On the method of optimal portfolio choice by cost-efficiency

Ludger Rüschendorf*, Viktor Wolf*

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*Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, 79104 Freiburg, Germany

Abstract

We develop the method of optimal portfolio choice based on the concept of cost-efficiency in two directions. First, instead of specifying a payoff distribution in an unique way we allow customer defined constraints resp. preferences for the choice of a distributional form of the payoff distribution. This leads to a class of possible payoff distributions. We determine upper and lower bounds for the corresponding strategies in stochastic order and describe related upper and lower price bounds for the induced class of cost-efficient payoffs. While the results for the cost-efficient payoff given so far in the literature in the context of Lévy models are based on the Esscher pricing measure we consider as alternative an empirical pricing measure leading to more precise pricing of cost-efficient options in the market. We show in some examples for real market data that this choice is numerically feasible and leads to more precise prices for the cost efficient payoffs and for values of the efficiency loss.

Keywords: cost-efficient strategies, Lévy models, Esscher transform, cost-efficiency, empirical pricing, optimal portfolio

1 The concept of cost-efficiency

This distributional analysis concept for portfolio choice has been introduced by Dybvig (1988a). The basic principle is to improve a given payoff X_T with distribution G, i.e. $X_T \sim G$, in a market model given by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ and an underlying price process $S = (S_t)_{0\le t\le T}$ by choosing a cheapest payoff $\underline{X}_T$ which has the same payoff distribution G as X_T, i.e. $\underline{X}_T \sim G$ that satisfies

$$c(\underline{X}_T) = \min_{X_T \sim G} c(X_T). \tag{1.1}$$

Here $c(X_T) = e^{-rT} E[Z_T X_T]$ is the cost of a strategy with terminal payoff X_T based on a pricing density Z_T used in the market such that $(e^{-rt} Z_t S_t)$ is a P-martingale. The cheapest payoff $\underline{X}_T$ with payoff distribution G in (1.1) has been named cost-efficient payoff in Bernard et al. (2012) and in the subsequent literature. Similarly, a payoff $\overline{X}_T \sim G$ is called *most expensive* if

$$c(\overline{X}_T) = \max_{X_T \sim G} c(X_T). \tag{1.2}$$

$$\ell(X_T) = c(X_T) - c(\underline{X}_T) \tag{1.3}$$

is called the *efficiency loss* of X_T.

The following characterization of cost-efficient payoffs has been stated in various generality in a series of papers including Dybvig (1988a,b), Jouini and Kallal (2001), Dana (2005), Schied (2004), Burgert and Rüschendorf (2006), Bernard and Boyle (2010), Bernard et al. (2012), Vanduffel et al. (2008, 2012) and Rüschendorf (2012).

Theorem 1.1 (Cost-efficient payoffs)
a) For a given payoff distribution G holds

$$c(\underline{X}_T) = e^{-rT} \int_0^1 G^{-1}(u) F_{Z_T}^{-1}(1-u) du \tag{1.4}$$

$$\text{and} \quad c(\overline{X}_T) = e^{-rT} \int_0^1 G^{-1}(u) F_{Z_T}^{-1}(u) du. \tag{1.5}$$

b) A payoff $\underline{X}_T \sim G$ is cost-efficient if and only if $\underline{X}_T$ and Z_T are antimonotonic. $\overline{X}_T \sim G$ is most expensive if and only if $\overline{X}_T$ and Z_T are comonotonic.

c) If F_{Z_T} is continuous then the cost efficient resp. most expensive payoffs are given by

$$\underline{X}_T = G^{-1}(1 - F_{Z_T}(Z_T)) \tag{1.6}$$

$$\text{resp.} \quad \overline{X}_T = G^{-1}(F_{Z_T}(Z_T)). \tag{1.7}$$

Theorem 1.1 has been applied in several papers to determine cost efficient payoffs, in particular in the context of the Samuelson model as well as in some classes of exponential Lévy processes (see Bernard et al. (2012), Vanduffel et al. (2012) and v. Hammerstein et al. (2014)) and has been applied to real market data.

In the context of Lévy models the results have been mainly based on the Esscher measure defined by the pricing density

$$Z_t^{\overline{\vartheta}} = \frac{e^{\overline{\vartheta} L_t}}{M_{L_t}(\overline{\vartheta})}, \tag{1.8}$$

where M_{L_t} denotes the moment generating function of L_t and $\overline{\vartheta}$, the Esscher parameter, is a solution to the equation

$$e^r = \frac{M_{L_1}(\overline{\vartheta}+1)}{M_{L_1}(\overline{\vartheta})}. \tag{1.9}$$

Condition (1.9) implies that the Esscher measure $Q^{\overline{\vartheta}} = Z_T^{\overline{\vartheta}} P$ is a risk neutral measure for the discounted stock price process $(e^{-rt} S_t)_{0 \le t \le T}$. It has the pleasant property that w.r.t. $Q^{\overline{\vartheta}}$ L remains a Lévy process with modified parameters.

In the context of Lévy models the market is bullish, i.e, $E\frac{S_t}{S_0} > e^{rt}$ iff $\overline{\vartheta} < 0$ and the market is bearish iff $\overline{\vartheta} > 0$ (see v. Hammerstein et al. (2014, Proposition 2.2)). Furthermore, for $\overline{\vartheta} < 0$ a payoff X_T is cost-efficient iff X_t is an increasing function in L_T and for $\overline{\vartheta} < 0$, X_T is cost-efficient iff X_T is a decreasing function of L_T. In particular a put is inefficient in increasing markets where $\overline{\vartheta} < 0$ and a call is inefficient in decreasing markets where $\overline{\vartheta} > 0$. It is shown (for certain examples) that the magnitude of efficiency loss is increasing in the magnitude of the trend in the market described by $|\overline{\vartheta}|$. As consequence one gets that path dependent options are not cost-efficient and thus can be improved by cost-efficient options.

In this paper, which is based on the dissertation Wolf (2014) we extend the distributional method of portfolio choice in two directions. In Section 2 we allow that a customer instead of specifying the wished payoff distribution completely determines a (finite) set of acceptable payoff distributions e.g. by posing constraints on the distribution functions at certain points and specifying customer preferences. For the resulting class of admissible payoff distributions we determine upper and lower bounds in stochastic order and determine upper and lower price bounds for the resulting cost-efficient payoffs.

In Section 3 we use a more realistic pricing method based on an empirical pricing measure while the cost-efficiency method applied to Lévy models has been based so far on pricing by the Esscher pricing measure. We show that this leads to more precise pricing of cost-efficient options and is numerically doable. We investigate some classes of basic options for real market data and determine cost efficient payoffs and the corresponding efficiency losses.

2 Customer specified payoff distributions

It is of particular interest for financial institutions or insurance companies to find accessible and economically priced strategies which fulfill customer-specified constraints and individual preferences. Typically, investors have little information on the strategy or option itself respectively the distribution function of the payoff of the option they actually seek for their current financial situation. In addition, it is hard to determine the entire distribution function of a suitable strategy or payoff for the customer.

It is however easier manageable and can be compiled via surveys or by interviewing clients to specify the aimed payoff distribution G at certain points α i.e. to specify that $P(X_T \leq \alpha) = \beta$ for $(\alpha, \beta) \in \mathcal{C}$ in some parameter class $\mathcal{C}$. The general class of in this way determined payoff distribution function

$$\mathbb{F}_{\mathcal{C}} := \{G \mid G \text{ is a df }, G(\alpha) = \beta, \quad \forall (\alpha, \beta) \in \mathcal{C}\} \tag{2.1}$$

may be a considerably large class.

Thus a second kind of restriction describing the individual preferences of a customer is introduced by specifying a certain (typically finite) subset of payoff distributions

$$\{F_1, \dots, F_N\} \subset \mathbb{F}_{\mathcal{C}}, \tag{2.2}$$

which are considered as acceptable. The first restriction in (2.1) specifies the payoff distribution G at certain levels of payoffs, the second restriction in (2.2) defines some individual preferences on the chosen payoff distribution. Note that the specification set $\mathcal{C}$ could be empty. In this case only some class $F_1, \dots, F_N$ of individually preferred payoff distributions is specified and no specified payoff levels are included.

Let G^* denote the supremum of $F_1, \dots, F_N$ in stochastic order and let G_* denote the infimum of $F_1, \dots, F_N$ in stochastic order. Then it is natural to consider all those payoff distributions G to be acceptable which satisfy $G_* \leq_{\mathrm{st}} G \leq_{\mathrm{st}} G^*$. Let

$$\mathbb{F}_{\mathcal{C},N} := \{G \in \mathbb{F}_{\mathcal{C}} |\, G_* \leq_{\mathrm{st}} G \leq_{\mathrm{st}} G^*\} \tag{2.3}$$

denote the set of all acceptable payoff distributions. We say that $\mathbb{F}_{\mathcal{C},N}$ is generated by $F_1, \dots, F_N$ and write $\mathbb{F}_{\mathcal{C},N} \sim F_1, \dots, F_N$. Then the following Proposition is obtained easily from the definition.

Proposition 2.1 *The elements G_* and G^* of the admissible class $\mathbb{F}_{\mathcal{C},N}$ in stochastic order are given by*

$$G^*(x) = \min\{F_1(x), \dots, F_N(x)\} \tag{2.4}$$

and

$$G_*(x) = \max\{F_1(x), \dots, F_N(x)\}. \tag{2.5}$$

Moreover G_* and G^* satisfy the specified constraints, i.e. $G_*, G^* \in \mathbb{F}_{\mathcal{C}}$.

Let for $G \in \mathbb{F}_{\mathcal{C},N}$ $\underline{X}_T^G$ be the cost-efficient strategy with payoff distribution G. Then we obtain the following price bounds for cost-efficient payoffs with payoff distribution in $\mathbb{F}_{\mathcal{C},N}$.

Proposition 2.2 *For any $G \in \mathbb{F}_{\mathcal{C},N}$ holds*

$$c(\underline{X}_T^{G_*}) \leq c(\underline{X}_T^G) \leq c(\underline{X}_T^{G^*}). \tag{2.6}$$

If F_{Z_T} is continuous, then

$$\underline{X}_T^{G^*} = \max_{1 \leq i \leq N} \underline{X}_T^{F_i}, \quad \underline{X}_T^{G_*} = \min_{1 \leq i \leq N} \underline{X}_T^{F_i}. \tag{2.7}$$

Proof: The inequalities in (2.6) are due to the representation of the lower cost bound in (1.4). If F_{Z_T} is continuous then the cost-efficient payoff is given by Theorem 1.1 as

$$\underline{X}_T^G = G^{-1}(1 - F_{Z_T}(Z_T)). \tag{2.8}$$

Using this representation and the stochastic ordering in the definition of $\mathbb{F}_{\mathcal{C},N}$ the result follows. Note that

$$\underline{X}_T^{G^*} = (G^*)^{-1}(1 - F_{Z_T}(Z_T)) \tag{2.9}$$

$$
\begin{aligned}
&= (\min\{F_1, \dots, F_N\})^{-1}(1 - F_{Z_T}(Z_T)) \\
&= \max\{F_1^{-1}(1 - F_{Z_T}(Z_T)), \dots, F_N^{-1}(1 - F_{Z_T}(Z_T))\}) \qquad (2.10) \\
&= \max_{1 \le i \le N} \underline{X}_T^{F_i}.
\end{aligned}
$$

The case with G_* is seen similarly. □

Remark 2.3 *1. In the case of an exponential Lévy model with Esscher parameter $\bar{\theta}$ the cost-efficient payoff can be represented in the following simplified form*

$$
\underline{X}_T^{G^*} := \begin{cases} G^{*-1}(1 - F_{L_T}(L_T)), & \text{if } \bar{\theta} > 0, \\ G^{*-1}(F_{L_T}(L_T)), & \text{if } \bar{\theta} < 0 \end{cases} \tag{2.11}
$$

and analogously for G_*.

2. As consequence of Propositions 2.1, 2.2 an investor may choose any payoff distribution function G between the lower and upper bound G_*, G^* matching his individual payoff constraints. Inequality (2.6) *shows the corresponding range of possible prices of the cost-efficient payoffs. If the cost of the stochastically largest payoff distribution G^* is still acceptable for the investor then this would be the choice. If the cost is the most important criterion, then $X_T^{G_*}$ would be the choice suitable for this customer. In general a choice of a payoff distribution $G \in \mathbb{F}_{\mathcal{C},N}$ might be based on a reward cost criterion of the form*

$$
E\underline{X}_T^G - \lambda c(\underline{X}_T^G) = \max_{G \in \mathbb{F}_{\mathcal{C},N}} ! \tag{2.12}
$$

Note that $E\underline{X}_T^G$ is identical to $\int x\, G(\mathrm{d}x)$ and thus easy to determine.

In the following we determine in some examples the optimal strategy corresponding to the upper payoff bound G^* and we explain how the optimal payoff is related to the generating optimal payoffs which fulfill customer-given constraints. To keep things simple we consider the case $N = 2$ and $\mathbb{F}_{\mathcal{C},N}$ to be generated by

$$
\{F_{\text{Put}}, F_{\text{Call}}\},\ \{F_{\text{sqC}}, F_{\text{sqP}}\},\ \{F_{\text{strdl}}, F_{\text{bfly}}\}
$$

where sqC, sqP denotes the self-quanto call and put with terminal payoffs $X_T^{\text{sqC}} = S_T(S_T - K)_+$, $X_T^{\text{sqP}} = S_T(K - S_T)_+$ while $\text{strdl}, \text{bfly}$ denotes the long straddle resp. long call butterfly spread options. We do not specify the payoff distribution function at certain levels in our examples but just specify the chosen payoff distributions. Since these intersect we implicitly also fix some constraint pairs (α, β). We evaluate the cost-efficient strategies for real market data of Volkswagen, Allianz, ThyssenKrupp and E.ON for the period October and November 2012 and 23 trading days. The initial stock price is $S_0 = 130.55$, the closing price at October 1, 2012. As distributional models we use the NIG, the VG and the normal model with parameters estimated from market data. We choose as pricing measure the Esscher martingale measure. For details on the data and the statistical analysis we refer to v. Hammerstein et al. (2014) and to Wolf (2014).

$\mathbb{F}_{\mathcal{C},2} \sim \{F_{\text{Put}}, F_{\text{Call}}\}$

We consider Volkswagen data for the period October and November 2012 and 23 trading days. The initial stock price is $S_0 = 133{,}55$, the closing price on October 1, 2012. The price $S_T = 153$ of Volkswagen at maturity November 1, 2012, is used to compute the payoffs $\omega^{\bullet}(S_T)$.

We consider first the standard long put X_T^{Put} and call X_T^{Call} with strike $K = 136$ and $K = 134$ respectively and $T = 23$ trading days, i.e. $\mathbb{F}_{\mathcal{C},2} = \{F_{\text{Put}}, F_{\text{Call}}\}$. This choice is consistent with the customer-given constraints, $(\alpha, \beta) = (19.4614, 0.9076)$.

Volkswagen	$c(X_T^{\text{Put}})$	$\omega^{\text{Put}}(S_T)$	$c(X_T^{\text{Call}})$	$\omega^{\text{Call}}(S_T)$
NIG	8.66	0.00	3.93	19.00
VG	8.63	0.00	3.90	19.00
Normal	8.65	0.00	3.92	19.00
Volkswagen	$c(\underline{X}_T^{G^*})$	$\underline{\omega}^{G^*}(S_T)$	$c(\underline{X}_T^{\text{Put}})$	$\underline{\omega}^{\text{Put}}(S_T)$
NIG	4.64	19.11	4.52	19.11
VG	4.60	19.13	4.49	19.13
Normal	4.63	19.07	4.51	19.07

Table 1: Comparison of prices and payoffs for standard put, call, cost-efficient put and the cost-efficient min-cost strategy $\underline{X}_T^{G^*}$.

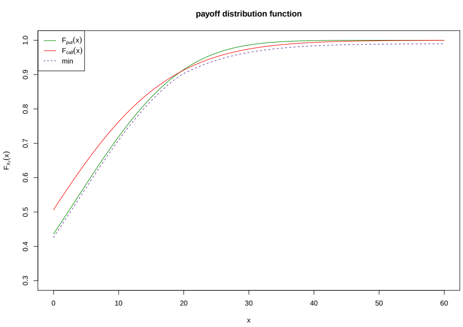

Figure 1: Distribution of standard put with strike $K = 136$ and call with strike $K = 134$ for Volkswagen. The dotted line marks the minimum G^*.

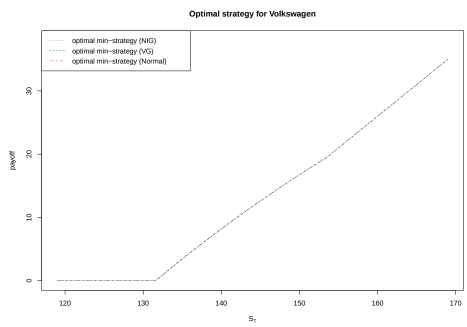

Figure 2: Optimal strategy $\underline{X}_T^{G^*}$ for Volkswagen in case of a standard put with strike $K = 136$ and call with strike $K = 134$.

Figure 1 shows the distribution functions of these strategies with a dotted line expressing $G^* = \min\{F_{\text{Put}}, F_{\text{Call}}\}$ which generates the optimal min-cost strategy $\underline{X}_T^{G^*}$ from equation (2.8). Figure 2 illustrates the corresponding optimal payoff of the min-cost strategy $\underline{X}_T^{G^*}$ for Volkswagen. When the price at maturity S_T ranges from about $[0, 153.50]$ the payoff of the corresponding cost-efficient min-cost strategy outperforms the standard call, which in this case is also cost-efficient. If at maturity the price S_T is greater than 153.50 the payoff of the corresponding cost-efficient min-cost strategy equals the payoff of the standard call and therefore outperforms the payoff of the corresponding cost-efficient put option. As can be seen from Table 1 resp. Figure 1 there is only a small influence of the different Lévy models chosen to model the data. Note that $\omega^X(S_T)$ denotes the payoff function of option X, i.e. in case of the standard call we have $\omega^{\text{Call}}(S_T) = (S_T - K)_+$.

Figure 2 shows that the optimal strategy $\underline{X}_T^{G^*}$ for Volkswagen is composed of a standard call and a cost-efficient put as shown in Proposition 2.2.

$\mathbb{F}_{\mathcal{C},2} \sim \{F_{\text{strdl}}, F_{\text{bfly}}\}$

The two options are chosen for ThyssenKrupp data as the standard long straddle X_T^{strdl} with exercise price $K = 16.5$ and the long call butterfly X_T^{bfly} with strike prices $K_1 = 12$ and $K_3 = 20$, that is, we consider the set $\mathbb{F}_{\mathcal{C},2} \sim \{F_{\text{strdl}}, F_{\text{bfly}}\}$. The payoff distribution function G^* is shown in Figure 3. The choice is consistent with a constraints specification to $\{(\alpha_1, \beta_2), (\alpha_2, \beta_2)\} = \{(0.1657, 0.0685), (3.8722, 0.9481)\}$. In this case we consider the ThyssenKrupp stock within the trading period $[0, T]$ with $T = 23$ days. The initial stock price is $S_0 = 16.73$, the closing price at October 1, 2012.

ThyssenKrupp	$c(\underline{X}_T^{\text{strdl}})$	$\underline{\omega}^{\text{strdl}}(S_T)$	$c(\underline{X}_T^{\text{bfly}})$	$\underline{\omega}^{\text{bfly}}(S_T)$	$c(\underline{X}_T^{G^*})$	$\underline{\omega}^{G^*}(S_T)$
NIG	1.538	0.692	2.40	1.834	2.448	1.834
VG	1.531	0.688	2.41	1.839	2.452	1.839
Normal	1.529	0.694	2.40	1.841	2.449	1.841

Table 2: Comparison of prices and payoffs for cost-efficient straddle, butterfly and the cost-efficient min-cost strategy $\underline{X}_T^{G^*}$. The terminal value $S_T = 17.74$ of ThyssenKrupp at maturity, November 1, 2012, is used to compute the payoffs $\omega^{\cdot}(S_T)$.

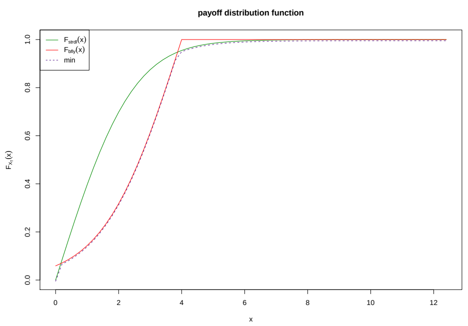

Figure 3: Distribution of standard long straddle with strike $K = 16.5$ and long call butterfly with strikes $K_1 = 12$ and $K_3 = 20$ for ThyssenKrupp. The dotted line marks the minimum G^*.

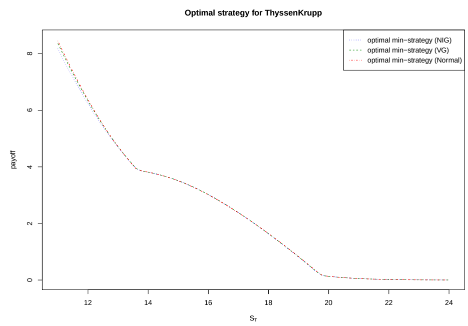

Figure 4: Optimal strategy $\underline{X}_T^{G^*}$ for ThyssenKrupp in case of a standard long straddle with strike $K = 16.5$ and long call butterfly with strikes $K_1 = 12$ and $K_3 = 20$.

Figure 4 illustrates the corresponding optimal payoff of the min-cost strategy $\underline{X}_T^{G^*}$ for ThyssenKrupp. When the price at maturity S_T ranges from about $[13.71, 19.72]$ the payoff of the corresponding cost-efficient min-cost strategy $\underline{X}_T^{G^*}$ outperforms the cost-efficient straddle. Otherwise, the payoff of the corresponding cost-efficient min-cost strategy equals the payoff of the cost-efficient straddle call and therefore outperforms the payoff of the corresponding cost-efficient butterfly option. As it is easily observed from Table 2 the cost-efficient min-cost strategy

$\underline{X}_T^{G*}$ performs almost as good as the cost-efficient long call butterfly option. The guarantee of having a higher payoff in mean here results in a marginally noticeable higher price compared to the optimal long call butterfly option and thus may be a good choice for an investor.

$\mathbb{F}_{\mathcal{C},2} \sim \{F_{\text{sqP}}, F_{\text{sqC}}\}$

Finally, for the Allianz stock we consider the the self-quanto call X_T^{sqC} and put X_T^{sqP} with strike $K = 98$ and maturity $T = 23$ days. Since the corresponding distribution functions intersect (see Figure 5) this choice is consistent with the customer constraints $(\alpha, \beta) = (1496.49, 0.95905)$.

Figure 6 illustrates the corresponding optimal payoff of the min-cost strategy $\underline{X}_T^{G*}$ for Allianz. When the price at maturity S_T ranges from about $[0, 111.50]$ the payoff of the corresponding cost-efficient min-cost strategy outperforms the cost-efficient self-quanto call. Otherwise, the payoff of the corresponding cost-efficient min-cost strategy equals the payoff of the cost-efficient self-quanto call and therefore outperforms the payoff of the corresponding cost-efficient self-quanto put.

Allianz	$c(\underline{X}_T^{\text{sqC}})$	$\underline{\omega}^{\text{sqC}}(S_T)$	$c(\underline{X}_T^{\text{sqP}})$	$\underline{\omega}^{\text{sqP}}(S_T)$	$c(\underline{X}_T^{G^*})$	$\underline{\omega}^{G^*}(S_T)$
NIG	205.40	0.00	452.85	557.10	468.61	557.10
VG	195.62	0.00	449.59	561.33	462.75	561.33
Normal	202.66	0.00	452.21	558.67	465.50	558.67

Table 3: Comparison of prices and payoffs for cost-efficient self-quanto call, self-quanto put and the cost-efficient min-cost strategy $\underline{X}_T^{G^*}$. The terminal value $S_T = 95.92$ of Allianz at maturity, November 1, 2012, is used to compute the payoffs $\omega^{\cdot}(S_T)$.

From Table 3 one can observe that the cost-efficient min-cost strategy $\underline{X}_T^{G*}$ performs almost as good as the cost-efficient self-quanto put option on the Allianz stock. The guarantee of having a higher payoff in mean here expresses in a marginally noticeable higher price compared to the optimal self-quanto put and thus may be a good choice for an investor.

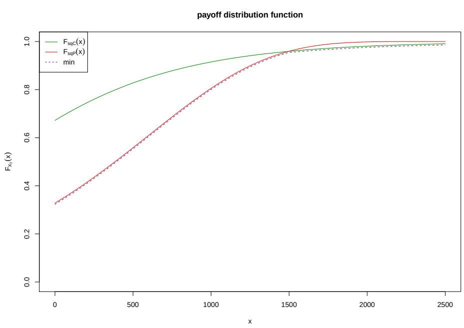

Figure 5: Distribution of self-quanto put and call with strike $K = 98$ for Allianz. The dotted line marks the minimum G^*. The initial stock price is $S_0 = 93.42$, the closing price at October 1, 2012.

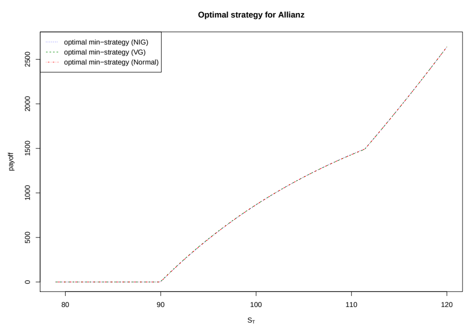

Figure 6: Optimal strategy $\underline{X}_T^{G^*}$ for Volkswagen in case of a standard put with strike $K = 136$ and call with strike $K = 134$.

3 An alternative pricing method

The applications of the method of portfolio selection based on cost-efficient payoffs have been applied so far in the context of Lévy models using the Esscher measure transform as pricing measure. This choice is mainly motivated by several mathematical simplifications induced by this pricing method. In the more recent research on option pricing it has however turned out that more precise pricing methods are available. A promising and well established alternative pricing method is pricing on the empirical pricing measure Q resp. the corresponding state price density $Z_T = \frac{\mathrm{d}Q}{\mathrm{d}P}|_{\mathcal{F}_T}$ which is provided by choosing a suitable large (but typically parametric) family of martingale measures and then choosing the parameters of the martingale measure by fitting the prices of a class of vanilla options observed in the market by the model prices. In this way market prices determine the choice of the martingale measure. In contrast to the Esscher pricing method the density Z_T of the empirical pricing measure in Lévy models is typically no longer a monotone function of L_T.

To start with a typically parametric class of Lévy models is chosen for the modeling of the underlying stock price process and the parameters of the model are estimated from daily log-returns of the data (under the physical measure P). Then for a given financial derivative X_T with maturity $T > 0$ the payoff distribution G is determined explicitly (under P). We want to emphasize that the payoff function $\underline{X}_T = G^{-1}(1 - F_{Z_T}(Z_T))$ of the cost-efficient version also depends on the payoff distribution G which is determined under P.

Then in the second step in the class of risk-neutral measures corresponding to the chosen Lévy model the parameters of the risk-neutral measure are estimated by a least square method over the squared differences of market and model prices for some basic derivatives are minimized over the Lévy parameters in the class of risk-neutral measures. Thus the risk neutral measure is modeled directly from observed market prices of some basic vanilla options as e.g. calls or puts. As a result we obtain the risk-neutral empirical pricing measure Q with density Z_T w.r.t. P. It is shown in the literature that this pricing measure leads to a more precise pricing of derivatives.

Hence, in a straightforward manner, the price of the cost-efficient claim $\underline{X}_T$ is given by

$$e^{rT} c(\underline{X}_T) = E[Z_T G^{-1}(1 - F_{Z_T}(Z_T))] = \int_{\mathbb{R}} G^{-1}(1 - F_{Z_T}(h_T(x)) d_{L_T^Q}(x) \,\mathrm{d}x \tag{3.13}$$

where $h_T := \frac{\mathrm{d}Q^{L_T}}{\mathrm{d}P^{L_T}}$ and $d_{L_T^Q}$ denotes the density of L_T under Q. This method takes all relevant past and present market occurrences into account and gives also a more accurate pricing of the cost-efficient versions of the derivatives. This allows us to compute the expectations, which arise as prices of cost-efficient strategies, as in equation (3.13).

3.1 Pricing under the equivalent martingale measure

We briefly recall the construction of the class of equivalent martingale measures in order to introduce the least squares estimation of the Lévy parameters of the risk neutral measure.

The pricing of derivatives which depends on the underlying price process given by $S_t = S_0 e^{L_t}$ requires $(S_t)_{t\geq 0}$ to be a martingale. The martingale property can be described alternatively by the following differential equation

$$\mathrm{d}S_t = S_{t-}\Big(\mathrm{d}L_t + \frac{c}{2}\,\mathrm{d}t + \int_{\mathbb{R}} (e^x - 1 - x)\mu^L(\mathrm{d}t, \mathrm{d}x)\Big) \tag{3.14}$$

where S_{t-} denotes the left limit at time point t. $(L_t)_{t\geq 0}$ possesses the triplet of local characteristics (b, c, F) with random measure of jumps denoted by μ^L Lévy measure F, drift b and volatility c. A necessary assumption for a martingale is that each variable has a finite expectation $E[S_t] = S_0 E[e^{L_t}] < \infty$. This is a consequence of assuming that there exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} e^{ux}\, F(\mathrm{d}x) < \infty \tag{3.15}$$

for all $u \in [-M, M]$ (see Sato (1999, Theorem 25.3)). Recall that all the common Lévy processes such as *hyperbolic*, *NIG*, *GH*, *VG*, and *CGMY* Lévy processes satisfy equation (3.15). From the stochastic differential equation (3.14) one can derive that $(S_t)_{t\geq 0}$ is a martingale if the drift parameter b coincides with the exponential compensator of the Gaussian and the pure jump part of L, i.e.

$$b + r = -\frac{c}{2} - \int_{\mathbb{R}} (e^x - 1 - x)F(\mathrm{d}x). \tag{3.16}$$

Note that because of the rich structure of the Lévy processes, the set of equivalent martingale measures is in general very large (see Eberlein and Jacod (1997)). We therefore consider a priori a martingale model which is determined by (3.16) or a sufficiently large subclass of it, which is described by a set of parameters $\eta = (\eta_1, \ldots, \eta_k)$ such that (3.16) is fulfilled.

In general we have to consider a functional φ of the whole price path which we write in the form $\varphi(S_0 e^{L_t}, 0 \leq t \leq T) = f(Y_T - s)$ where $s = -\log S_0$ and the driving process Y can be L or other functions of the path of the underlying Lévy process. The time-0-price of this option as a function of the process Y and the value s is given by

$$c_0(Y; s) = E_Q[\varphi(S_t, 0 \leq t \leq T)] = E_Q[f(Y_T - s)]. \tag{3.17}$$

The expectation is taken with respect to the empirical martingale measure Q which is given by estimating the Lévy parameters from current prices of common financial derivatives. The least squares estimation of the Lévy parameter is determined as follows: We use the notation $c_0(\eta; s) = c_0(Y; s)$ for the time-0-price in order to emphasize the dependence on the Lévy parameters $\eta = (\eta_1, \ldots, \eta_k)$, $k \in \mathbb{N}$. The martingale measure Q respectively the Lévy parameters under Q are determined by a least squares estimation technique. Hereto, we denote by $c_0(\eta; s)$ the time-0-price of a standard long call (long put) with maturity $T > 0$ and strike K for a chosen Lévy model with parameters η. Let c_0^m be the quoted market price with maturity T and strike K_m, $1 \leq m \leq m_0$. Then the solution $\hat{\eta}$ of the minimization problem

$$\min_{\eta} \Big[\sum_{m=1}^{m_0} \big(c_0^m(\eta; s) - c_0^m\big)^2\Big] \tag{3.18}$$

yield the Lévy parameters of $Y_1 = L_1$ for the empirical risk-neutral measure Q and $c_0^m(\eta; s)$ is the model price of a call with strike K_m. Alternatively, a summation over different maturities T could be done.

3.2 Cost-efficient strategies under empirical martingale measure pricing

We next calibrate a *NIG* model to real market data from a ThyssenKrupp stock. Table 4 lists the estimated parameters $\eta(L_1)$ from the daily log-returns from ThyssenKrupp from March 14, 2011 to March 13, 2014 as well as the parameters of the calibrated empirical risk-neutral pricing measure. Also the Esscher parameter $\bar{\theta}$ is displayed in Table 4.

ThyssenKrupp	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\delta}$	$\boldsymbol{\mu}$	$\bar{\boldsymbol{\theta}}$
$\eta(L_1)$	49.24615	−3.5086	0.028	0.0015	0.449252
$\eta(L_1^Q)$	116.1038	−7.2793	0.03452	0.00	–

Table 4: Estimated NIG parameters under Q from long call prices on ThyssenKrupp at March 14, 2014 with maturity $T = 444$ days and under P from daily log-returns of ThyssenKrupp from March 14, 2011 to March 13, 2014.

Then in the second step we use least squares to calibrate the prices of long calls on ThyssenKrupp on March 14, 2014 with expiration date December 15, 2015 within the *NIG* model. Figure 7 gives the resulting call price curve in dependence of the strike K, varying form 10 to 48 which shows a good coincidence with the observed market prices. For the least squares estimation of the Lévy parameters the continuously compounded 1-Day-Euribor rate of March 14, 2014, which equals $r = 4.388879 \cdot 10^{-6}$, is used.

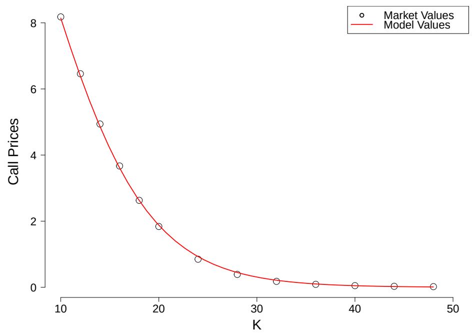

Figure 7: Calibration of NIG parameters for prices of a long call on ThyssenKrupp at March 14, 2014 with expiration date December 15, 2015.

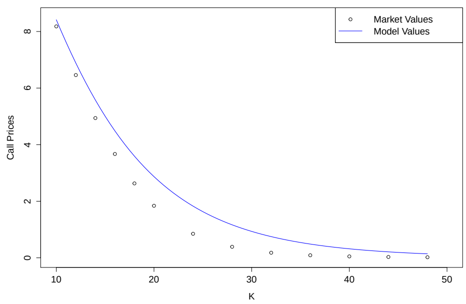

Figure 8: Long call prices on ThyssenKrupp at March 14, 2014 with maturity $T = 444$ days and computed prices via the Esscher martingale measure with estimated parameters under P from daily log-returns of ThyssenKrupp from March 14, 2011 to March 13, 2014.

Figure 8 shows that the long call prices computed by the Esscher martingale measure depart considerably from the market prices in the central range $16 \leq K \leq 35$ which confirms a weakness of this pricing method. The estimated parameters $\eta(L_1^Q)$ are listed in Table 4 for the ThyssenKrupp stock in the *NIG* model. For the least squares estimation long call prices[1] on ThyssenKrupp at March 14, 2014 with expiration date December 15, 2015 are used, that is, the maturity equals $T = 444$ days. The estimated Lévy parameters $\eta(L_1)$ from the daily log-returns of ThyssenKrupp from March 14, 2011 to March 13, 2014 can be found in Table 4 and are utilized to compute the payoff distribution G_{Call} and the payoff of the cost-efficient $\underline{X}_T^{\text{Call}}$.

In the next step we compute the payoff distribution G_{Call} of the long call and the payoff of the cost-efficient call $\underline{X}_T^{\text{Call}}$ (based on the parameters $\eta(L_1)$) for the strikes $K = 16$ and $K = 24$. The interest rate used to calculate the Esscher parameter $\bar{\theta}$ is the continuously compounded 1-Day-Euribor rate[2] of March 14, 2014, which is $r = 4.388879 \cdot 10^{-6}$.

In Table 5 we compare the two introduced pricing methods. The cost of a long call option on ThyssenKrupp and the corresponding cost-efficient payoffs in the *NIG* Lévy model are presented for the Esscher pricing method and for pricing by the empirical pricing measure Q with parameters $\eta(L_1^Q)$ based on formula (3.13). We obtain a correction of the cost of the cost-efficient call. In both cases the more precise price of the efficient call is reduced compared to the Esscher price which leads to a correction of the relative efficiency loss in the magnitude of 8% resp. 27%.

$K = 16$	$c(X_T^{\text{Call}})$	$c(\underline{X}_T^{\text{Call}})$	**Efficiency loss in %**
$\eta(L_1)$	4.4864	2.2597	49.63
$\eta(L_1^Q)$	3.67	1.5434	57.95
$K = 24$	$c(X_T^{\text{Call}})$	$c(\underline{X}_T^{\text{Call}})$	**Efficiency loss in %**
$\eta(L_1)$	1.8271	0.7316	59.96
$\eta(L_1^Q)$	0.85	0.5779	32.01

Table 5: The pricing methods, Esscher transform versus pricing under the equivalent martingale measure Q, are compared. The initial stock price at March 14, 2014 is $S_0 = 18$ and the maturity is chosen to be $T = 444$ days. The other parameters needed for the calculations are taken from Table 4.

Remark 3.4 (Numerical issues) *Since the empirical pricing density $Z_T = h(L_T)$ typically is not monotone in L_T – in the NIG model it is the quotient of two NIG densities – the expression $1 - F_{Z_T}(h(x))$ in formula (3.13) must be estimated or simulated. The results established in Table 5 for $c(\underline{X}_T^{\text{Call}})$ in the market model are achieved by simulating the distribution function F_{Z_T} for $Z_T = h_T(L_T)$ where*

$$h_T = \frac{\mathrm{d}Q^{L_T}}{\mathrm{d}P^{L_T}} = \frac{\mathrm{d}Q^{L_T}}{\mathrm{d}\lambda\!\!\!\lambda} \frac{\mathrm{d}\lambda\!\!\!\lambda}{\mathrm{d}P^{L_T}}.$$

In order to obtain precise results we computed the distribution function F_{Z_T} by simulating the corresponding Lévy process over $9 \cdot 10^6$ times. For such an estimation it appears that the absolute error has the scale of 10^{-5}. A higher number of simulations leads to a better value for the absolute error but also to an increase of the timespan to compute the price.

[1]Historical call prices are listed, e.g. on `www.eurexchange.com`.

[2]The 1-Day-Euribor equals the 1-day interbank interest rate for the Euro zone (*Eonia* rate).

4 Conclusion

In our paper we propose a way to include realistically available information of a customer on the specification of the payoff distribution aimed at and also to include preferences of the customer given in terms of a finite number of acceptable payoff curves. This information is combined with the construction of cost-efficient payoffs which allows to interpolate between admissible costs and high payoff.

As a second innovation we use a more precise pricing method based on the empirical pricing measure compared to pricing of the mathematically simpler Esscher pricing measure in order to obtain a more precise value of the price of cost-efficient payoffs. We show that this approach is numerically feasible and that it may lead in examples to considerably corrected prices for the cost-efficient strategies and thus to more precise values for the efficiency loss.

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Address: Department of Mathematical Stochastics, University of Freiburg,
e-mail: ruschen@stochastik.uni-freiburg.de, wolf@stochastik.uni-freiburg.de;
FAX: +49 761 203 5661; Phone: +49 761 203 5665