Optimal Payoffs under State-dependent Constraints

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Abstract

Most decision theories including expected utility theory, rank dependent utility theory and the cumulative prospect theory assume that investors are only interested in the distribution of returns and not about the states of the economy in which income is received. Optimal payoffs have their lowest outcomes when the economy is in a downturn, and this is often at odds with the needs of many investors. We introduce a framework for portfolio selection that permits to deal with state-dependent preferences. We are able to characterize optimal payoffs in explicit form. Some applications in security design are discussed in detail. We extend the classical expected utility optimization problem of Merton to the state-dependent situation and also give some some stochastic extensions of the target probability optimization problem.

Key-words: Optimal portfolio selection, state-dependent preferences, conditional distribution, hedging, state-dependent constraints.

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Introduction

The study of optimal investment strategies is usually based on the optimization of an expected utility, a target probability or some other (increasing) *law-invariant* measure. Assuming that investors have law-invariant preferences is equivalent to supposing that they care only about the distribution of returns and not about the states of the economy in which the returns are received. This is the case for example with Expected Utility Theory, Yaari's Dual Theory, Rank-dependent Utility Theory, Mean-Variance Optimization and Cumulative Prospect Theory. Clearly, an optimal strategy has some distribution of terminal wealth and it must be the cheapest possible strategy which attains this distribution. Otherwise it is possible to strictly improve the objective and to contradict its optimality. Dybvig (1988) was first to study strategies that reach a given return distribution at lowest possible cost. Bernard, Boyle and Vanduffel (2011) call these strategies cost-efficient and provide sufficient conditions for cost-efficiency. In a fairly general market setting they show that the cheapest way to generate a given distribution is obtained by a contract whose payoff has perfect negative correlation with the pricing kernel (see also Carlier and Dana (2011)). The basic intuition is that investors consume less in states of an economic recession because these are more expensive to insure. This feature is also explicit in a Black-Scholes framework where optimal payoffs at time horizon T are shown to be an *increasing* function of the price of the risky asset (as a representation of the economy) at time T. In particular, such payoffs are path-independent.

An important issue with the optimization criteria and the resulting payoffs under most standard frameworks, is that their worst outcomes are obtained when the market declines. Arguably, this property of optimal payoffs does not fit with the aspirations of investors who may seek protection against declining markets or, more generally, who may consider sources of background risk when making investment decisions. In other words, two payoffs with the same distribution do not necessarily present the same "value" for an investor. Bernard and Vanduffel (2012) show that insurance contracts can usually be substituted by financial contracts that have the same payoff distribution but are cheaper. The existence of insurance contracts that provide protection against specific events show that they must present more value for an investor than financial payoffs which do not have this feature. This observation supports the general observation that investors are more inclined to receive income in a "crisis" (for example when their property burns down or when the economy is in recession) than in "normal" conditions.

This paper makes several further theoretical contributions and highlights valuable applications in portfolio management and security design. First, we clarify the setting under which optimal investment strategies must necessarily exhibit path-independence. These findings complement Cox and Leland (1982), (2000) and Dybvig's (1988) seminal results and further enforces the important role of path-independence in traditional optimal portfolio selection.

As main contribution, we introduce a framework for portfolio selection that permits to also consider the states in which income is received. More precisely, it is assumed that investors target some distribution for their terminal wealth and *additionally* aim at maintaining a certain (desired) interaction with a random benchmark. For example, the investor may want his strategy to be unrelated to the benchmark when it decreases while following it under normal conditions. We are able to characterize optimal payoffs explicitly (Theorems 3.2 and 3.4) and show that they become *conditionally increasing* functions of the terminal value of the underlying risky asset.

This setting is also well-suited to solving several investment problems of interest. A main contribution in this part of the paper is the extension of the classical result of portfolio optimization under expected utility (Cox and Huang (1989)). More precisely, we determine the optimal payoff for an expected utility maximizer under a dependence constraint, reflecting a desired interaction with the benchmark (or background risk) (Theorem 5.2). The proof builds on isotonic approximations and their properties (Barlow *et al.* (1972)). We also solve two stochastic generalizations of Browne (1999) and Cvitanič and Spivak (1999) classical target optimization problem in the given state-dependent context.

Finally, we show how these theoretical results are useful in security design and can help to simplify (and improve) payoffs commonly used in the financial market. We show how to substitute highly path-dependent products by weakly path-dependent and less involved contracts that we call "twins". This result is illustrated with an extensive discussion of the optimality of Asian options. We also construct alternative payoffs with appealing properties.

The paper is organized as follows. Section 1 outlines the setting of the investment problem under study. In Section 2, we restate basic optimality results for path-independent payoffs for investors with law-invariant preferences. We also discuss in detail the sufficiency of path-independent payoffs when allocating wealth. In Section 3, we point out drawbacks of optimal path-independent payoffs and introduce the concept of state-dependence used in the following sections. "Twins" are defined as payoffs that depend on two asset values only. We show that they are optimal for state-dependent preferences. In Section 4, we discuss applications to improving security designs. In particular, we propose several improvements in the design of geometric Asian options. In Section 5, we give an extension of the fundamental result on the optimal design of portfolios under the expected utility paradigm where we specify the interaction with the benchmark. In this context, we also generalize Browne (1999) and Cvitanič and Spivak (1999) results on target probability maximization. Final remarks are presented in Section 6. Most proofs are given in the Appendix.

1 Framework and notations

Consider investors with a given finite investment horizon T and no intermediate consumption. We model the financial market on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It consists of a bank account B paying a constant risk-free rate $r > 0$, so that B_0 invested in a bank account at time 0 yields $B_t = B_0 e^{rt}$ at time t. Furthermore, there is a risky asset (say, an investment in stock), whose price process is denoted by

$S=(S_t)_{0\leqslant t\leqslant T}$. We assume that S_t $(0<t<T)$ has a continuous distribution F_{S_t}. The no-arbitrage price[1] at time 0 of a payoff X_T paid at time $T>0$ is given by

$$c_0(X_T)=\mathbb{E}[\xi_T X_T], \tag{1}$$

where $(\xi_t)_t$ is the state-price process ensuring that $(\xi_t S_t)_t$ is a martingale. Based on standard economic theory, we assume throughout this paper that state prices are decreasing with asset prices[2], i.e.

$$\xi_t=g_t(S_t),\ t\geqslant 0, \tag{2}$$

where g_t is decreasing. The functional form (2) for $(\xi_t)_t$ allows presenting our results on optimal portfolios using $(S_t)_t$ as a reference which is practical[3]. Assumption (2) is satisfied by many popular pricing models, including the CAPM, the consumption-based models and by exponential Lévy markets where the market participants use Esscher pricing.

The Black–Scholes model can be seen as a special case of the latter setting. As we use it to illustrate our theoretical results we recall its main properties. In the Black–Scholes market, the price process $(S_t)_t$ satisfies

$$\frac{dS_t}{S_t}=\mu dt+\sigma dZ_t,$$

with solution $S_t=S_0\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma Z_t\right)$. Here $(Z_t)_t$ is a standard Brownian motion, μ $(>r)$ the drift and $\sigma>0$ the volatility. The distribution (cdf) of S_T is given as

$$F_{S_T}(x)=\mathbb{P}(S_T\leqslant x)=\Phi\left(\frac{\ln\left(\frac{x}{S_0}\right)-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \tag{3}$$

where Φ is the cdf of a standard normal random variable. In the Black–Scholes market, the state-price process $(\xi_t)_t$ is unique and $\xi_t=e^{-rt}e^{-\theta Z_t-\frac{\theta^2 t}{2}}$ where $\theta=\frac{\mu-r}{\sigma}$. Consequently, ξ_t can also be expressed as a decreasing function of the stock price S_t,

$$\xi_t=\alpha_t\left(\frac{S_t}{S_0}\right)^{-\beta} \tag{4}$$

where $\alpha_t=\exp\left(\frac{\theta}{\sigma}\left(\mu-\frac{\sigma^2}{2}\right)t-\left(r+\frac{\theta^2}{2}\right)t\right)$, $\beta=\frac{\theta}{\sigma}$. Extending the current setting to a multidimensional market is straightforward and is discussed in the Final Remarks (Section 6).

[1]The payoffs we consider are all tacitly assumed to be square integrable, to ensure that all expectations mentioned in the paper exist. In particular, $c_0(X_T)<+\infty$ for any payoff X_T considered throughout this paper.

[2]See for instance Cox, Ingersoll and Ross (1985) as well as Bondarenko (2003) who shows that property (2) must hold if the market does not allow for statistical arbitrage opportunities, where a statistical arbitrage opportunity is defined as a zero-cost trading strategy delivering at T a positive expected payoff unconditionally, and non-negative expected payoffs conditionally on ξ_T.

[3]However, a deeper feature of the results and characterizations we present is that optimality of a payoff X_T is tied to its (conditional) anti-monotonicity with ξ_T. See also the final remarks in Section 6.

2 Law-invariant preferences and optimality of path-independent payoffs

In this section, investors have *law-invariant* (state-independent) preferences. This means that they are indifferent between two payoffs having the same payoff distribution (under $\mathbb{P}$). In this case, any random payoff X_T (that possibly depends on the path of the underlying asset price) admits a path-independent alternative with the same price, which is at least as good for these investors. Recall that a payoff is *path-independent* if there exists some function f such that $X_T = f(S_T)$ holds almost surely. Hence, investors with law-invariant preferences only need to consider path-independent payoffs when making investment decisions. Under the additional (usual) assumption that preferences are *increasing*, any path-dependent payoff can be strictly dominated by a path-independent one which is increasing in the risky asset[4].

Note that results in this section are closely related to original work of Cox and Leland (1982), Dybvig (1988), Bernard, Boyle and Vanduffel (2011) and Carlier and Dana (2011). These overview results are recalled here to facilitate the exposition of the extensions that are developed in the following sections.

2.1 Sufficiency of path-independent Payoffs

Proposition 2.1 shows that for any given payoff there is a path-independent alternative with the same price, which is at least as good for investors with law-invariant preferences. Thus, such an investor needs to consider path-independent payoffs only. All other payoffs are indeed redundant in the sense that they are not needed to optimize the investor's objective. The proof of Proposition 2.1 provides an explicit construction of an equivalent path-independent payoff.

Proposition 2.1 (Sufficiency of path-independent payoffs)**.** *Let X_T be a payoff with price c and having a cdf F. Then, there exists at least one path-independent payoff $f(S_T)$ with price $c := c_0(f(S_T))$ and cdf F.*

The proof of Proposition 2.1 is given in Appendix A.1. □

Proposition 2.1, however, does not conclude that a given path-dependent payoff can be strictly dominated by a path-independent one. The next section shows that the dominance becomes strict as soon as preferences are increasing.

2.2 Optimality of path-independent payoffs

The following basic result originally due to Dybvig (1988) and presented more generally in Bernard, Boyle and Vanduffel (2011) shows that any path-dependent payoff

[4]This dominance can easily be implemented in practice as all path-independent payoffs can be statistically replicated with European call and put options as shown for example by Carr and Chou (1997) and Breeden and Litzenberger (1978).

admits a path-independent alternative which is strictly cheaper[5] This result implies that investors with *increasing* law-invariant preferences may restrict their optimization *strictly* to the set of path-independent payoffs, when making investment decisions.

Theorem 2.2 (Cost optimality of path-independent payoffs)**.** *Let F be a cdf. The following optimization problem*

$$\min_{X_T \sim F} c_0\left(X_T\right) \tag{5}$$

has an almost surely unique solution X_T^* that is path-independent, almost surely increasing in S_T and given by

$$X_T^* = F^{-1}(F_{S_T}(S_T)), \tag{6}$$

where F^{-1} is the (left-continuous) inverse of the cdf F defined as

$$F^{-1}(p) = \inf\left\{x \mid F(x) \geqslant p\right\}. \tag{7}$$

This theorem can be seen as an application of the Hoeffding–Fréchet bounds recalled in Lemma A.1 presented in the appendix. A given payoff X_T can be optimized by redistributing wealth levels $X_T(\omega)$ across the different states ω of the economy such that they become ordered with the stock price $S_T(\omega)$. The payoff (6) generating a given terminal distribution F at minimal price is called *cost-efficient* by Bernard, Boyle and Vanduffel (2011). It is obviously *increasing* in S_T. Theorem 2.2 implies that both properties are actually equivalent.

Corollary 2.3 (Cost-efficient payoffs)**.** *A payoff is cost-efficient if and only if it is almost surely increasing in S_T.*

Theorem 2.2 also implies that investors with increasing law-invariant preferences only invest in path-independent payoffs that are increasing in S_T. This is consistent with the literature on optimal investment problems where optimal payoffs, derived using various techniques, turn out to always exhibit this property.

Corollary 2.4 (Optimal payoffs for increasing law-invariant preferences)**.** *For any payoff Y_T at price c which is not almost surely increasing in S_T there exists a path-independent payoff Y_T^* at price c which is a strict improvement for any investor with increasing and law-invariant preferences. A possible choice is given by $Y_T^* := F^{-1}(F_{S_T}(S_T)) + (c - c_0^*)e^{rT}$ where c_0^* denotes the price of* (6)*. Y_T^* has price c and is almost surely increasing in S_T.*

Indeed assume that the investor considers some payoff Y_T (with price c and cdf F) which is *not* almost surely increasing in S_T. Then the payoff $Y_T^* = F^{-1}(F_{S_T}(S_T)) + (c - c_0^*)e^{rT}$ is strictly better than Y_T. This is because it consists in investing an amount $c_0^* < c$ in the cost-efficient payoff (distributed also with F) and placing the remaining funds $c - c_0^* > 0$ in the bank account which is a strict improvement of the payoff Y_T.

[5] Similar optimality results as in Theorem 2.2 have been given in the class of admissible claims X_T which are smaller than F in convex order in Dana and Jeanblanc (2005) and Burgert and Rüschendorf (2006) and Rüschendorf (2012).

3 Optimal payoffs under state-dependent preferences.

Optimal contracts chosen by law-invariant investors do not offer protection in times of economic scarcity. In fact, due to the observed monotonicity property with S_T the lowest outcomes for an optimal (thus cost-efficient) payoff exactly occur when the stock price S_T reaches its lowest levels. More precisely, denote by $f(S_T)$ a cost-efficient payoff (with a increasing function f) and by X_T another payoff such that both are distributed with F at maturity. Then, $f(S_T)$ delivers low outcomes when S_T is low and it holds[6] for all $a \geqslant 0$ that

$$\mathbb{E}[f(S_T)|S_T < a] \leqslant \mathbb{E}[X_T|S_T < a]. \tag{8}$$

Let F be the distribution of a put option with payoff $X_T := (K - S_T)^+$. Bernard, Boyle and Vanduffel (2011) show that the payoff of the cheapest strategy with cdf F can be computed as in (6). It is given by $X_T^* = \max(K - a\, S_T^{-1}, 0)$ with $a := S_0^2 \exp(2\,(\mu - \sigma^2/2)\,T)$ and is a power put option (with power -1). X_T^* is the cheapest way to achieve the distribution F whereas the first "ordinary" put strategy (with payoff X_T) is actually the most expensive way to do so. These payoffs interact with S_T in a fundamentally different way, as one payoff is increasing while the other is decreasing in S_T. A put option protects the investor against a declining market where consumption is more expensive, whereas the cost-efficient counterpart X_T^* provides no protection at all but rather emphasizes the effect of a market deterioration on the wealth received.

The use of put options is a signal that many investors do care about states of the economy in which income of investment strategies is received. In particular, they may seek strategies that resist against declining markets or, more generally, that exhibit a desired dependence with some source of background risk. More evidence of state-dependent preferences can be found in the mere existence of insurance contracts. As explained in the introduction, for each insurance contract there exists a financial contract that has the same payoff distribution but is cheaper to buy (Bernard and Vanduffel (2012)). Yet people buy insurance indicating that insurance payments, which provide protection conditionally upon the occurrence of adverse events, present more value for an investor than financial payoffs which do not have this feature.

Hence, in the rest of the paper, we consider investors who exhibit state-dependent preferences in the sense that they seek a payoff a payoff X_T with a desired distribution*and* additionally a desired dependence with a benchmark asset A_T. In other words, they fix the joint distribution G of the random couple (X_T, A_T). The optimal strategy is the one that solves for

$$\min_{(X_T, A_T) \sim G} c_0\,(X_T) \tag{9}$$

[6] We give here a short proof of (8). It is clear that the couple $(f(S_T), \mathbb{1}_{S_T<a})$ has the same marginals as $(X_T, \mathbb{1}_{S_T<a})$ but $\mathbb{E}[f(S_T)\mathbb{1}_{S_T<a}] \leqslant \mathbb{E}[X_T \mathbb{1}_{S_T<a}]$ because $f(S_T)$ and $\mathbb{1}_{S_T<a}$ are anti-monotonic (from Lemma A.1).

Note that the setting also includes law-invariant preferences as a special (limiting) case when A_T is deterministic. In this case, we effectively revert to the framework of state-independent preferences that we discussed in the previous section. In what follows we consider as benchmark (or background risk) the underlying risky asset or any other asset in the market, considered at final or intermediate time(s). We voluntarily organize the rest of this section similarly as Section 2 so that the impact of state-dependent preferences on the structure of optimal payoffs is clear.

Remark 3.1. One can use a copula as a device to model the interaction between payoffs and benchmarks. The joint distribution G of the couple (X_T, A_T) can be written using a copula C. From Sklar's theorem, $G(x,a) = C(F_{X_T}(x), F_{A_T}(a))$, where C is a copula (this representation is unique for continuously distributed random variables). It is then clear that the optimization problem (9) can also be formulated as

$$\min_{\substack{X_T \sim F,\\ \mathcal{C}_{(X_T,A_T)}=C}} c_0\left(X_T\right)$$

where "$\mathcal{C}_{(X_T,A_T)} = C$" means that the copula between the payoff X_T and the benchmark A_T is C.

3.1 Sufficiency of twins

In this paper any payoff that writes as $f(S_T, A_T)$ or $f(S_T, S_t)$ is called a twin. We first show that in our state-dependent setting for any payoff there exists a twin, which is at least as good. When also assuming that preferences are increasing we find that optimal payoffs writes as twins and we are able to characterize them explicitly. Conditionally on H_T, optimal twins are increasing in the terminal value of the risky asset.

The next theorems show that for any given payoff there is a twin that is at least as good for investors with state-dependent preferences.

Theorem 3.2. (Twins as payoffs with a given joint distribution with a benchmark A_T and price c)**.** *Let X_T be a payoff with price c having joint distribution G with some benchmark A_T, where (S_T, A_T) is assumed to have a joint density with respect to the Lebesgue measure. Then, there exists at least one twin $f\left(S_T, A_T\right)$ with price $c := c_0\left(f\left(S_T, A_T\right)\right)$ having the same joint distribution G with A_T.*

This theorem does not cover the important case where S_T is playing the role of the benchmark (because (S_T, S_T) has no density). This interesting case is considered in the next theorem.

Theorem 3.3 (Twins as payoffs with a given joint distribution with S_T and price c)**.** *Let X_T be a payoff with price c having joint distribution G with the benchmark S_T , where (S_T, S_t) is assumed to have a joint density with respect to the Lebesgue measure. Then, for any $0 < t < T$ there exists at least one twin $f(S_t, S_T)$ with price $c = c_0\left(f\left(S_t, S_T\right)\right)$ having a joint distribution G with S_T. For example, for some $t \in (0, T)$,*

$$f(S_t, S_T) := F^{-1}_{X_T|S_T}(F_{S_t|S_T}(S_t)). \tag{10}$$

The proofs for Theorems 3.2 and 3.3 are in Appendix A.3 and A.4. In particular an explicit construction of $f(S_T, A_T)$ can be found in the proof of Theorem 3.2. $\square$

Theorems 3.2 and 3.3 essentially state that all investors who care about the *joint distribution of terminal wealth with some benchmark*[7] only need to consider twins. This result is a direct extension to Proposition 2.1. All other payoffs are useless in the sense that they are not needed for these investors per se.

Note that in Theorem 3.3, t can be chosen freely between 0 and T and hence there is an infinite number of twins $f(S_t, S_T)$ all having the joint distribution G with S_T and the same price[8]. The question then raises how to select the optimal one among them. A natural possibility is to determine the optimal twin $X_T = f(S_t, S_T)$ by imposing some additional criterion. For example, one could then define the best twin X_T as the one that minimizes

$$\mathbb{E}\left[(X_T - A_T)^2\right]. \tag{11}$$

Because all marginal distributions are fixed, the criterion (11) is equivalent to maximizing the correlation between X_T and A_T. We use this criterion in one of the applications.

3.2 Optimality of twins

Next we investigate the optimality of twins. The following main result of Section 3 extends Theorem 2.2 to the state-dependent case.

Theorem 3.4 (Cost optimality of twins). *Assume that (S_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a bivariate cumulative distribution function. The following optimization problem*

$$\min_{(X_T, A_T)\sim G} c_0\left(X_T\right) \tag{12}$$

has an almost surely unique solution X_T^* which is a twin of the form $f(S_T, A_T)$, almost surely increasing in S_T conditionally on A_T, and given by

$$X_T^* := F^{-1}_{X_T|A_T}(F_{S_T|A_T}(S_T)). \tag{13}$$

The proof of Theorem 3.4 is given in Appendix A.5. $\square$

Recall from Section 2 that when preferences are law-invariant optimal payoffs are path-independent and increasing in S_T. When preferences are state-dependent we observe from expression (13) that optimal payoffs become path-dependent and *conditionally* (on A_T) increasing in S_T We end this section with a corollary derived from Theorem 3.4. The result echoes the one established for investors with law-invariant preferences in the previous section (Corollary 2.4)

[7] The benchmark can be A_T where (A_T, S_T) is continuously distributed as in Theorem 3.2 or it can be S_T as in Theorem 3.3.

[8] To see this, recall that the joint distribution between the twin $f(S_t, S_T)$ and S_T is fixed and thus also the joint distribution between the twin and ξ_T (as ξ_T is a decreasing function of S_T due to (2)). All twins $f(S_t, S_T)$ with such a property have the same price $\mathbb{E}[\xi_T f(S_t, S_T)]$.

Corollary 3.5 (Cheapest twin)**.** *Assume that (S_T, A_T) has joint density with respect to the Lebesgue measure. Let G be a bivariate cumulative distribution function. Let X_T be a payoff such that $(X_T, A_T) \sim G$. Then X_T is a cheapest payoff if and only if, conditionally on A_T, X_T is (almost surely) increasing in S_T.*

The proof of Corollary 3.5 is given in Appendix A.6. □

4 Improving security design

In this section, we show the results are useful in designing balanced and transparent investment policies for retail investors as well as financial institutions.

1. If the investor who buys the financial contract has law-invariant preferences and if the contract is not increasing in S_T, then there exists a strictly cheaper derivative (cost-efficient contract) that is strictly better for this investor by applying Theorem 2.2.
2. If the investor buys the contract because of the interaction with the market asset S_T, and the contract depends on another asset, then we can apply Theorem 3.3 to simplify its design while keeping it "at least as good". The contract then depends for example on S_T and S_t for some $t \in (0, T)$.
3. If the investor buys the contract because he likes the dependence with a benchmark A_T, which is not S_T, and if the contract does not only depend on A_T and S_T, then we use Theorem 3.2 to construct a simpler contract which is "at least as good"and writes as a function of S_T and A_T. Finally, if the obtained contract is not increasing in S_T conditionally on A_T, then it is also possible to construct a strictly cheaper alternative using Theorem 3.4 and Corollary 3.5.

We now use the Black–Scholes market to illustrate these three situations with various examples. We start with the Asian option with fixed strike followed by the one with floating strike.

4.1 The geometric Asian twin with fixed strike

Consider a fixed strike (continuously monitored) geometric Asian call with payoff given by

$$Y_T := (G_T - K)^+ . \tag{14}$$

Here K denotes the fixed strike and G_T is the geometric average of stock prices from 0 to T defined as,

$$\ln(G_T) := \frac{1}{T}\int_0^T \ln(S_s)\, ds. \tag{15}$$

We can now apply the material exposed above to design products that improve upon Y_T.

Use of cost-efficiency payoff for investors with increasing law-invariant preferences. By applying Theorem 2.2 to the payoff Y_T (14), one finds that the cost-efficient payoff associated with a fixed strike (continuously monitored) geometric Asian call is

$$Y_T^* = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+ \tag{16}$$

where $d = S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$. This is also the payoff of a power call option, whose pricing is well-known. One obtains:

$$c_0\left(Y_T^*\right) = S_0 e^{(\frac{1}{\sqrt{3}}-1)rT+(\frac{1}{2}-\frac{1}{\sqrt{3}})\mu T-\frac{\sigma^2 T}{12}}\Phi(h_1) - Ke^{-rT}\Phi(h_2) \tag{17}$$

where

$$h_1 = \frac{\ln\left(\frac{S_0}{K}\right) + (\frac{1}{2}-\frac{1}{\sqrt{3}})\mu T + \frac{r}{\sqrt{3}}T + \frac{1}{12}\sigma^2 T}{\sigma\sqrt{\frac{T}{3}}}, \quad h_2 = h_1 - \sigma\sqrt{\frac{T}{3}}.$$

While the above results can also be found in Bernard *et al.* (2011) they are worth considering here for the purpose of comparison with what follows. Note that letting K to zero provides the cost-efficient payoff equivalent to the geometric average G_T.

A twin, useful for investors who care about the dependence with S_T. By applying Theorem 3.3 to the payoff G_T, we can find a twin payoff $R_T(t) = f(S_t, S_T)$ such that

$$\left(S_T, R_T(t)\right) \sim \left(S_T, G_T\right). \tag{18}$$

By definition, this twin preserves existing dependence between G_T and S_T. Compared to the original contract, however, it is weakly path-dependent. Interestingly, both the call option written on $R_T(t)$ and the call option written on G_T have thus the same joint distribution with S_T. More formally, one has

$$\left(S_T, (R_T(t)-K)^+\right) \sim \left(S_T, (G_T-K)^+\right). \tag{19}$$

$(R_T(t)-K)^+$ is therefore a twin equivalent to the fixed strike (continuously monitored) geometric Asian call (as of Theorem 3.3). We can compute $R_T(t)$ by applying Theorem 3.3 and we find that

$$R_T(t) = S_0^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} \tag{20}$$

where t is freely chosen in $(0, T)$. Details on how (10) becomes (20) are given in Appendix B.1 [9]. Equality of joint distributions exposed in (19) implies that the call option written on $R_T(t)$ has the same price as the original fixed strike (continuously monitored) geometric Asian call (14). The time−0 price of both contracts is therefore

$$c_0((R_T(t)-K)^+) = S_0 e^{-\frac{rT}{2}-\frac{\sigma^2 T}{12}}\Phi(\tilde{d}_1) - Ke^{-rT}\Phi(\tilde{d}_2) \tag{21}$$

where $\tilde{d}_1 = \frac{\ln(S_0/K)+rT/2+\sigma^2 T/12}{\sigma\sqrt{T/3}}$ and $\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{T/3}$ (see Kemna and Vorst (1990)).

[9] The formula (20) is based on the expression (10) for a twin depending on S_t and S_T. Note that there is no uniqueness. For example, $1 - F_{S_t|S_T}(S_t)$ is also independent of S_T we can thus also consider $H_T(t) := F^{-1}_{X_T|S_T}(1 - F_{S_t|S_T}(S_t))$ as suitable twin $(0 < t < T)$ satisfying the joint distribution as in (18). In this case one obtains $H_T(t) = S_0^{\frac{1}{2}+\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}} S_t^{-\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} S_T^{\frac{1}{2}+\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}$.

Choosing among twins. The construction in Theorem 3.3 depends on t. Maximizing the correlation between $\ln(R_T(t))$ and $\ln(G_T)$ is nevertheless a possible way to select a specific t. The covariance between $\ln(R_T(t))$ and $\ln(G_T)$ is given by

$$\operatorname{cov}\left(\ln\left(R_T\left(t\right)\right), \ln\left(G_T\right)\right) = \frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right)$$

and, by construction of $R_T(t)$, standard deviations of $\ln(R_T(t))$ and $\ln(G_T)$ are similar and equal to $\sigma\sqrt{\frac{T}{3}}$. So maximizing the correlation coefficient is equivalent to maximizing the covariance and thus of $f(t) = (T-t)\,t$. This is obtained for $t^* = \frac{T}{2}$ and the maximal correlation $\rho_{\max}$ between $\ln(R_T(t))$ and $\ln(G_T)$ is

$$\rho_{\max} = \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t^*)\,t^*}}{4T} = \frac{3}{4} + \frac{\sqrt{3}}{8} \approx 0.9665,$$

reflecting that the optimal twin is highly correlated to the initial Asian, while being considerably simpler. Note that both the maximum correlation and the optimum $R_T(\frac{T}{2})$ are robust to changes in market parameters.

4.2 The geometric Asian twin with floating strike

Consider now a floating strike (continuously monitored) Asian put option defined by

$$Y_T = \left(G_T - S_T\right)^+. \tag{22}$$

For increasing law-invariant preferences, Corollary 2.4 may be used to find a cheaper contract that depends on S_T only. The cheapest contract with cdf F_{Y_T} is known to be $F_{Y_T}^{-1}\left(\Phi\left(\frac{\ln\left(\frac{S_T}{S_0}\right)-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right)$. Notice that $F_{Y_T}^{-1}$ can only be approximated numerically because the distribution of the difference between two lognormal distributions is unknown.

If investors care about the dependence with S_T, by applying Theorem 3.3, one finds twins $F_{Y_T|S_T}^{-1}(F_{S_t|S_T}(S_t))$ as functions of S_t and S_T. Similarly as in the previous subsection, it is equal to

$$\left(S_0^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}}\, S_t^{\frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}}\, S_T^{\frac{1}{2}-\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}} - S_T\right)^+. \tag{23}$$

Details can be found in Appendix B.2.

Finally, if investors care about the dependence with G_T, then it is possible to construct a cheaper twin because the payoff (22) is not conditionally increasing in S_T and therefore can strictly be improved using Theorem 3.4 and computing the expression (13). The reason is that we can improve the payoff (22) by making it cheaper while keeping dependence with G_T. Hence, we invoke Theorem 3.4 to exhibit another payoff $X_T = F_{Y_T|G_T}^{-1}\left(F_{S_T|G_T}(S_T)\right)$ such that

$$(Y_T, G_T) \sim (X_T, G_T)$$

but so that X_T is strictly cheaper. After some calculations, we find that X_T writes as

$$X_T = \left(G_T - a\frac{G_T^3}{S_T} \right)^+ \tag{24}$$

where $a = \frac{e^{\left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{2}}}{S_0}$. Details can be found in Appendix B.3.

Finally, one can easily assess to which extent the twin (24) is cheaper than the initial payoff Y_T. To do so, we recall the price of a geometric Asian option with floating strike (the no-arbitrage price of Y_T),

$$c_0(Y_T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(G_T - S_T\right)^+ = S_0 e^{-\frac{rT}{2}} \left(\Phi\left(f\right) e^{-\frac{\sigma^2 T}{12}} - e^{\frac{rT}{2}} \Phi\left(f - \sigma\sqrt{\frac{T}{3}} \right) \right) \tag{25}$$

where $f = \frac{\frac{\sigma^2}{12}T - r\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$. Similarly, one finds that

$$c_0(X_T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T} \right)^+ = S_0 e^{-\frac{rT}{2}} \left(\Phi\left(d\right) e^{-\frac{\sigma^2 T}{12}} - e^{\frac{\mu T}{2}} \Phi\left(d - \sigma\sqrt{\frac{T}{3}} \right) \right) \tag{26}$$

where $d = \frac{\frac{\sigma^2 T}{12} - \mu\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$ which we need to compare numerically to (25). For example, when $\mu = 0.06$, $r = 0.02$, $\sigma = 0.3$ and $T = 1$ one has $c_0(Y_T) = 6.74$ and $c_0(X_T) = 5.86$, indicating that cost savings can be substantial. Also note the close correspondence between the formulas (25) and (26). The proofs for these are given in Appendix B.4.

5 Portfolio management

This section provides several contributions to portfolio management. We first derive the optimal investment for an expected utility maximizer who has a constraint on the dependence with a given benchmark. Next, we revisit optimal strategies for target probability maximizers (see Browne (1999) and Cvitanič and Spivak (1999)), and we extend this problem in two different directions by adding dependence constraints and by considering a random target. In both cases, we derive analytical solutions that are given by twins.

5.1 Expected utility maximization with dependence constraints

The most prominent decision theory used in various fields of economics is the expected utility theory (EUT) of von Neumann & Morgenstern (1947). In the expected utility framework investors assign a utility $u(x)$ to each possible level of wealth x. Increasing preferences are equivalent to a increasing utility function $u(\cdot)$. Assuming $u(\cdot)$ is concave is equivalent to assuming investors are risk averse in the sense that for a given budget they prefer a sure income above a random one with the same mean. In their seminal paper on optimal portfolio selection, Cox and Huang (1989) showed how to obtain the optimal strategy for a risk averse expected utility maximizer; see also Merton (1971) and He and Pearson (1991a),(1991b). We recall this classical result in the following theorem.

Theorem 5.1 (Optimal payoff in EUT)**.** *Consider a utility function $u(\cdot)$ defined on (a,b) such that $u(\cdot)$ is continuously differentiable and strictly increasing, $u'(\cdot)$ is strictly decreasing, $\lim_{x\searrow a} u'(x) = +\infty$ and $\lim_{x\nearrow b} u'(x) = 0$. Consider the following portfolio optimization problem*

$$\max_{\mathbb{E}[\xi_T X_T]=W_0} \mathbb{E}[u(X_T)]$$

The optimal solution to this problem is given by

$$X_T^* = [u']^{-1}(\lambda\xi_T) \tag{27}$$

where λ is such that $\mathbb{E}\left[\xi_T [u']^{-1}\left(\lambda X_T^*\right)\right] = W_0$.

Note that the optimal EUT payoff X_T^* is decreasing in ξ_T and thus increasing in S_T (illustration of the results in Section 2), also pointing out the lack of protection of optimal portfolios when markets go down. To account for this, we give the investor the opportunity to maintain a desired dependence with a benchmark portfolio (for example representing the financial market). This extends earlier results on expected utility maximization with constraints, for example by Brennan and Solanki (1981), Brennan and Schwartz (1989), He and Pearson (1991a),(1991b), Basak (1995), Grossman and Zhou (1996), Sorensen (1999) and Jensen and Sorensen (2001). They mostly study expected utility maximization problem when investors want a lower bound on their optimal wealth either at maturity or throughout some time interval. When this bound is deterministic this corresponds to classical portfolio insurance. Boyle and Tian (2007) extend and unify the different results by allowing the benchmark to be beaten with some confidence. They consider the following maximization problem over all payoffs X_T,

$$\max_{\substack{\mathbb{P}(X_T \geqslant A_T)\geqslant\alpha, \\ c_0(X_T)=W_0}} \mathbb{E}[u(X_T)] \tag{28}$$

where A_T is a benchmark (which could be for instance the portfolio of another manager in the market). In Theorem 2.1 (page 327) of Boyle and Tian (2007), the optimal contract X_T^* is derived explicitly (under some regularity conditions ensuring feasibility of the stated problem) and it is an optimal twin.

This also follows from our results. Assume that the solution to (28) exists, and denote it by X_T^*. Then let G be the bivariate cdf of (X_T^*, A_T). The cheapest way to preserve this joint bivariate cdf is obtained by a twin $f(A_T, S_T)$ which is increasing in S_T conditionally on A_T (see Corollary 3.5). Hence, X_T^* must also be of this form, otherwise one can easily contradict the optimality of X_T^* to the problem. Thus, the solution to optimal expected utility maximization with the additional probability constraint (when it exists) is an optimal twin.

With a similar reasoning, this result also holds when there are several probability constraints involving the joint distribution of terminal portfolio value X_T and the benchmark A_T. The next theorem extends Theorem 5.1 and the above literature by considering an expected utility maximization problem where the investor is fixing the dependence with a benchmark, entirely. This amounts to specifying upfront the joint copula of (X_T, A_T). Assume that the copula between X_T and A_T is specified to be C, i.e. $\mathcal{C}_{(X_T,A_T)} = C$. Let us denote by $C_{1|A_T}$ the conditional distribution of the first component given A_T (or equivalently given $F_{A_T}(A_T)$).

Theorem 5.2 makes use of the projection on the convex cone

$$M_{\downarrow} := \{f \in L^2[0,1]; f \text{decreasing}\}, \tag{29}$$

which is a subset of $L^2[0,1]$ equipped with the Lebesgue measure and the standard $||\cdot||_2$ norm. For an element $\varphi \in L^2[0,1]$, we denote by $\widehat{\varphi} = \pi_{M_{\downarrow}}(\varphi)$ the projection of φ on $M_{\downarrow}$. $\widehat{\varphi}$ can be interpreted as the best approximation of φ by a decreasing function for the $||\cdot||_2$ norm.

Theorem 5.2 (Optimal payoff in EUT with dependence constraint)**.** *Given a utility function $u(\cdot)$ as in Theorem 5.1, consider the following portfolio optimization problem*

$$\max_{\substack{c_0(X_T)=W_0 \\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{E}\left(u(X_T)\right). \tag{30}$$

Assume that (A_T, S_T) has a joint density. With $U_T = F_{S_T|A_T}(S_T)$ and $Z_T = C^{-1}_{1|A_T}(U_T)$, let $H_T = \mathbb{E}(\xi_T|Z_T) = \varphi(Z_T)$, and define $\widehat{H}_T = \widehat{\varphi}(Z_T)$, where $\widehat{\varphi}$ be the projection of φ on $M_{\downarrow}$. Then the solution to the optimization problem (30) *is given by*

$$\widehat{X}_T = [u']^{-1}\left(\lambda \widehat{H}_T\right) \tag{31}$$

where λ is such that $\mathbb{E}\left[\xi_T\,[u']^{-1}\left(\lambda\widehat{H}_T\right)\right] = W_0$.

The proof of Theorem 5.2 is given in Appendix C.1. □

Remark 5.3. In the case that $H_T = \mathbb{E}(\xi_T|Z_T)$ is decreasing in Z_T, we obtain as solution to (30)

$$\widehat{X}_T = [u']^{-1}(\lambda H_T). \tag{32}$$

In this case the proof of Theorem 5.2 can be simplified and reduced to the classical optimization result in Theorem 5.1 since by Theorem 3.4 an optimal solution X_T is unique and satisfies

$$X_T = F^{-1}_{X_T|A_T}(F_{S_T|A_T}(S_T)).$$

By Lemma A.2 one can conclude that $X_T = F^{-1}_{X_T}(Z_T)$, i.e. X_T is an increasing function of Z_T. Theorem 5.1 then allows finding the optimal element in this class.

Remark 5.4. The determination of the isotonic approximation $\widehat{\varphi}$ of φ is a well-studied problem (see Theorem 1.1 in Barlow *et al.* (1972)). $\widehat{\varphi}$ is the slope of the smallest concave majorant $SCM(\varphi)$ of φ, i.e. $\widehat{\varphi} = (SCM(\varphi))'$. In Barlow *et al.* (1972) the projection on $M_{\uparrow}$ is given as slope of the greatest convex minorant $GCM(\varphi)$ of φ. Fast algorithms are known to determine $\widehat{\varphi}$.

Example
We illustrate Theorem 5.2 by an example. Let W_0 denote the initial wealth and let $A_T = S_t$ for some $(t < T)$. Consider for C the Gaussian copula with correlation coefficient $\rho \in \left[-\sqrt{1-\frac{t}{T}}, 1\right)$ Furthermore, denote by J the copula between S_t and S_T. Then we find that

$$\widehat{X}_T = [u']^{-1}\left(\lambda e^{-\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)W_T}\right) \tag{33}$$

where λ is such that $\mathbb{E}\left[\xi_T \widehat{X}_T\right] = W_0$ and W_T is a function of S_T and S_t given by

$$W_T = \sqrt{1-\rho^2}\left(\frac{\ln(\frac{S_T}{St}) - (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) + \rho\left(\frac{\ln(\frac{S_t}{S_0}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right).$$

The proof of (33) can be found in Appendix C.2.

5.2 Target probability maximization

Target probability maximizers are investors who, for a given budget (initial wealth) and given time frame, want to maximize the probability that the final wealth achieves some fixed target b. In a Black–Scholes financial market model, Browne (1999) and Cvitanič and Spivak (1999) derive the optimal investment strategy for these investors using stochastic control theory, and shows that it is optimal to buy a digital option written on the risky asset. We show that his results follow from Theorem 2.2 in a more straightforward way.

Proposition 5.5 (Browne's original problem)**.** *Let W_0 be the initial wealth and let $b > W_0 e^{rT}$ be the desired target*[10]*. The solution to the following target probability maximization problem*

$$\max_{X_T \geqslant 0, c_0(X_T)=W_0} \mathbb{P}[X_T \geqslant b]$$

is given by the payoff

$$X_T^* = b\,\mathbb{1}_{\{S_T>\lambda\}}$$

where λ is given by $\mathbb{E}\left(\xi_T X_T^*\right) = W_0$.

The proof of this proposition is given in Appendix C.3. In a Black–Scholes market one easily verifies that $\lambda = \exp\left((r - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\Phi^{-1}\left(\frac{W_0 e^{rT}}{b}\right)\right)$. $\square$

A target probability maximizing strategy is essentially an all-or-nothing strategy. Intuitively, investors would perhaps not be so attracted by the design of the optimal payoff maximizing the probability to beat a fixed target. The obtained wealth solely

[10] If $b \leqslant W_0 e^{rT}$ then the problem is not interesting since an investment in the risk-free asset allows reaching a 100% chance to beat the target b.

depends on the ultimate value of the underlying risky asset making it highly dependent on final market behavior, and thus prone to unexpected and brutal changes in the value of the underlying. A first extension concerns the case of a stochastic benchmark, so that preferences become state-dependent.

Theorem 5.6 (Target probability maximization with a random target)**.** *Let W_0 be the initial wealth. Consider the following portfolio optimization problem,*

$$\max_{X_T \geqslant 0, c_0(X_T)=W_0} \mathbb{P}[X_T \geqslant A_T]$$

Let A_T be a random target such that (A_T, S_T) has a density. The solution to the problem is given by

$$X_T^* = A_T \mathbb{1}_{\{A_T \xi_T < \lambda\}}$$

where λ is implicitly given by $\mathbb{E}\left[\xi_T A_T \mathbb{1}_{\{A_T \xi_T < \lambda\}}\right] = W_0$.

The proof of this proposition is given in Appendix C.4. □

A second extension assumes a fixed dependence with a benchmark in the financial market. We now consider the problem of an investor who for a given budget aims at maximizing the probability that the final wealth achieves some fixed target while preserving a certain dependence with a benchmark.

Theorem 5.7 (Target probability maximization with a random benchmark)**.** *Let W_0 be the initial wealth and $b > W_0 e^{rT}$ the desired target for final wealth. Consider the problem,*

$$\max_{\substack{X_T \geqslant 0, c_0(X_T)=W_0, \\ \mathcal{C}_{(X_T, A_T)}=C}} \mathbb{P}[X_T \geqslant b]$$

Assume the pair (A_T, S_T) has a density and define as in Theorem 5.2

$$U_T = F_{S_T|A_T}(S_T) \quad \textit{and} \quad Z_T = C^{-1}_{1|A_T}(U_T).$$

Then the solution to the optimization problem is given by

$$X_T^* = b\mathbb{1}_{\{Z_T > \lambda\}}$$

where λ is determined by $b\mathbb{P}(Z_T > \lambda) = W_0$.

The proof of this result is given in Appendix C.5. □

Example The result in Theorem 5.7 holds in particular when $A_T = S_t$ $(0 < t < T)$ and when C is a Gaussian copula with correlation coefficient ρ. Then, the optimal solution is explicit and equal to

$$X_T^* = b\mathbb{1}_{\{S_t^\alpha S_T > \lambda\}}, \tag{34}$$

with $\alpha = \sqrt{\frac{T-t}{t(1-\rho^2)}}\rho - 1$ and $\ln(\lambda) = \left(r - \frac{\sigma^2}{2}\right)(\alpha t + T) - \sigma\sqrt{k}\Phi^{-1}\left(\frac{W_0 e^{rT}}{b}\right)$ with $k = (\alpha+1)^2 t + (T-t)$. The proof for this example is given in Appendix C.6.

6 Final remarks

In this paper we characterize optimal strategies for investors who target a known wealth distribution at maturity (as in the traditional setting) and *additionally* aim at maintaining a desired interaction with a random benchmark. We show that optimal contracts are weakly path-dependent and we are able to characterize them explicitly. Throughout the paper we have assumed that the state-price process ξ_T is a decreasing functional of the risky asset price S_T and that there is a single risky asset. It is possible to relax these assumptions and to still provide explicit representations of optimal payoffs. However, the optimality is then no longer related to path-independence properties.

In a Black–Scholes market the state-price ξ_t is the inverse of the value of a unit investment in a constant-mix strategy at time t, where a fraction $\frac{\theta}{\sigma}$ is invested in the risky asset and the remaining fraction $1-\frac{\theta}{\sigma}$ in the bank account. Indeed the value process $(S_t^*)_t$ of such strategy evolves according to

$$\frac{dS_t^*}{S_t^*} = \left(\frac{\theta}{\sigma}\mu + \left(1-\frac{\theta}{\sigma}\right)r\right)dt + \theta dZ_t.$$

Hence, for $S_0^* = 1$ one finds that $S_t^* = e^{\frac{\theta}{\sigma}\mu t + (1-\frac{\theta}{\sigma})rt - \frac{\theta^2}{2}t + \theta Z_t}$ and $\xi_t = 1/S_t^*$. It is easy to prove that this strategy is optimal for an expected log-utility maximizer and in the literature it is also referred to as the growth-optimal portfolio[11] (GOP). Using a milder notion of arbitrage, Platen and Heath (2006) argue that, in general, the price of (non-negative) payoffs could be achieved using the pricing rule (1) where the role of ξ_T is now played by the inverse of the growth-optimal portfolio. Hence our results are also valid in their setting where now the growth optimal portfolio is taken as the reference. Path-independence properties will then be expressed in terms of the GOP.

The case of multidimensional markets described by a price process $(S_t^{(1)}, \ldots, S_t^{(d)})_t$ is essentially included in the results given in this paper assuming that the state-price process $(\xi_t)_t$ of the risk-neutral measure chosen for pricing is of the form $\xi_t = g_t\left(h_t(S_t^{(1)}, \ldots, S_t^{(d)})\right)$ with some real functions g_t , h_t. All results in the paper apply by replacing the one-dimensional stock price process S_t by the one-dimensional process $h_t(S_t^{(1)}, \ldots, S_t^{(d)})$. Dealing with this case in more concrete form and establishing the results in this paper under weaker regularity assumptions are left for future research.

[11]Its origins can be traced back to Kelly (1956).

A Proofs

Throughout the paper and the different proofs, we make repeatedly use of the following lemmas. The first lemma gives a restatement of the classical Hoeffding–Fréchet bounds going back to the early work of Hoeffding (1940) and Fréchet (1940), (1951).

Lemma A.1 (Hoeffding–Fréchet bounds)**.** *Let (X, Y) be a random pair and U uniformly distributed on $(0, 1)$. Then*

$$\mathbb{E}\left[F_X^{-1}(U)F_Y^{-1}(1-U)\right] \leqslant \mathbb{E}\left[XY\right] \leqslant \mathbb{E}\left[F_X^{-1}(U)F_Y^{-1}(U)\right]. \tag{35}$$

The upper bound for $\mathbb{E}[XY]$ is attained if and only if (X, Y) is comonotonic, i.e. $(X, Y) \sim (F_X^{-1}(U), F_Y^{-1}(U))$. Similarly, the lower bound for $\mathbb{E}[XY]$ is attained if and only if (X, Y) is anti-monotonic, i.e. $(X, Y) \sim (F_X^{-1}(U), F_Y^{-1}(1-U))$.

The following lemma combines special cases of two classical construction results. The Rosenblatt transformation describes a transform of a random vector to iid uniformly distributed random variables (see Rosenblatt (1952)). The second result is a special form of the standard recursive construction method for a random vector with given distribution out of iid uniform random variables due to O'Brien (1975), Arjas and Lehtonen (1978) and Rüschendorf (1981).

Lemma A.2 (Construction method)**.** *Let (X, Y) be a random pair and assume that $F_{Y|X=x}(\cdot)$ is continuous $\forall x$. Denote $V = F_{Y|X}(Y)$. Then V is uniformly distributed on $(0, 1)$ and independent of X. It is also increasing in Y conditionally on X. Furthermore, for every variable Z, $(X,\ F_{Z|X}^{-1}(V)) \sim (X, Z)$.*

For the proof of the first part note that by the continuity assumption on $F_{Y|X=x}$ we get from the standard transformation

$$(V \mid X = x) \sim \left(F_{Y|X=x}(Y) \mid X = x\right) \sim U(0, 1), \quad \forall x.$$

Clearly $V \sim U(0, 1)$. Furthermore, the conditional distribution $F_{V|X=x}$ does not depend on x and thus V and X are independent. For the second part one gets by the usual quantile construction that $F_{Z|X=x}^{-1}(V)$ has distribution function $F_{Z|X=x}$. This implies that $(X, F_{Z|X}^{-1}(V)) \sim (X, Z)$ since both sides have the same first marginal distribution and the same conditional distribution.

Lemma A.3. *Let (X, Y) be jointly normally distributed. Then, conditionally on Y, X is normally distributed and,*

$$\begin{aligned} \mathbb{E}(X|Y) &= \mathbb{E}(X) + \frac{\operatorname{cov}(X, Y)}{\operatorname{var}(Y)}(Y - \mathbb{E}(Y) \\ var(X|Y) &= (1-\rho^2)\operatorname{var}(X). \end{aligned}$$

Denote the density of Y by $f_Y(y)$. One has,

$$\int_{-\infty}^{c} e^{a+by} f_Y(y)dy = e^{a+b\mathbb{E}(Y)+\frac{b^2}{2}\operatorname{var}(Y)} \frac{1}{\sqrt{2\pi \operatorname{var}(Y)}} \int_{-\infty}^{c} e^{-\frac{1}{2}\left(\frac{y-(\mathbb{E}(Y)+b\operatorname{var}(Y))}{\sqrt{\operatorname{var}(Y)}}\right)^2} dy.$$

The results in this lemma are well-known and we omit its proof. □

A.1 Proof of Proposition 2.1

Let $U = F_{S_T}(S_T)$ a uniformly distributed variable on $(0,1)$. Consider a payoff X_T. One has,

$$c_0(X_T) = \mathbb{E}\left[X_T\xi_T\right] \geqslant \mathbb{E}\left[F^{-1}_{X_T}(U)\xi_T\right] = c_0(X_T^*),$$

where the inequality follows from the fact that $F^{-1}_{X_T}(U)$ and ξ_T are anti-monotonic and using the Hoeffding–Fréchet bounds in Lemma A.1. Hence, $X_T^* = F^{-1}(F_S(S_T))$ is the cheapest payoff with cdf F. Similarly, the most expensive payoff with cdf F writes as $Z_T^* = F^{-1}(1 - F_S(S_T))$. Since c is the price of a payoff X_T with cdf F, one has

$$c \in [c_0(X_T^*), c_0(Z_T^*)].$$

If $c = c_0(X_T^*)$ then X_T^* is a solution. Similarly, if $c = c_0(Z_T^*)$ then Z_T^* is a solution. Next, let $c \in (c_0(X_T^*), c_0(Z_T^*))$ and define the payoff $f_a(S_T)$ with $a \in \mathbb{R}$,

$$f_a(S_T) = F^{-1}\left[(1 - F_{S_T}(S_T))\mathbb{1}_{S_T \leqslant a} + (F_{S_T}(S_T) - F_{S_T}(a))\mathbb{1}_{S_T > a}\right].$$

Then $f_a(S_T)$ is distributed with cdf F. The price $c_0(f_a(S_T))$ of this payoff is a continuous function of the parameter a. Since $\lim_{a\to 0^+} c_0(f_a(S_T)) = c_0(X_T^*)$ and $\lim_{a\to+\infty} c_0(f_a(S_T)) = c_0(Z_T^*)$, using the theorem of intermediary values for continuous functions, there exists a^* such that $c_0(f_{a^*}(S_T)) = c$. This ends the proof. $\square$

A.2 Proof of Corollary 2.3

Let $X_T \sim F$ be cost-efficient. Then X_T solves (5) and Theorem 2.2 implies that $X_T = F^{-1}(F_{S_T}(S_T))$ almost surely. Reciprocally, let $X_T \sim F$ be increasing in S_T. Then, by our continuity assumption, $X_T = F^{-1}(F_{S_T}(S_T))$ almost surely and thus X_T is cost-efficient. $\square$

A.3 Proof of Theorem 3.2

The idea of the proof is very similar to the proof of Proposition 2.1. Let U be given by $U = F_{S_T|A_T}(S_T)$. It is uniformly distributed over $(0,1)$ and independent of A_T (see Lemma A.2). Furthermore, conditionally on A_T, U is increasing in S_T. Consider next a payoff X_T and note that $F^{-1}_{X_T|A_T}(U) \sim X_T$. We find that

$$\begin{aligned} c_0(X_T) &= \mathbb{E}\left[X_T\xi_T\right] = \mathbb{E}[\mathbb{E}[X_T\xi_T|\, A_T]] \\ &\geqslant \mathbb{E}\left[\mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\middle|\, A_T\right]\right] = \mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\right], \end{aligned} \tag{36}$$

where the inequality follows from the fact that $F^{-1}_{X_T|A_T}(U)$ and ξ_T are conditionally (on A_T) anti-monotonic and using (35) in Lemma A.1 for the conditional expectation (conditionally on A_T). Similarly, one finds that

$$c_0(X_T) \leqslant \mathbb{E}\left[F^{-1}_{X_T|A_T}(1-U)\,\xi_T\right].$$

Next we define the uniform $(0,1)$ distributed variable,

$$g_a(S_T) = (1 - F_{S_T}(S_T))\mathbb{1}_{S_T \leqslant a} + (F_{S_T}(S_T) - F_{S_T}(a))\mathbb{1}_{S_T > a}.$$

We observe that thanks to Lemma A.2, $F_{g_a(S_T)|A_T}(g_a(S_T))$ is independent of A_T and also that $f_a(S_T, A_T)$ given as

$$f_a(S_T, A_T) = F^{-1}_{X_T|A_T}(F_{g_a(S_T)|A_T}(g_a(S_T)))$$

is a twin with the desired joint distribution G with A_T. Denote by $X_T^* = F^{-1}_{X_T|A_T}(U)$ and by $Z_T^* = F^{-1}_{X_T|A_T}(1-U)$. Note that $X_T^* = f_0(S_T, A_T)$ and $Z_T^* = f_1(S_T, A_T)$ almost surely. The same discussion as in the proof of Proposition 2.1 applies here. When $c = c_0(X_T^*)$ then X_T^* is a twin with the desired properties. Similarly, when $c = c_0(Z_T^*)$ then Z_T^* is a twin with the desired properties. Otherwise, when $c \in (c_0(X_T^*), c_0(Z_T^*))$ then the continuity of $c_0(f_a(S_T, A_T))$ with respect to a ensures that there exists a^* such that $c := c_0(f_{a^*}(S_T, A_T))$. Thus, $f_{a^*}(S_T, A_T)$ is a twin with the desired joint distribution G with A_T and with cost c. This ends the proof. □

A.4 Proof of Theorem 3.3

Let $0 < t < T$. It follows from Lemma A.2 that $F_{S_t|S_T}(S_t)$ is uniformly distributed on $(0,1)$ and independent of S_T. Let the twin $f(S_t, S_T)$ be given as

$$f(S_t, S_T) := F^{-1}_{X_T|S_T}(F_{S_t|S_T}(S_t)).$$

Using Lemma A.2 again, one finds that $(f(S_t, S_T), S_T) \sim (X_T, S_T) \sim G$. This also implies,

$$c_0(f(S_t, S_T)) = \mathbb{E}\left[f(S_t, S_T)\xi_T\right] = \mathbb{E}[X_T\xi_T] = c_0(X_T),$$

and this ends the proof. □

A.5 Proof of Theorem 3.4

It follows from Lemma A.2 that $U = F_{S_T|A_T}(S_T)$ is uniformly distributed on $(0,1)$, stochastically independent of A_T and increasing in S_T conditionally on A_T. Let the twin X_T^* be given as

$$X_T^* = F^{-1}_{X_T|A_T}(U).$$

Invoking Lemma A.2 again, $(X_T^*, A_T) \sim (X_T, A_T) \sim G$. Moreover,

$$\begin{aligned} c_0(X_T) = \mathbb{E}\left[X_T\xi_T\right] &= \mathbb{E}[\mathbb{E}[\,X_T\xi_T|\,A_T]] \\ &\geqslant \mathbb{E}\left[\mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\middle|\,A_T\right]\right] \\ &= \mathbb{E}\left[F^{-1}_{X_T|A_T}(U)\xi_T\right] = c_0(X_T^*) \end{aligned}$$

where the inequality follows from the fact that $F^{-1}_{X_T|A_T}(U)$ and S_T are conditionally (on A_T) comonotonic and using (35) in Lemma A.1 for the conditional expectation (conditionally on A_T). □

A.6 Proof of Corollary 3.5

Let us first assume that X_T is a cheapest twin. By Theorem 3.4, X_T is (almost surely) equal to X_T^* as defined by (13) which is, conditionally on A_T, increasing in S_T. Reciprocally, we now assume that $X_T = f(S_T, A_T)$ is conditionally on A_T increasing in S_T. Hence $X_T = F^{-1}_{X_T|A_T}\left(F_{S_T|A_T}(S_T)\right)$ almost surely, which means it is a solution to (12) and thus a cheapest twin. □

B Security design

B.1 Twin of the fixed strike (continuously monitored) geometric Asian call option

Expression (10) allows us to find twins satisfying the constraint (18) on the dependence with the benchmark S_T. Using Lemma A.3 we find that

$$\ln(S_t/S_0)|\ln(S_T/S_0) \sim \mathcal{N}\left(\frac{t}{T}\ln\left(\frac{S_T}{S_0}\right), \sigma^2 t\left(1-\frac{t}{T}\right)\right),$$

and thus

$$F_{S_t|S_T}(S_t) = \Phi\left(\frac{\ln\left(\frac{S_t S_0^{\frac{t}{T}-1}}{S_T^{\frac{t}{T}}}\right)}{\sigma\sqrt{\frac{tT-t^2}{T}}}\right).$$

Furthermore, the couple $(\ln(G_T), \ln(S_T))$ is bivariate normally distributed with mean and variance for the marginals that are given as $\mathbb{E}[\ln(G_T)] = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{T}{2}$, $\text{var}[\ln(G_T)] = \frac{\sigma^2 T}{3}$ and $\mathbb{E}[\ln(S_T)] = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T$, $\text{var}[\ln(S_T)] = \sigma^2 T$. For the correlation coefficient one has $\rho(\ln(S_T), \ln(G_T)) = \frac{\sqrt{3}}{2}$. Applying Lemma A.3 again one finds that,

$$\ln(G_T)|\ln(S_T) \sim \mathcal{N}\left(\ln\left(S_0^{1/2} S_T^{1/2}\right), \frac{\sigma^2 T}{12}\right), \tag{37}$$

and thus,

$$F_{G_T|S_T}(x) = \Phi\left(\frac{\ln(x) - \ln\left(S_0^{1/2} S_T^{1/2}\right)}{\frac{\sigma\sqrt{T}}{2\sqrt{3}}}\right).$$

Therefore,

$$F^{-1}_{G_T|S_T}(y) = \exp\left(\ln\left(S_0^{1/2} S_T^{1/2}\right) + \frac{\sigma\sqrt{T}}{2\sqrt{3}}\Phi^{-1}(y)\right).$$

The expression of $R_T(t)$ given in (20) is then straightforward to derive.

For choosing a specific twin among others, we suggest to maximize $\rho(\ln R_T(t), \ln G_T)$. First, we calculate,

$$\begin{aligned}
\operatorname{cov}\left(\ln S_T, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) &= \frac{1}{T}\int_0^T \operatorname{cov}(\ln S_T, \ln(S_s))\,ds \\
&= \frac{\sigma^2}{T}\int_0^T (s\wedge T)\,ds = \frac{\sigma^2 T}{2}.
\end{aligned}$$

Furthermore, by denoting $a = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{T-t}{t}}$, $b = \frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$ and $c = \frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}$, equation (20) may be rewritten as $\ln R_T(t) = a\ln S_0 + b\ln S_t + c\ln S_T$. The covariance being bilinear, one then has,

$$\begin{aligned}
&\operatorname{cov}(\ln R_T(t), \ln G_T) \\
&\quad= b\operatorname{cov}\left(\ln S_t, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) + c\operatorname{cov}\left(\ln S_T, \frac{1}{T}\int_0^T \ln(S_s)\,ds\right) \\
&\quad= b\frac{1}{T}\int_0^T \operatorname{cov}(\ln S_t, \ln S_s)\,ds + c\frac{\sigma^2 T}{2} \\
&\quad= b\frac{\sigma^2}{T}\int_0^T (s\wedge t)\,ds + c\frac{\sigma^2 T}{2} \;=\; b\frac{\sigma^2}{T}\left(\int_0^t s ds + \int_t^T t ds\right) + c\frac{\sigma^2 T}{2} \\
&\quad= b\frac{\sigma^2}{T}\left(\frac{t^2}{2} + t(T-t)\right) + c\frac{\sigma^2 T}{2} \;=\; b\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) + c\frac{\sigma^2 T}{2} \\
&\quad= \frac{T}{t}\frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\frac{\sigma^2 t}{T}\left(T - \frac{t}{2}\right) + \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\right)\frac{\sigma^2 T}{2} \\
&\quad= \frac{\sigma^2}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\left(T - \frac{t}{2}\right) + \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\sqrt{\frac{t}{T-t}}\right)\frac{\sigma^2 T}{2} \\
&\quad= \frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right).
\end{aligned}$$

Denote by $\sigma_{\ln R_T(t)}$ and by $\sigma_{\ln G_T}$ the respective standard deviations. For the correlation we find that

$$\begin{aligned}
\rho(\ln R_T(t), \ln G_T) = \frac{\operatorname{cov}(\ln R_T(t), \ln G_T)}{\sigma_{\ln R_T(t)}\sigma_{\ln G_T}} &= \frac{\frac{\sigma^2}{2}\left(\frac{T}{2} + \frac{\sqrt{t}\sqrt{T-t}}{2\sqrt{3}}\right)}{\frac{\sigma^2 T}{3}} \\
&= \frac{3}{4} + \frac{\sqrt{3}\sqrt{(T-t)\,t}}{4T}.
\end{aligned}$$

Hence $\rho(\ln R_T(t), \ln G_T)$ is maximized for $t = \frac{T}{2}$. □

B.2 Twin of the floating strike (continuously monitored) geometric Asian put option

We first recall from equation (37) that,

$$\ln(G_T)|\ln(S_T) \sim \mathcal{N}\left(\ln\left(S_0^{\frac{1}{2}} S_T^{\frac{1}{2}}\right), \frac{\sigma^2 T}{12}\right).$$

Therefore $Y_T = (G_T - S_T)^+$ has the following conditional cdf

$$\mathbb{P}(Y_T \leqslant y | S_T = s) = \Phi\left(\frac{\ln(s+y) - \ln\left(S_0^{1/2} s^{1/2}\right)}{\frac{\sigma\sqrt{T}}{2\sqrt{3}}}\right) \mathbb{1}_{y \geqslant 0}$$

Then

$$F^{-1}_{Y_T|S_T}(z) = \left(S_0^{\frac{1}{2}} S_T^{\frac{1}{2}} e^{\frac{\sigma}{2}\sqrt{\frac{T}{3}}\Phi^{-1}(z)} - S_T\right)^+.$$

Therefore $F^{-1}_{Y_T|S_T}\left(F_{S_t|S_T}(S_t)\right)$ can then easily be computed and after some calculations it simplifies to (23). □

B.3 Cheapest Twin of the floating strike (continuously monitored) geometric Asian put option

Applying Lemma A.3 we find,

$$\ln(S_T)|\ln(G_T) \sim \mathcal{N}\left(\ln\left(\frac{G_T^{3/2}}{S_0^{\frac{1}{2}}}\right) + \frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T,\ \frac{\sigma^2 T}{4}\right).$$

Hence,

$$F_{S_T|G_T}(S_T)) = \Phi\left(\frac{\ln\left(\frac{S_T S_0^{\frac{1}{2}}}{G_T^{\frac{3}{2}}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{4}}{\frac{\sigma\sqrt{T}}{2}}\right). \tag{38}$$

Furthermore, $Y_T = (G_T - S_T)^+$ has the following conditional cdf,

$$P(Y_T \leqslant y | G_T = g) = \begin{cases} 1 & \text{if } y \geqslant g, \\ \Phi\left(\frac{\ln\left(\frac{g^{3/2}}{S_0^{\frac{1}{2}}}\right) + \frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T - \ln(g-y)}{\frac{\sigma\sqrt{T}}{2}}\right) & \text{if } 0 \leqslant y \leqslant g, \\ 0 & \text{if } y < 0. \end{cases}$$

Then

$$F^{-1}_{Y_T|G_T}(z) = \left(G_T - \frac{G_T^{\frac{3}{2}}}{S_0^{\frac{1}{2}}} e^{\frac{1}{4}\left(\mu - \frac{\sigma^2}{2}\right)T - \frac{\sigma}{2}\sqrt{T}\Phi^{-1}(z)}\right)^+.$$

Replacing z by the expression (38) for $F_{S_T|G_T}(S_T))$ derived above, then gives rise to expression (24). □

B.4 Derivation of the prices (25) and (26)

Price (25)
Let us observe that,

$$(G_T - S_T)^+ = G_T\left(1 - \frac{S_T}{G_T}\right)^+ = S_0 e^Y \left(1 - e^Z\right)^+,$$

where $Z = X - Y$, $Y = \ln\left(\frac{G_T}{S_0}\right)$, $X = \ln\left(\frac{S_T}{S_0}\right)$. We find, with respect to the *risk neutral measure* $\mathbb{Q}$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[(G_T - S_T)^+\right] &= S_0 \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}\left[e^Y|Z\right]\left(1-e^Z\right)^+\right) \\
&= S_0 \mathbb{E}_{\mathbb{Q}}\left[\left(e^{\mathbb{E}_{\mathbb{Q}}(Y|Z)+\frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)} - e^{\mathbb{E}_{\mathbb{Q}}(Y|Z)+\frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)+Z}\right)^+\right].
\end{aligned}$$

We now compute (still with respect to $\mathbb{Q}$),

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(Y|Z) &= \mathbb{E}_{\mathbb{Q}}(Y) + \frac{\operatorname{cov}_{\mathbb{Q}}(Y,Z)}{\operatorname{var}_{\mathbb{Q}}(Z)}(Z - \mathbb{E}_{\mathbb{Q}}(Z)) = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{4} + \frac{1}{2}Z \\
\operatorname{var}_{\mathbb{Q}}(Y|Z) &= (1-\rho^2)\operatorname{var}_{\mathbb{Q}}(Y) = \frac{3}{4}\frac{\sigma^2 T}{3} = \frac{\sigma^2 T}{4}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(G_T - S_T\right)^+ &= S_0 \mathbb{E}_{\mathbb{Q}}\left(e^{r\frac{T}{4}+\frac{1}{2}Z} - e^{r\frac{T}{4}+\frac{3}{2}Z}\right)^+ \\
&= S_0 \int_{-\infty}^{0} e^{r\frac{T}{4}+\frac{1}{2}Z} f_Z(z)dz - S_0 \int_{-\infty}^{0} e^{r\frac{T}{4}+\frac{3}{2}Z} f_Z(z)dz,
\end{aligned}$$

where $f_Z(z)$ is now denoting the density of Z under $\mathbb{Q}$. Here Z is normally distributed with parameters $(r - \frac{\sigma^2}{2})\frac{T}{2}$ and variance $\frac{\sigma^2 T}{3}$. Hence, taking into account Lemma A.3,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left(G_T - S_T\right)^+ \\
&= S_0 e^{r\frac{T}{2}-\sigma^2\frac{T}{12}}\mathbb{Q}\left(N\left(\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma^2 T}{6}, \frac{\sigma^2 T}{3}\right) \leqslant 0\right) \\
&\qquad - S_0 e^{rT}\mathbb{Q}\left(N\left(\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}+\frac{\sigma^2 T}{2}, \frac{\sigma^2 T}{3}\right) \leqslant 0\right) \\
&= S_0 e^{r\frac{T}{2}-\sigma^2\frac{T}{12}}\Phi\left(\frac{-\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}-\frac{\sigma^2 T}{6}}{\sqrt{\frac{\sigma^2 T}{3}}}\right) - S_0 e^{rT}\Phi\left(\frac{-\left(r-\frac{\sigma^2}{2}\right)\frac{T}{2}-\frac{\sigma^2 T}{2}}{\sqrt{\frac{\sigma^2 T}{3}}}\right)
\end{aligned}$$

Choose $f = \frac{-r\frac{T}{2}+\frac{\sigma^2 T}{12}}{\sigma\sqrt{\frac{T}{3}}}$ to obtain (25).

Price (26)
One has,

$$\left(G_T - a\frac{G_T^3}{S_T}\right)^+ = G_T\left(1 - a\frac{G_T^2}{S_T}\right)^+ = S_0 e^Y\left(1 - ce^Z\right)^+$$

where $Z = 2Y - X$, $Y = \ln\left(\frac{G_T}{S_0}\right)$, $X = \ln\left(\frac{S_T}{S_0}\right)$, $c = e^{\left(\mu-\frac{\sigma^2}{2}\right)\frac{T}{2}}$. Hence, with respect to the *risk neutral measure* $\mathbb{Q}$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ &= S_0 \mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(e^Y|Z)\left(1-ce^Z\right)^+\right) \\
&= S_0 \mathbb{E}_{\mathbb{Q}}\left(e^{\mathbb{E}_{\mathbb{Q}}(Y|Z)+\frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)} - ce^{\mathbb{E}_{\mathbb{Q}}(Y|Z)+\frac{1}{2}\operatorname{var}_{\mathbb{Q}}(Y|Z)+Z}\right)^+.
\end{aligned}$$

We now compute,

$$\mathbb{E}_{\mathbb{Q}}(Y|Z) = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{1}{2}Z \quad \text{and} \quad \mathrm{var}_{\mathbb{Q}}(Y|Z) = \frac{\sigma^2 T}{4}.$$

Hence,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ &= S_0\mathbb{E}_{\mathbb{Q}}\left(e^{r\frac{T}{2}-\frac{\sigma^2 T}{8}+\frac{1}{2}Z} - ce^{r\frac{T}{2}-\frac{\sigma^2 T}{8}+\frac{3}{2}Z}\right)^+ \\ &= S_0\int_{-\infty}^{\ln(c)} e^{r\frac{T}{2}-\frac{\sigma^2 T}{8}+\frac{1}{2}Z} f_Z(z)dz - S_0 c\int_{-\infty}^{\ln(c)} e^{r\frac{T}{2}-\frac{\sigma^2 T}{8}+\frac{3}{2}Z} f_Z(z)dz,\end{aligned}$$

where $f_Z(z)$ is the density of Z, under $\mathbb{Q}$. Note that Z is normally distributed with parameters 0 and variance $\frac{\sigma^2 T}{3}$. Taking into account Lemma A.3, we find,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(G_T - a\frac{G_T^3}{S_T}\right)^+ &= S_0\mathbb{Q}\left(N\left(\frac{\sigma^2 T}{6}, \frac{\sigma^2 T}{3}\right) \leqslant -\ln(c)\right)\exp\left(r\frac{T}{2} - \frac{\sigma^2 T}{12}\right) \\ &\quad -c.S_0\mathbb{Q}\left(N\left(\frac{\sigma^2 T}{2}, \frac{\sigma^2 T}{3}\right) \leqslant -\ln(c)\right)\exp\left(r\frac{T}{2} + \frac{\sigma^2 T}{4}\right) \\ &= S_0 e^{\frac{rT}{2}}\left(\Phi\left(d\right)e^{-\frac{\sigma^2 T}{12}} - e^{\frac{\mu T}{2}}\Phi\left(d - \frac{\sigma\sqrt{T}}{\sqrt{3}}\right)\right)\end{aligned}$$

where $d = \frac{-\ln(c) - \frac{\sigma^2 T}{6}}{\sigma\sqrt{\frac{T}{3}}} = \frac{\frac{\sigma^2 T}{12} - \mu\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}$. □

C Portfolio Management

C.1 Proof of Theorem 5.2

Let $H_T = \mathbb{E}(\xi_T|Z_T) = \varphi(Z_T)$ and let $\widehat{\varphi}$ denote the projection of φ on the cone $M_{\downarrow}$ defined as in (29) with respect to $L^2(\lambda_{[0,1]})$. Then we define $\widehat{X}_T$ and $k(\cdot)$ by

$$u'(\widehat{X}_T) := \lambda\widehat{\varphi}(Z_T),$$

i.e. $\widehat{X}_T = [u']^{-1}(\lambda\widehat{\varphi}(Z_T)) =: k(Z_T)$ with λ such that $\mathbb{E}\left[\xi_T\widehat{X}_T\right] = \mathbb{E}\left[\varphi(Z_T)k(Z_T)\right] = \int_0^1 \varphi(t)k(t)dt = \varphi \cdot k = W_0$.

By definition, $\widehat{X}_T$ is increasing in Z_T since $[u']^{-1}$ is decreasing and $\widehat{\varphi}$ is decreasing (it belongs to $M_{\downarrow}$). As a consequence $\widehat{X}_T$ is increasing in S_T, conditionally on A_T. For any $Y_T = h(Z_T)$ with a increasing function h, we have by concavity of u

$$u(Y_T) - u(\widehat{X}_t) \leqslant u'(\widehat{X}_T)(Y_T - \widehat{X}_T) = \lambda\widehat{\varphi}(Z_T)(h(Z_T) - k(Z_T)).$$

Thus, we obtain

$$\mathbb{E}[u(Y_T)] - \mathbb{E}[u(\widehat{X}_T)] \leqslant \lambda\int_0^1 \widehat{\varphi}(t)(h(t) - k(t))dt = \lambda\widehat{\varphi}\cdot(h - \Psi(\widehat{\varphi})), \tag{39}$$

where $\Psi(\widehat{\varphi}) = [u']^{-1}(\lambda\widehat{\varphi}) = k$ is increasing and $\Psi(t) = [u']^{-1}(\lambda t)$ is decreasing.

Now we use some properties of isotonic approximations (see Barlow *et al.* (1972)) and obtain

$$
\begin{aligned}
\widehat{\varphi}\cdot(h-\Psi(\widehat{\varphi})) &= \widehat{\varphi}\cdot((-\Psi)(\widehat{\varphi})-(-h)) \\
&= \varphi\cdot(-\Psi)(\widehat{\varphi}) - \widehat{\varphi}\cdot(-h) \qquad \text{(cf. Theorem 1.7 in Barlow } et\ al. \text{ (1972))} \\
&= \varphi\cdot(-h) - \widehat{\varphi}\cdot(-h) \qquad \text{both claims have price } W_0 \\
&= (\varphi-\widehat{\varphi})\cdot(-h) \leqslant 0
\end{aligned}
$$

by the projection equation (cf. Theorem 7.8 in Barlow *et al.* (1972)) using that $-h \in M_{\downarrow}$. As a result we obtain from (39) that

$$\mathbb{E}[u(Y_T)] \leqslant \mathbb{E}[u(\widehat{X}_T)],$$

i.e. $\widehat{X}_T$ is an optimal claim. □

C.2 Proof of formula (33)

We apply Theorem 5.2. Let W_0 denote the initial wealth and let $A_T = S_t$ for some $(t < T)$. From Lemma A.3 we find,

$$\ln(S_T)|\ln(S_t) \sim \mathcal{N}\left(\ln\left(\frac{S_T}{St}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t),\ \sigma^2(T-t)\right).$$

Hence,

$$F_{S_T|S_t}(S_T) = \Phi\left(\frac{\ln\left(\frac{S_T}{S_t}\right) - \left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right). \tag{40}$$

Since C is a Gaussian copula,

$$C_{1|S_t}(x) = \left[\frac{\Phi^{-1}[x] - \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right)}{\sqrt{1-\rho^2}}\right].$$

This implies,

$$Z_T = C^{-1}_{1|S_t}(F_{S_T|S_t}(S_T)) = \Phi\left[W_T\right],$$

where W_T is a function of S_T and S_t given by

$$W_T = \sqrt{1-\rho^2}\left(\frac{\ln\left(\frac{S_T}{St}\right) - \left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) + \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu-\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right).$$

Recall that from (4), $\xi_T = \alpha_T \left(\frac{S_T}{S_0}\right)^{-\beta}$ where $\alpha_T = \exp\left(\frac{\theta}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)T - \left(r + \frac{\theta^2}{2}\right)T\right)$, $\beta = \frac{\theta}{\sigma}$, $\theta = \frac{\mu - r}{\sigma}$. It follows that

$$H_T = \mathbb{E}(\xi_T|Z_T) = \mathbb{E}(\xi_T|W_T) = \delta e^{-\beta \operatorname{cov}(\ln(S_T),W_T)},$$

for some $\delta > 0$. Hence we find that

$$H_T = \delta e^{-\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)W_T}.$$

Note that the conditions on the correlation coefficient imply that H_T is decreasing in W_T and thus H_T is decreasing in Z_T. The optimal contract thus writes as

$$X_T^* := [u']^{-1}\left(\lambda e^{-\theta\left(\rho\sqrt{t}+\sqrt{(1-\rho^2)(T-t)}\right)W_T}\right) \tag{41}$$

where λ is such that $\mathbb{E}\left[\xi_T X_T^*\right] = W_0$. $\square$

C.3 Proof of Proposition 5.5

Assume that there exists an optimal solution to the Target Probability Maximization problem. It is a maximization of a law-invariant objective and therefore it is path-independent. Denote it by $X_T^* := f^*(S_T)$. Define $A_0 = \{x \mid f^*(x) = 0\}$, $A_1 = \{x \mid f^*(x) = b\}$, $A_2 = \{x \mid f^*(x) \in]0, b[\}$ and $A_3 = \{x \mid f^*(x) > b\}$. We show that $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$ must hold. Assume $\mathbb{P}(S_T \in A_0 \cup A_1) < 1$ so that $\mathbb{P}(S_T \in A_2 \cup A_3) > 0$. Define

$$Y = \begin{cases} f^*(S_T) & \text{for } S_T \in A_0 \cup A_1, \\ 0 & \text{for } S_T \in A_2, \\ b & \text{for } S_T \in A_3. \end{cases}$$

Then we observe that $Y = f^*(S_T)$ on $A_0 \cup A_1$ and $Y < f^*(S_T)$ on $A_2 \cup A_3$. Since $\mathbb{P}(S_T \in A_2 \cup A_3) > 0$ also $\mathbb{Q}(S_T \in A_2 \cup A_3) > 0$ because $\mathbb{P}$ and the risk neutral probability $\mathbb{Q}$ are equivalent. Hence $c_0(Y) < W_0$. Next we define $Z = b\mathbb{1}_{S_T \in C} + Y$ where we have chosen $C \subseteq A_2 \cup A_0$ such that $c_0(b\mathbb{1}_{S_T \in C}) = W_0 - c_0(Y)$. Since $\mathbb{P}(S_T \in C) > 0$ one has that $\mathbb{P}(Z \geqslant b) > \mathbb{P}(Y \geqslant b) = \mathbb{P}(f^*(S_T) \geqslant b)$. Hence Z contradicts the optimality of $f^*(S_T)$. Therefore $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Hence $f^*(S_T)$ can take only the values 0 or b. Since it is increasing in S_T almost surely (by cost-efficiency) it must write as

$$f^*(S_T) = b\mathbb{1}_{S_T > a},$$

where a is chosen such that the budget constraint is satisfied. $\square$

C.4 Proof of Theorem 5.6

The target probability maximization problem is given by

$$\max_{X_T \geqslant 0, c_0(X_T) = W_0} \mathbb{P}[X_T \geqslant A_T].$$

Assume that there exists an optimal solution X_T^* to this optimization problem. There are three steps in the proof.

1. The optimal payoff is of the form $f(S_T, A_T)$.
2. The optimal payoff is of the form $A_T \mathbb{1}_{h(S_T,A_T)\in A}$.
3. The optimal payoff is of the form $A_T \mathbb{1}_{A_T\xi_T<\lambda^*}$ for $\lambda^* > 0$.

Step 1: We observe that X_T^* has some joint distribution G with A_T. Theorem 3.2 implies there exists a twin $f(A_T, S_T)$ such that $(f(A_T, S_T), A_T) \sim (X_T^*, A_T) \sim G$ and $c_0(f(A_T, S_T)) = c_0(X_T^*) = W_0$. Therefore $\mathbb{P}(f(A_T, S_T) \geqslant A_T) = \mathbb{P}(X_T^* \geqslant A_T)$ and $\mathbb{P}(f(A_T, S_T) \geqslant 0) = \mathbb{P}(X_T^* \geqslant 0) = 1$. Thus $f(A_T, S_T)$ is also an optimal solution.

Step 2: This is similar to the proof of Proposition 5.5, applied conditionally on A_T. Define the sets $A_0 = \{s, f(A_T, s) = 0\}$, $A_1 = \{s, f(A_T, s) = A_T\}$, then $\mathbb{P}(S_T \in A_0 \cup A_1 | A_T) = 1$ and therefore $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Thus there exists a set A and a function h such that

$$f(A_T, S_T) = A_T \mathbb{1}_{h(S_T,A_T)\in A}.$$

Step 3: Define $\lambda > 0$ such that

$$\mathbb{P}(h(S_T, A_T) \in A) = \mathbb{P}(A_T\xi_T < \lambda).$$

Observe that $\mathbb{1}_{h(S_T,A_T)\in A}$ and $\mathbb{1}_{A_T\xi_T<\lambda}$ have the same distribution and that in addition, $A_T\xi_T$ is anti-monotonic with $\mathbb{1}_{A_T\xi_T<\lambda}$. Therefore by applying Lemma A.1 one has that

$$c_0(A_T \mathbb{1}_{A_T\xi_T<\lambda}) = \mathbb{E}[A_T\xi_T \mathbb{1}_{A_T\xi_T<\lambda}] \leqslant \mathbb{E}[A_T\xi_T \mathbb{1}_{h(S_T,A_T)\in A}]$$

and therefore the optimum must be of the form $A_T \mathbb{1}_{A_T\xi_T<\lambda^*}$ where $\lambda^* > \lambda$ is determined such that $c_0(A_T \mathbb{1}_{A_T\xi_T<\lambda^*}) = W_0$. □

C.5 Proof of Theorem 5.7

The target probability maximization problem is given by

$$\max_{\substack{X_T \geqslant 0,\ c_0(X_T)=W_0, \\ \mathcal{C}_{(X_T,A_T)}=C}} \mathbb{P}[X_T \geqslant b]$$

Assume that there exists an optimal solution X_T^* to this optimization problem. There are three steps in the proof.

1. The optimal payoff is of the form $f(S_T, A_T)$.
2. The optimal payoff is of the form $b\mathbb{1}_{h(S_T,A_T)\in A}$.
3. The optimal payoff is of the form $A_T \mathbb{1}_{Z_T>\lambda^*}$ for $\lambda^* > 0$.

Step 1: We observe that X_T^* has some joint distribution G with A_T. Theorem 3.2 implies there exists a twin $f(S_T, A_T)$ such that $(f(S_T, A_T), A_T) \sim (X_T^*, A_T) \sim G$ and $c_0(f(S_T, A_T)) = c_0(X_T^*) = W_0$. Therefore $\mathbb{P}(f(S_T, A_T) \geqslant b) = \mathbb{P}(X_T^* \geqslant b)$ and $P(f(S_T, A_T) \geqslant 0) = \mathbb{P}(X_T^* \geqslant 0) = 1$. Thus $f(S_T, A_T)$ is also an optimal solution.

Step 2: This is similar to the proof of Proposition 5.5. Define the sets $A_0 = \{(s,t), f(s,t) = 0\}$, $A_1 = \{(s,t), f(s,t) = b\}$, then $\mathbb{P}(S_T \in A_0 \cup A_1) = 1$. Thus there exists a set A and a function h such that

$$f(S_T, A_T) = b\mathbb{1}_{h(S_T,A_T)\in A}.$$

Step 3: Define $\lambda > 0$ such that

$$\mathbb{P}(h(S_T, A_T) \in A) = \mathbb{P}(Z_T > \lambda).$$

Observe that $b\mathbb{1}_{h(S_T,A_T)\in A}$ and $b\mathbb{1}_{Z_T>\lambda}$ have the same joint distribution G with distribution A_T. Therefore, Theorem 3.4 shows that,

$$c_0(b\mathbb{1}_{Z_T>\lambda}) \leqslant c_0(b\mathbb{1}_{h(S_T,A_T)\in A}).$$

Hence, $b\mathbb{1}_{Z_T>\lambda^*}$ where λ^* such that $c_0(b\mathbb{1}_{Z_T>\lambda^*}) = W_0$ is the optimum. $\square$

C.6 Proof of formula (34)

We know that $b\mathbb{1}_{Z_T>\lambda^*}$ where λ^* is such that $c_0(b\mathbb{1}_{Z_T>\lambda^*}) = W_0$ is the optimal solution. We find that (see also the proof for (33)),

$$\begin{aligned} Z_T &= C_{1|S_t}^{-1}(F_{S_T|S_t}(S_T)) \\ &= \Phi\left[\sqrt{1-\rho^2}\left(\frac{\ln\left(\frac{S_T}{St}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) + \rho\left(\frac{\ln\left(\frac{S_t}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right)\right]. \end{aligned}$$

It is then straightforward that $X_T^* = b\mathbb{1}_{\{S_t^\alpha S_T > \lambda^*\}}$ is the optimal solution, with α and λ given by

$$\begin{aligned} \alpha &= \sqrt{\frac{T-t}{t(1-\rho^2)}}\rho - 1 \\ \lambda &= \exp\left(\left(r - \frac{\sigma^2}{2}\right)(\alpha t + T) - \sigma\sqrt{(\alpha+1)^2 t + (T-t)}\,\Phi^{-1}\left(\frac{W_0 e^{rT}}{b}\right)\right). \end{aligned}$$

$\square$

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