mathrm{epi}_s(g) \subseteq \mathrm{epi}(f) + \mathrm{epi}(g)$$

provides a contradiction to the minimality of r^*. Thus (3.14) holds.

b) Since $(x, (f \wedge g)(x)) \in \mathrm{epi}(f) + \mathrm{epi}(g)$ it follows that the infimum in (3.14) is attained for $(f \wedge g)(x)$.

3) $\Rightarrow$ 1) Because the minimum is attained, there exist $(x_1, r_1) \in \operatorname{epi}(f)$ and $(x_2, r_2) \in \operatorname{epi}(g)$ with $x = x_1 + x_2$ and $r_1 + r_2 = (f \wedge g)(x)$. From $r_1 \geqslant f(x_1)$ and $r_2 \geqslant g(x_2)$ it follows $f(x_1) + g(x_2) \leqslant (f \wedge g)(x)$ and thus $f \wedge g$ is exact in $x \in E$. □

In general the epigraph condition (EC) for f^*, g^* in (3.9) is not easy to check. Various interior point conditions have been stated in [KR] (2010) to imply the epigraph condition. These conditions are also extended there to the case of more than two functions (see [KR] (2010, Proposition 5.10)) and thus imply some general checkable conditions for the existence of optimal allocations also in the general Banach space case. We restate the main result in this direction. Except for the *strong intersection condition* (SIS), each interior point condition has to be stated in a system of $(n-1)$ conditions.

Proposition 3.8 *For lower semicontinuous functions $g_1, \dots, g_n$ any of the following interior point conditions implies the epigraph condition*

$$\operatorname{epi}\left(\left(\sum_{i=1}^{n} g_i^*\right)^*\right) = \sum_{i=1}^{n} \operatorname{epi} g_i. \tag{3.17}$$

$$(SIS) \quad \bigcap_{i=1}^{n-1} \operatorname{int}\operatorname{dom} g_i^* \cap \operatorname{dom} g_n^* \neq \emptyset; \tag{3.18}$$

$$0 \in \operatorname{core}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right), \quad k \in \{2, \dots, n\}; \tag{3.19}$$

$$0 \in \operatorname{sqri}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right), \quad k \in \{2, \dots, n\}; \tag{3.20}$$

$$\begin{gathered} 0 \in \operatorname{icr}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right) \textit{ and} \\ \operatorname{aff}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right) \textit{ is a closed subspace}, \quad k \in \{2, \dots, n\}. \end{gathered} \tag{3.21}$$

4 Uniqueness of optimal allocations

In this section we extend some uniqueness results given in Jouini et al. (2006), Acciaio (2007), Filipović and Svindland (2008), and [KR] (2008; 2010) to the general frame of Banach spaces. It is shown in these papers that in the case of cash invariance, optimal allocations are unique up to rebalancing the cash, i.e. if $(\eta_1, \dots, \eta_n)$

is optimal then any allocation $(\eta_1 + c_1, \dots, \eta_n + c_n)$ with $(c_1, \dots, c_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n c_i = 0$ is optimal, too. This is a direct consequence of the cash invariance which allows to transfer cash freely between the individuals and thus without changing the overall risk. In the multivariate case the same is true. Let $\Psi_1, \dots \Psi_n$ be multivariate cash invariant risk functionals, let $(\eta_1, \dots, \eta_n)$ be a optimal allocation and $(c_1, \dots, c_n) \in \mathbb{R}^{n\times d}$ with $c_i = (c_i^1, \dots, c_i^d)$ such that $\sum_{i,j} c_i^j = 0$ then we get for the new allocation $(Z_1, \dots, Z_n) := (\eta_1 + c_1, \dots, \eta_n + c_n)$

$$\sum_{i=1}^n \Psi_i(Z_i) = \sum_{i=1}^n \Psi_i(X_i + c_i) = \sum_{i=1}^n \Psi_i(X_i) + \sum_{j=1}^d c_i^j = \sum_{i=1}^n \Psi_i(X_i).$$

Hence it is optimal as well and thus optimality of one allocation goes hand in hand with a whole class of rebalanced optimal allocations. Uniqueness is closely coupled with strict convexity. A function is called strictly convex if the inequality in the definition of convexity is strict for any $x, y \in E$, $x \neq y$ and $\alpha \in (0,1)$, i.e. $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$. In Filipović and Svindland (2008) the main argument for the uniqueness under strict convexity is that any convex combination of two optimal allocations has an aggregated risk which is strictly smaller than the value of the infimal convolution and thus contradicts the optimality assumption. We use a different approach which gives us more information on the structure of the unique allocations.

Lemma 4.1 *Let $\varrho : E \longrightarrow \overline{\mathbb{R}}$ be a strictly convex risk functional. Then for any $x^* \in E^*$ holds*

$$|\partial \varrho^*(x^*)| \leqslant 1. \tag{4.1}$$

Proof: The conjugate of ϱ is defined by

$$\varrho^*(x^*) = \sup_{x\in E}\{\langle x^* \mid x\rangle - \varrho(x)\}, \quad x^* \in E^*.$$

Strict convexity of ϱ implies that for any $x^* \in E^*$ there is at most one maximizer $\bar{x} \in E$ of $\langle x^* \mid x\rangle - \varrho(x)$, such that $\varrho^*(x^*) = \langle x^* \mid \bar{x}\rangle - \varrho(\bar{x})$. Since this exactly characterizes subgradients, it follows that $|\partial\varrho^*(x^*)| \leqslant 1$. □

Due to the characterization (2.8′) of optimal allocations in Theorem 2.2, we conclude that the strict convexity of the j-th risk functional ϱ_j of the infimal convolution $\bigwedge \varrho_i$ implies the uniqueness of j-th risk contribution $\xi_j \in E$ of an optimal allocation $(\xi_1, \dots, \xi_n) \in E^n$.

Proposition 4.2 (Uniqueness and strict convexity) *Let $\varrho_i : E \longrightarrow \overline{\mathbb{R}}, i \in \{1, \dots, n\}$ be risk functionals, with ϱ_j strictly convex and the minimal total risk problem well posed then in any optimal allocation $(\xi_1, \dots, \xi_n) \in E^n$ the j-th risk contribution ξ_j is unique.*

Proof: Since optimal allocations $(\xi_1, \ldots, \xi_n) \in E^n$ are characterized by the existence of a dual element $V \in E^*$ such that $\xi_i \in \partial\varrho^*(V)$ for all $i \in \{1, \ldots, n\}$, the strict convexity of ϱ_j implies by Lemma 4.1 that $\partial\varrho_j^*(V)$ is a singleton for any $V \in E^*$ and the claim follows. □

Hence any optimal allocation is unique if at least $(n-1)$ risk functionals are strictly convex.

Corollary 4.3 (Uniqueness and strict convexity) *Let at least $(n-1)$ of the risk functionals $\varrho_i : E \longrightarrow \overline{\mathbb{R}}, i \in \{1, \ldots, n\}$ be strictly convex, such that the minimal total risk problem is well-posed. Then an optimal allocation is unique.*

For cash invariant risk functionals these uniqueness result does not apply, since they are not strictly convex on the affine subspace generated by addition of constants, i.e. on $E+\mathbb{R}$ in the univariate case and on $E^d+\mathbb{R}^d$ in the multivariate case (see [KR] (2008)). Thus strict convexity for cash invariant risk functionals makes only sense if the inequality in the definition of convexity is strict for $x, y \in E$ with $x - y \notin \mathbb{R}$. We call these *strict convex cash invariant* risk functionals. Before adapting the uniqueness results to cash invariant risk functionals, we give a characterization of the strict convexity in the case of cash invariance.

Definition 4.4 *Let E be a real Banach space containing the real constants $\mathbb{R}$. Then $(E, E^*, \langle \cdot \mid \cdot \rangle)$ is called a dual pairing* **with dual unit** *if the dual space E^* contains an element $\mathbf{1} \in E^*$, such that*

$$\langle \mathbf{1} \mid c \rangle = c, \quad \forall c \in \mathbb{R}.$$

In the multivariate case $(E^d, (E^*)^d, \langle \cdot \mid \cdot \rangle_d)$ for the dual unit $\mathbf{1}^d \in (E^*)^d$ we postulate

$$\langle \mathbf{1}^d \mid \mathbf{c} \rangle_d = \sum_{i=1}^{d} c_i \quad \forall \mathbf{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d.$$

Example 4.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $1 \leqslant p < \infty$. Then $(L^p(P), L^q(P), \langle \cdot \mid \cdot \rangle)$ with $\langle X^* \mid X \rangle := \mathbf{E}[X^* X]$ is a dual pairing with dual unit $\mathbf{1}(\omega) = 1$ P-a.s..*

Further we introduce the equivalence relation $\sim_+$ on E^d by

$$x \sim_+ y :\Longleftrightarrow\ x - y \in \mathbb{R}^d \tag{4.2}$$

and the sets

$$E^d_\kappa := \{x \in E^d \mid \langle \mathbf{1} \mid x_i \rangle = \kappa, \forall 0 \leqslant i \leqslant d\}, \quad \kappa \in \mathbb{R}. \tag{4.3}$$

Lemma 4.5 *For any fixed $\kappa \in \mathbb{R}$ the set E^d_κ is a representation set of the equivalence classes in the quotient space $E^d/_{\sim_+}$.*

Proof: We refer to [K] (2013). □

Example 4.2 *In the $L^p(P)$–spaces E_0 is the set of all centered random variables, i.e. $E_0 = \{X \in L^p(P) \mid \mathbf{E}_P[X] = 0\}$. In the spaces $L^p_d(P)$ holds $E^d_0 = \{X \in L^p_d(P) \mid \mathbf{E}_P[X_i] = 0, \forall i\}$*

Proposition 4.6 *For a cash invariant risk functional $f : E^d \longrightarrow \overline{\mathbb{R}}$ on a real Banach space with constants whose dual E^* contains a dual unit, the following statements are equivalent:*

1) $$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y), \ \forall x, y \in E^d \text{ with } x - y \notin \mathbb{R}^d. \tag{4.4}$$

2) *f is strictly convex on the subspace E^d_0.*

Proof: we refer to [K] (2013). □

With this result we are able to extend Proposition 4.2 and Corollary 4.3 to the case where f_i are cash invariant and strictly convex on E^d_0. Hence for f_j strictly convex on E^d_0 the corresponding risk contribution ξ_j of an optimal allocation $(\xi_1, \dots, \xi_n)$ is unique in E^d_0. Since ξ_j is a representative for the equivalence class $\{\xi_j + \mathbb{R}^d\}$ and the cash invariance of f_j implies

$$\partial f_j(\xi_j) = \partial f_j(\xi_j + \mathbf{c}), \quad \forall \mathbf{c} \in \mathbb{R}^d, \tag{4.5}$$

we get uniqueness of ξ_j up to additive vectors.

Corollary 4.7 (Uniqueness of cash invariant risk functionals) *If at least $(n-1)$ cash invariant risk functionals are strictly convex on E^d_0 an optimal allocation is unique up to rebalancing the cash.*

5 Optimal allocations for expected risks in Orlicz hearts

In the following part of the paper we discuss in detail the optimal allocation problem for expected risk functionals. For the expected risk as in Example 2.1 we need a polynomial growth condition (see (2.20)). Thus for example in case $d = 1$ the exponential risk functional

$$r(x) = \exp(\alpha x), \quad \alpha > 0 \tag{5.1}$$

is not included. We show that the general frame of Banach spaces as used in this paper allows to deal with the allocation problem for general expected risk functionals.

In the following the risk functions $r : \mathbb{R} \longrightarrow \mathbb{R}$ are considered to be strictly convex, continuously differentiable on $\mathbb{R}$ and increasing on $\mathbb{R}^+$. Further $(\Omega, \mathcal{F}, P)$ is a non-atomic probability space. In the first part of this section we remind some basic notations and study the Gâteaux differentiability of these functions on Orlicz hearts.

5.1 Gâteaux differentiability on Orlicz Hearts

A function $\Phi : [0, \infty) \longrightarrow [0, \infty)$ is called a Young function if it is left continuous, convex, normed, i.e. $\Phi(0) = 0$, and non-trivial. These properties imply that Φ is strictly increasing and that it is continuous except at possibly one point, where it jumps to ∞. It follows that Φ is also continuously differentiable almost everywhere with derivative $\Phi' = \varphi$ satisfying

$$\Phi(x) = \int_0^x \varphi(u)\,\mathrm{d}u.$$

If $\Phi(x) = \infty$ for some $x \in \mathbb{R}^+$ we set $\varphi(x) = \infty$. The derivative $\Phi' = \varphi$ is called Young derivative of the Young function Φ.

Lemma 5.1 *For a risk function r, $\Phi_r : [0, \infty) \longrightarrow [0, \infty)$ defined by*

$$\Phi_r(x) := r(x) - r(0)$$

is a strict convex Young function. We call it the Young function associated with r. Particularly the Young derivative is $\varphi_r(x) = r'(x)$, $\forall x \in \mathbb{R}^+$.

The Orlicz spaces resp. Orlicz hearts are function spaces of real valued functions defined for a measure space $(\Omega, \mathcal{F}, P)$, where functions are identified if they coincide P-almost surely.

Definition 5.2 *The Orlicz space L_Φ corresponding to the Young function Φ is defined by*

$$L_\Phi = L_\Phi(\Omega, \mathcal{F}, P) := \Big\{X \in L^0(\Omega) \mid \int_\Omega \Phi\left(\frac{|X|}{c}\right) \mathrm{d}P < \infty \text{ for some } c > 0\Big\}.$$

The Orlicz heart H_Φ corresponding to the Young function Φ is defined by

$$H_\Phi = H_\Phi(\Omega, \mathcal{F}, P) := \left\{X \in L^0(\Omega) \mid \int_\Omega \Phi\left(\frac{|X|}{c}\right) \mathrm{d}P < \infty \textit{ for all } c > 0\right\}.$$

Equipped with the Luxemburg norm

$$\|X\|_\Phi := \inf\left\{c > 0 \mid \int \Phi\left(\frac{|X|}{c}\right) \mathrm{d}P < \infty\right\} \tag{5.2}$$

L_Φ and H_Φ are Banach spaces. Note that the Orlicz space L_{Φ_r} contains any random variable X which has finite expected risk in the following sense:

For a random variable $X \in L^0$ and $c > 0$ holds

$$\mathbf{E}[r(c|X|)] < \infty \text{ for some } c > 0 \text{ if and only if } X \in L_{\Phi_r}.$$

Furthermore,

$$\mathbf{E}[r(c|X|)] < \infty \text{ for all } c > 0 \text{ if and only if } X \in H_{\Phi_r}. \tag{5.3}$$

It is possible that an optimal allocation $(\xi_1, \ldots, \xi_n)$ may contain components ξ_i which are proportional to the initial risk variable X, i.e. $\xi_i = \lambda_i X$ for an $\lambda_i > 0$ (see (2.25)).Thus the Orlicz heart is a sensible restriction of the domain of the minimal total risk problem and of the underlying risk functionals, as well.

From now on we define for a risk function r the expected risk functional ϱ_r on the Orlicz heart, which corresponds to the Young function Φ_r associated to r, i.e.

$$\varrho_r : H_{\Phi_r} \longrightarrow \mathbb{R},\ \varrho_r(X) := \mathbf{E}[r(X)].$$

Corollary 5.3 *For a risk function $r : \mathbb{R} \longrightarrow \mathbb{R}$ the induced expected risk functional $\varrho_r(\cdot) := \mathbf{E}[r(\cdot)]$ is continuous and subdifferentiable on H_{Φ_r}.*

Proof: We know that r is either strictly increasing on $\mathbb{R}$ or is strictly decreasing on R^- and strictly increasing on R^+. Further generally it holds $\operatorname{intdom}(\varrho_r) = H_{\Phi_r}$. Then the claim follows in the first case directly from the extended Namioka–Klee theorem (see Biagini and Frittelli (2009)). For the second case we split r into the two functions r_+^0 and r_-^0 defined by

$$r_+^0(x) := \begin{cases} r(x) - r(0), & x \geqslant 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad r_-^0(x) := \begin{cases} 0, & x > 0, \\ r(x) - r(0), & x \leqslant 0. \end{cases}$$

Then it holds

$$r(x) = r_+^0(x) + r_-^0(x) + r(0),$$

where r_+^0 is increasing and r_-^0 is decreasing on $\mathbb{R}$. Now the extended Namioka–Klee theorem states, that the finite and increasing functions $\varrho_+(X) := \mathbf{E}[r_+^0(X)]$ and $\varrho_-(X) := \mathbf{E}[r_-^0(-X)]$ are continuous and subdifferentiable on H_{Φ_r}. Thus the continuity of ϱ_r follows since $\varrho_-(0) = \varrho_+(0)$. The subdifferentiability follows from the identity

$$\partial\big(\mathbf{E}[r_-^0(\cdot)]\big)(X) = -\partial\varrho_-(-X).$$

and from the generally true inclusion of the subdifferential sum formula

$$\partial\varrho_r(X) \supseteq \partial\varrho_+(X) - \partial\varrho_-(-X). \qquad \square$$

Next we show that $\varrho_r : H_{\Phi_r} \longrightarrow \mathbb{R}$ is Gâteaux differentiable everywhere by showing that the operator

$$F(Y) := \mathbf{E}[r'(X)Y],\ Y \in H_{\Phi_r} \tag{5.4}$$

is linear and continuous on H_{Φ_r} for every $X \in H_{\Phi_r}$. We remind that the dual space of H_{Φ_r} is the Orlicz space L_{Ψ_r}, which corresponds to the complementary Young function Ψ_r of Φ_r given by

$$\Psi_r(y) = \sup_{x\in\mathbb{R}^+}\{xy - \Phi_r(x)\}.$$

Lemma 5.4 *The operator $F : H_{\Phi_r} \longrightarrow \mathbb{R}$, $F(Y) := \mathbf{E}[r'(X)Y]$ is linear and continuous on H_{Φ_r} for every $X \in H_{\Phi_r}$.*

Proof: The linearity is clear. It thus remains to prove the continuity for which we show that $r'(X) \in L_{\Psi_r}$ for all $X \in L_{\Phi_r}$. Note that the form of r implies $r' > 0$ and φ_r strictly increasing. Using the Young (in)equality, the right directional derivative of ϱ_r and the inequality

$$f(x) - f(x-y) \leqslant \mathcal{D}(f,x)(y) \leqslant f(x+y) - f(x) \tag{5.5}$$

we derive for $X \in H_{\Phi_r}$:

$$\begin{aligned}
\mathbf{E}[\Psi_r(1|r'(X)|)] &\leqslant \mathbf{E}[\Psi_r(r'(|X|))] = \mathbf{E}[\Psi_r(\varphi_r(|X|))] \\
&= \mathbf{E}[r'(|X|)|X|] - \mathbf{E}[\Phi_r(|X|)] \\
&= \mathcal{D}(\varrho_r, |X|)(|X|) - \mathbf{E}[\Phi(|X|)] \\
&\leqslant \varrho_r(2|X|) - \varrho_r(|X|) - \mathbf{E}[\Phi(|X|)] \\
&< \infty.
\end{aligned}$$

The last inequality follows from (5.3). Thus $r'(X) \in L_{\Psi_r}$, $\forall X \in H_{\Phi_r}$.
Hence the operator $G : L_{\Phi_r} \longrightarrow \mathbb{R}$ defined by

$$G(Y) := \mathbf{E}[r'(X)Y]$$

is linear and continuous on L_{Φ_r} for all $X \in H_{\Phi_r}$ due to the general inequality

$$\frac{1}{2}\|r'(X)\|_\Psi \leqslant \|G\| \leqslant \|r'(X)\|_\Psi, \tag{5.6}$$

where $\|G\|$ is the norm of G in $(L_\Phi)^*$. The restriction $G_{|H_{\Phi_r}}$ of G to H_{Φ_r} coincides with F, which therefore is linear and continuous itself. Additionally we get from a Riesz representation theorem for Orlicz hearts that $r'(X)$ is the uniquely determined element of L_{Ψ_r} such that

$$F(Y) := \mathbf{E}[r'(X)Y].$$

□

Corollary 5.5 *The risk functional $\varrho_r : H_{\Phi_r} \longrightarrow \mathbb{R}$ defined by*

$$\varrho_r(X) := \mathbf{E}[r(X)]$$

is Gâteaux differentiable, with Gâteaux gradient $\nabla\varrho_r(X) = r'(X)$ for all $X \in H_{\Phi_r}$.

Proof: Since the right directional derivative of ϱ_r at X in direction Y has the form

$$\mathcal{D}(\varrho_r, X)(Y) = \mathbf{E}[r'(X)Y]$$

we get by Lemma 5.4 that it is linear and continuous in Y for all $X \in H_{\Phi_r}$ and thus is Gâteaux differentiable everywhere, with Gâteaux gradient $\nabla\varrho_r(X) = r'(X)$. □

5.2 Matching Orlicz hearts

in this subsection we assume that the risk functions $r_1, \dots, r_n$ are such that the Orlicz hearts coincide, i.e. $H_{\Phi_{r_i}} =: H$, for all i, for a Banach space H.

Corollary 5.6 (Matching Orlicz hearts) *The infimal convolution $\bigwedge \varrho_{r_i}$ defined on H with respect to $\varrho_{r_1}, \dots, \varrho_{r_n}$ is well posed for all $X \in H$.*

Proof: Recall that the infimal convolution is well posed at an $X \in H$ if the mapping $\bar{\varrho} : H^n \longrightarrow \mathbb{R}$ defined for $\bar{X} = (X_1, \dots, X_n)$ by

$$\bar{\varrho}(\bar{X}) := \sum_{i=1}^n \varrho_{r_i}(X_i)$$

fulfills

$$\operatorname{domc}(\bar{\varrho}) \cap \mathcal{A}(X) \neq \emptyset.$$

The continuity of $\bar{\varrho}$ follows from the continuity of the expected risk functionals ϱ_{r_i}, $1 \leqslant i \leqslant n$ shown in Corollary 5.3. Let $X \in H$, then the allocation $(X, 0, \ldots, 0) \in \mathcal{A}(X)$ lies in $\operatorname{domc}(\bar{\varrho})$ and therefore the infimal convolution is well posed. $\square$

Thus the characterization of optimal allocations in the sense of Section 2 applies to the setup of expected risk, where the Orlicz hearts of the Young functions associated to the underlying risk functions coincide.

Example 5.7 Let $r_1, \ldots, r_n : \mathbb{R} \longrightarrow \mathbb{R}$ a family of risk functions of the form

$$r_i(x) := \exp(\alpha_i x),\ \alpha_i > 0,\ \forall i.$$

Then the Orlicz hearts $H_{\Phi_{r_i}}$ coincide for all i. The induced risk functionals are finite and Gâteaux differentiable on $H_{\Phi_{r_1}}$ with Gâteaux gradient $\nabla \varrho_{r_i}(X) = r_i'(X)$, $\forall i$ and $X \in H_{\Phi_{r_1}}$. Since the infimal convolution with respect to $\varrho_{r_1}, \ldots, \varrho_{r_n}$ is well posed, we get from the characterization result Theorem 2.2, that the allocation $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$ is optimal for an $X \in H_{\Phi_{r_1}}$ if and only if

$$\xi_1 = \left(1 + \alpha_1 \sum_{j=2}^{d} \frac{1}{\alpha_i}\right)^{-1} \left(X - \sum_{j=2}^{d} \frac{1}{\alpha_j} \log\left(\frac{\alpha_1}{\alpha_j}\right)\right), \tag{5.7}$$

and from (2.23)

$$\xi_i = \frac{1}{\alpha_i} \log\left(\frac{\alpha_1}{\alpha_i}\right) + \frac{\alpha_1}{\alpha_i} \xi_1,\ 2 \leqslant i \leqslant n. \tag{5.8}$$

5.3 Nonidentical Orlicz Hearts

For the application of Section 2 to allocation problems the risk functionals have to be defined on the same Banach space. In Example 5.7 we have no issue with this matter, because we have chosen the risk functions in such a way that the Orlicz hearts corresponding to the associated Young functions coincide. In the following we study under which conditions on the risk functions it is still possible to find a joint Banach space even if the risk functions do not generate identical Orlicz hearts.

We introduce two orders on the class of Young functions.

Definition 5.8 *Let Φ_1 and Φ_2 be two Young functions.*

1. *We write $\Phi_1 \prec \Phi_2$, if there exist $c, T > 0$ such that*

$$\Phi_1(t) \leqslant \Phi_2(ct),\ \text{for } t \geqslant T.$$

If $\Phi_1 \prec \Phi_2$ and $\Phi_1 \succ \Phi_2$ we say that Φ_1, Φ_2 are equivalent.

2. *We write* $\Phi_1 \lll \Phi_2$, *if*

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(ct)} = 0, \textit{ for all } c > 0.$$

The following theorem shows how the Orlicz spaces of two Young functions relate, if they are ordered in the sense above (see Rao and Ren (1991, Section V)).

Theorem 5.9 *Let* Φ_1, Φ_2 *be two Young functions and let* $(\Omega, \mathcal{F}, P)$ *be a non–atomic probability space.*

1) *The inclusion*

$$L_{\phi_1} \subseteq L_{\Phi_2} \textit{ holds if and only if } \Phi_2 \prec \Phi_1.$$

Furthermore, if $L_{\phi_1} \subseteq L_{\Phi_2}$, *then there exists a* $k > 0$ *such that* $\|X\|_{\Phi_2} \leqslant k\|X\|_{\Phi_1}$. *Hence, we have the embedding* $L_{\phi_1} \hookrightarrow L_{\Phi_2}$.

2) *The inclusion*

$$L_{\phi_1} \subseteq H_{\Phi_2} \textit{ holds if } \Phi_2 \lll \Phi_1.$$

For Orlicz hearts a similar result is true.

Theorem 5.10 *Let* Φ_1, Φ_2 *be two Young functions and* $(\Omega, \mathcal{F}, P)$ *a finite measure space. The inclusion*

$$H_{\phi_1} \subseteq H_{\Phi_2} \textit{ holds if } \Phi_2 \prec \Phi_1.$$

Proof: If $\Phi_2 \prec \Phi_1$ then there exist $c > 0$ and $T \geqslant 0$ such that

$$\Phi_2(t) \leqslant \Phi_1(ct) \text{ for } t \geqslant T. \tag{5.9}$$

Let $X \in H_{\Phi_1}$. Then by Definition 5.2

$$\int \Phi_1\left(\frac{|X|}{k}\right) \mathrm{d}P < \infty \text{ for all } k > 0, \text{ i.e. } \frac{X}{k} \in Y_{\Phi_1}, \forall k > 0.$$

We define a sequence of subsets $\Omega_k \subset \Omega$ by

$$\Omega_k := \{\omega \in \Omega \mid |X(\omega)| < kT\}.$$

From (5.9) we get

$$\Phi_2\left(\frac{|X(\omega)|}{k}\right) \leqslant \Phi_1\left(\frac{|cX(\omega)|}{k}\right), \text{ for } \omega \in \Omega\backslash\Omega_k,\ k > 0.$$

Hence we obtain

$$\begin{aligned}\mathbf{E}\left[\Phi_2\left(\frac{|X|}{k}\right)\right] =& \mathbf{E}\left[\Phi_2\left(\frac{|X|}{k}\right)\mathbb{1}_{\Omega_k}\right] + \mathbf{E}\left[\Phi_2\left(\frac{|X|}{k}\right)\mathbb{1}_{\Omega\setminus\Omega_k}\right] \\ \leqslant& \Phi_2(T)P(\Omega_k) + \mathbf{E}\left[\Phi_1\left(\frac{c|X|}{k}\right)\right] \\ \leqslant& \Phi_2(T)P(\Omega) + M_{\Phi_1}\left(\frac{c}{k}X\right) \\ <& \infty, \; \forall k > 0.\end{aligned}$$

Thus $X \in H_{\Phi_2}$. □

Remark: If the Young functions are equivalent, i.e. if $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$, then the Orlicz spaces L_{Φ_i} (resp. the Orlicz hearts H_{Φ_i}) coincide for all i. □

Corollary 5.11 *Let r_1, r_2 be two risk functions such that the associated Young functions Φ_1, Φ_2 are ordered by*

$$\Phi_2 \prec \Phi_1.$$

Then the mapping $\widehat{\varrho}_{r_2}$ defined as the restriction of ϱ_{r_2} to H_{Φ_1} is a finite, continuous, subdifferentiable increasing risk functional on H_{Φ_1}. In particular, it is Gâteaux differentiable on H_{Φ_1} with gradient $\nabla\widehat{\varrho}_{r_2}(X) = r'(X)$ for all $X \in H_{\Phi_2}$.

Remark: The continuity of $\widehat{\varrho}_{r_2}$ on H_{Φ_1} is understood with respect to the topology induced by the norm $\|\cdot\|_{L_{\Phi_1}}$. □

Proof: The properness and convexity are obvious. In addition, from Theorem 5.10, we get $H_{\Phi_1} \subseteq H_{\Phi_2}$, hence the finiteness is also obvious. The continuity follows from Corollary 5.3 and the fact that H_{Φ_1} is embedded in H_{Φ_2}, see Theorem 5.9. Therefore, it remains to show that $\widehat{\varrho}_{r_2}$ is Gâteaux differentiable for all $X \in H_{\Phi_1}$.

The proof of Lemma 5.4 provides

$$r_2'(X) \in L_{\Psi_2}, \; \forall X \in H_{\Phi_2}.$$

Thus by Theorem 5.10 it holds

$$r_2'(X) \in L_{\Psi_2}, \; \forall X \in H_{\Phi_1}.$$

This implies by the embedding result and the equivalence of the orderings for Young functions and their complementary Young functions

$$r_2'(X) \in L_{\Psi_1}, \; \forall X \in H_{\Phi_1}.$$

Following the arguments in the proof of Lemma 5.4 we therefore get that

$$F : H_{\Phi_1} \longrightarrow \mathbb{R} \text{ defined by } F(Y) := \mathbf{E}[r_2'(X)Y]$$

is continuous and linear for all $X \in H_{\Phi_1}$. Finally the Gâteaux differentiability follows as in Corollary 5.5. The continuity of $\widehat{\varrho}_{r_2}$ with respect to the topology induced by the norm $\|\cdot\|_{L_{\Phi_1}}$ is guaranteed, since H_{Φ_1} is embedded in H_{Φ_2}. $\square$

Following the line of arguments in Corollary 5.6, the infimal convolution is well posed.

Corollary 5.12 *Let $r_1, \ldots, r_n$ be risk functions such that the associated Young functions $\Phi_1, \ldots, \Phi_n$ are ordered in the following sense*

$$\Phi_i \prec \Phi_1, \; \forall i \in \{2, \ldots, n\}. \tag{5.10}$$

Then the infimal convolution with respect to $\widehat{\varrho}_{r_i} : H_{\Phi_1} \longrightarrow \mathbb{R}$ defined by $\widehat{\varrho}_{r_i}(X) := \mathbf{E}[r_i(X)]$, $i \in \{1, \ldots, n\}$ is well posed for all $X \in H_{\Phi_1}$. Further the characterization of optimal allocations in the sense of Section 2 applies.

We close this section by giving an existence criterion. Here the function $h : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ defined by

$$h(y) := y + \sum_{i=2}^{n} (r_i')^{-1}(r_1'(y)) \tag{5.11}$$

plays a major role. We assume that the image $\operatorname{Im}(h) \neq \emptyset$.

Theorem 5.13 (Existence) *Let $r_1, \ldots, r_n$ be risk functions such that the associated Young functions $\Phi_1, \ldots, \Phi_n$ are ordered in the sense of (5.10) and let $X \in H_{\Phi_1}$. Then the minimal total risk problem $\bigwedge \widehat{\varrho}_{r_i}$ is exact at X if and only if*

$$X(\Omega) \subseteq \operatorname{Im}(h) \quad P\text{-a.s.} \tag{5.12}$$

Proof: Let $\bigwedge \widehat{\varrho}_{r_i}$ be exact at $X \in H_{\Phi_1}$. Then there exists $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$ such that $r_i'(\xi_i) = r_j'(\xi_j)$ P-a.s. for all $i, j \in \{1, \ldots, n\}$. Since the r_i are strictly convex their derivatives $r_i' : \mathbb{R} \longrightarrow \operatorname{Im}(r_i') \subset \mathbb{R}^+$ are invertible and thus

$$\xi_i = (r_i')^{-1}(r_1'(\xi_1)) \quad P\text{-a.s.} \forall i.$$

Since $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$ we get for P-a.s. all $\omega \in \Omega$

$$X(\omega) = \xi_1 + \sum_{i=2}^{n} (r_i')^{-1}(r_1'(\xi_1(\omega))).$$

Hence $X(\Omega) \subseteq \mathrm{Im}(h)$ P-a.s..

For the converse, assume $X(\Omega) \subseteq \mathrm{Im}(h)$ P-a.s.. Then for P-a.s. every $\omega \in \Omega$ there exists a real number $z_1(\omega)$ such that

$$X(\omega) = z_1(\omega) + \sum_{i=2}^{n} (r_i')^{-1}(r_1'(z_1(\omega))). \tag{5.13}$$

For every such real number $z_1(\omega)$ we define for the remaining i

$$z_i(\omega) := (r_i')^{-1}(r_1'(z_1(\omega))). \tag{5.14}$$

Note that this expression is well defined, i.e. $r_1'(z_1(\omega)) \in \mathrm{Im}(r_i')$ P-a.s. $\forall i$. Otherwise (5.13) would not be well defined and thus (5.12) would not hold. Therefore, we get for all $i, j \in \{1, \ldots, n\}$ and P-a.s. every $\omega \in \Omega$

$$r_i'(z_i(\omega)) = r_j'(z_j(\omega)).$$

The mappings $z_i : \Omega \longrightarrow \mathbb{R}$ defined by $\omega \mapsto z_i(\omega)$ are measurable for all $i \in \{1, \ldots, n\}$, because h and $(r_i')^{-1} \circ r_1'$ are continuous and invertible for all i. Therefore, if we can show that it holds $z_i \in H_{\Phi_1}$ for all i, it follows from Theorem 2.2 that $(z_1, \ldots, z_n) \in \mathcal{A}(X)$ and that this allocation minimizes the total risk problem $\bigwedge \widehat{\varrho}_{r_i}$ at X.

Since $X \in H_{\Phi_1}$, Φ_1 is increasing and

$$|z_i| \leqslant \sum_{j=1}^{n} |z_j| = |X|,$$

which follows from the construction of the z_i, we get

$$\mathbf{E}\left[\Phi_1\left(\frac{|z_i|}{c}\right)\right] \leqslant \mathbf{E}\left[\Phi_1\left(\frac{|X|}{c}\right)\right] < \infty \ \forall c > 0 \ i \in \{1, \ldots, n\}$$

and thus $z_i \in H_{\Phi_1}$ for all $i \in \{1, \ldots, n\}$. The proof is complete. □

Remark: The strict convexity of the risk functions r implies strict convexity of the corresponding expected risk functionals $\widehat{\varrho}_r$. Therefore, Corollary 4.3 implies the uniqueness of any optimal allocation. □

6 Numerical results for expected risks

The calculation of optimal allocations is not always as explicit as in Example 5.7. On the contrary, many examples (as in (2.26)) do not admit explicit presentations. In this section we numerically derive functions $f_1, \ldots, f_n : \mathbb{R} \longrightarrow \mathbb{R}$ such that $(f_1(X), \ldots, f_n(X)) \in \mathcal{A}(X)$ is an optimal allocation of X. We use the software MATHEMATICA 8.0 for these calculations.

Definition 6.1 *For an optimal allocation* $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$ *the corresponding optimal redistribution rules are real functions* $f_1, \ldots f_n : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\xi_i = f_i(X) \quad P\text{-a.s.}, \forall i \in \{1, \ldots, n\}.$$

Lemma 6.2 *Let* $r_1, \ldots, r_n$ *be risk functions such that the associated Young functions* $\Phi_1, \ldots, \Phi_n$ *are ordered in the sense of* (5.10). *Then for* $X \in H_{\Phi_1}$ *with* $X(\Omega) \subseteq \mathrm{Im}(h)$ P*-a.s. and* $x \in \mathbb{R}$ *we have:*

1) *the root* $f_1(x)$ *of the real function* $y \mapsto y + \sum_{i=2}^{n}(r_i')^{-1}(r_1'(y)) - x$*, and*

2) $f_2, \ldots, f_n$*, defined by* $f_i(x) := (r_i')^{-1}(r_1'(f_1(x)))$, $i \in \{2, \ldots, n\}$,

are optimal redistribution rules corresponding to the optimal allocation $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$.

Proof: The claims follow directly from the construction made in the proof of Theorem 5.13. □

By form $\xi_i = f_i(X)$ we are able to derive the value of the minimal total risk for an $X \in H_{\Phi_1}(P)$ by evaluating the right hand side of

$$\bigwedge \widehat{\varrho}_{r_i}(X) = \mathbf{E}[\sum_{i=1}^{n}(r_i \circ f_i)(X)].$$

A first calculation deals with the polynomial example in (2.26).

Example 6.3 We consider the risk functions $r_1(x) := \frac{1}{4}x^6$, $r_2(x) := \frac{1}{5}x^4$ and $r_3(x) := 3x^2$. For the associated Young functions Φ_1, Φ_2 and Φ_3 it holds $\Phi_3 \prec \Phi_2 \prec \Phi_1$ and therefore we define the risk functionals $\widehat{\varrho}_{r_i}, i = 1, 2, 3$ on $H_{\Phi_1} = L^6(P)$. Further the function $h : \mathbb{R} \to \mathbb{R}$ defined in (5.11) is bijective and takes the form

$$h(y) = y + \left(\frac{15}{8}\right)^{1/3} y^{5/3} + \frac{1}{4}y^5.$$

Thus we get from Theorem 5.13 that $\bigwedge \widehat{\varrho}_{r_i}$ is exact for all $X \in L^6(P)$. Using Lemma 6.2 we are able to derive the optimal redistributions f_1, f_2 and f_3, which are plotted in the following figure for positive $X \in L^6(P)$.

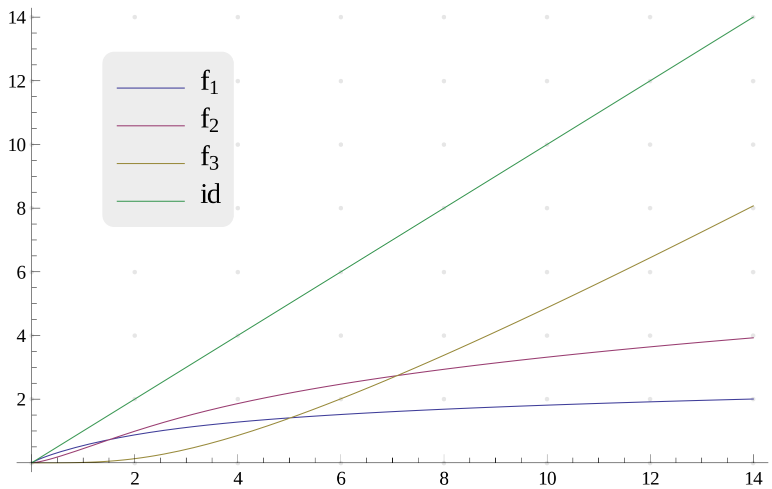

Figure 6.1: Optimal redistributions for $r_1(x) := \frac{1}{4}x^6$, $r_2(x) := \frac{1}{5}x^4$ and $r_3(x) := 3x^2$.

The green diagonal line in Figure 6.1 serves as a reference since we know that the sum of the optimal redistributions equals the identity. Particularly, this is a way to determine the error of the algorithm we used to derive the root of the function h. We distinguish between two errors. The first is the absolute error which is described by the *absolute error function* $e_{abs} : \mathbb{R} \longrightarrow \mathbb{R}$, ${}_{abs}(x) := \sum_{i=1}^{3} f_i(x) - x$.

And the second is the *relative error* which is described by the *relative error function* $e_{rel} : \mathbb{R} \longrightarrow \mathbb{R}$, $e_{rel}(x) := \frac{1}{x}\sum_{i=1}^{3} f_i(x) - 1$.

Figure 6.2 resp. Figure 6.3 show the absolute resp. relative error of the computations for $0 \leqslant x \leqslant 10^6$.

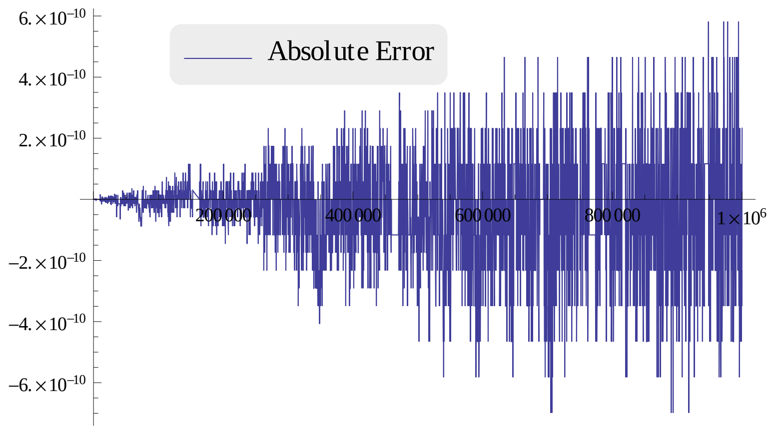

Figure 6.2: Absolute error produced by the algorithm.

Figure 6.3: Relative error produced by the algorithm.

The next example shows the computation of the optimal redistributions with respect to non-polynomial risk functions.

Example 6.4 We consider the risk functions $r_1(x) := \frac{1}{10}\exp(\frac{1}{5}x)$, $r_2(x) := 5x\log(x+1)$ and $r_3(x) := x^2$. For the associated Young functions Φ_1, Φ_2, and Φ_3 holds $\Phi_3 \prec \Phi_2 \prec \Phi_1$ and therefore we define the risk functionals $\widehat{\varrho}_{r_i}, i = 1,2,3$ on $H_{\Phi_1}(P)$. There is no explicit representation of the function h, since there is no explicit form of the inverse of r_2'. However, it is not difficult to show that the Image of h is $\mathbb{R}$ and thus $\bigwedge \widehat{\varrho}_{r_i}$ is exact for all $X \in H_\Phi$. The optimal redistributions and and the errors e_{abs} and e_{rel} are displayed below.

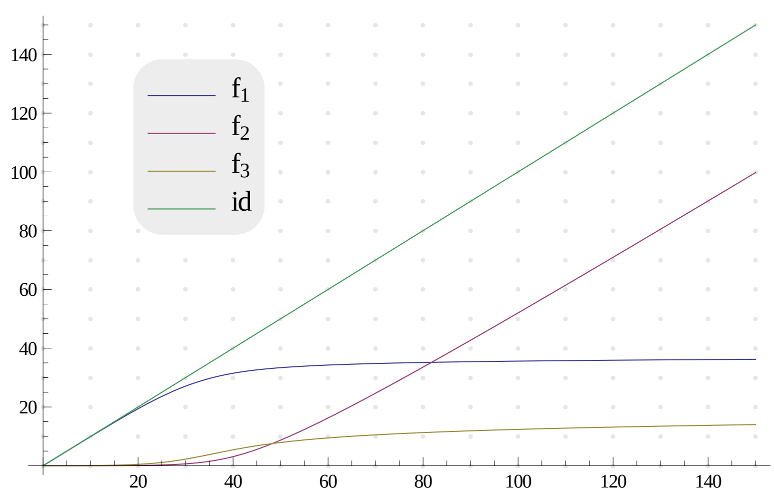

Figure 6.4: Optimal redistributions for $r_1(x) := \frac{1}{10}\exp(\frac{1}{5}x)$, $r_2(x) := 5x\log(x+1)$ and $r_3(x) := x^2$.

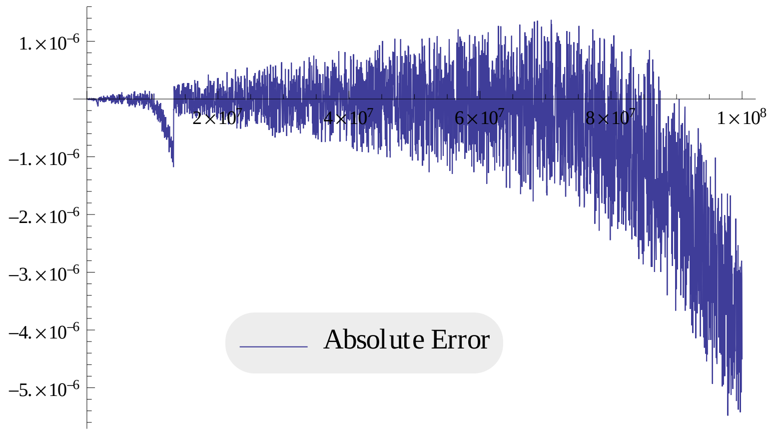

Figure 6.5: Absolute error produced by the algorithm.

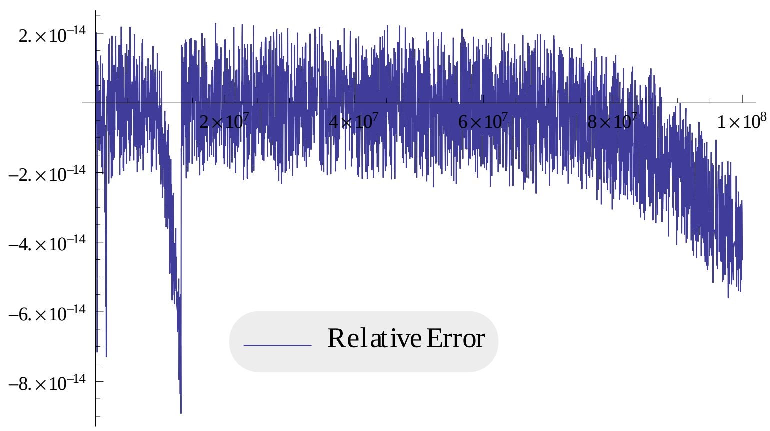

Figure 6.6: Relative error produced by the algorithm.

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