On optimal allocation of risk vectors

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Abstract

In this paper we extend results on optimal risk allocations for portfolios of real risks w.r.t. convex risk functionals to portfolios of risk vectors. In particular we characterize optimal allocations minimizing the total risk as well as Pareto optimal allocations. Optimal risk allocations are shown to exhibit a worst case dependence structure w.r.t. some specific max-correlation risk measure and they are comonotone w.r.t. a common worst case scenario measure. We also derive a new existence criterion for optimal risk allocations and discuss some examples.

Key words: Optimal risk allocations, worst case portfolio, comonotonicity

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JEL classification: G32, C43

1 Introduction

In this paper we consider an extension of the optimal risk allocation problem resp. the risk exchange problem to the case of risk vectors. This extension allows to include the effects of dependence in a portfolio as measured by multivariate risk measures ϱ_i in the risk allocation problem. On a basic probability space $(\Omega, \mathcal{A}, P)$ we consider convex, proper, normed lower semicontinuous (lsc) risk functions, called in the following *risk functionals* $\varrho_i : L_d^p(P) \to (-\infty, \infty]$, $1 \le i \le n$, defined on risk vectors $X = (X_1, \dots, X_d)$ with $X_i \in L^p(P) = L^p$, i.e. $L_d^p(P) = L_d^p$ is the d-fold product of $L^p(P)$. The risk functionals ϱ_i describe the risk evaluation of n traders in the market. Here ϱ_i are *normed* means that $\varrho_i(0) = 0$ and ϱ_i are *proper* means that dom $\varrho_i \neq \emptyset$ and $\varrho_i(X) \neq -\infty$ for all X. We allow unbounded risks and assume that $1 \le p \le \infty$.

For a given portfolio of d risks described by a risk vector $X = (X^1, \dots, X^d) \in L_d^p$ we define the set $\mathcal{A}(X) = \mathcal{A}^n(X)$ of *n-allocations* of the portfolio X by

$$\mathcal{A}(X) := \left\{ (\xi_1, \dots, \xi_n) \mid \xi_i \in L_d^p, \sum_{i=1}^n \xi_i = X \right\}. \tag{1.1}$$

For an allocation $(\xi_1, \ldots, \xi_n) \in \mathcal{A}(X)$ trader i is exposed to the risk ξ_i which is evaluated by the risk functional ϱ_i. ξ_i may contain some zero components and thus trader i may only be exposed to some of the d components of risk in our formulation. Let

$$\mathcal{R} := \{(\varrho_i(\xi_i)) \mid (\xi_i) \in \mathcal{A}(X)\} \tag{1.2}$$

denote the corresponding risk set. Our aim is to characterize Pareto-optimal (PO) allocations $(\xi_i) \in \mathcal{A}(X)$, i.e. allocations such that the corresponding risk vectors are minimal elements of the risk set $\mathcal{R}$ in the pointwise ordering. A related optimization problem is to characterize allocations $(\eta_i) \in \mathcal{A}(X)$ which minimize the total risk, i.e.

$$\begin{aligned} \sum_{i=1}^n \varrho_i(\eta_i) &= \inf \left\{ \sum_{i=1}^n \varrho_i(\xi_i) \mid (\xi_i) \in \mathcal{A}(X) \right\} \\ &=: \wedge \varrho_i(X). \end{aligned} \tag{1.3}$$

The optimal allocation problem of risks is a classical problem in mathematical economics and insurance and is of considerable practical and theoretical interest. It has been studied in the case of real risks, i.e. in the case $d = 1$ in the classical papers of Borch (1962), Gerber (1979), Bühlmann and Jewell (1979), Deprez and Gerber (1985) and others in the context of risk sharing in insurance and reinsurance contracts. In more recent years this problem has also been studied in the context of financial risks as in risk exchange, assignment of liabilities to daughter companies individual hedging problems and others (see the papers of Heath and Ku (2004), Barrieu and El Karoui (2005a,b), Burgert and Rüschendorf (2006, 2008), Jouini et al. (2007), Acciaio (2007), Filipović and Svindland (2007), [KR][1] (2008), and others).

The aim of this paper is to extend the risk allocation results to the case of multivariate risks resp. the case of risk portfolios. The main motivation for considering multivariate risk measures is to include the influence of (positive) dependence on the risk of a portfolio. In recent papers several of the aspects of multivariate risks like worst case portfolios, diversification effects or strong coherence have been studied (see e.g. Ekeland et al. (2009), [R] (2009), Carlier et al. (2009)). As we will see the optimal risk allocation problem has some close ties to these developments.

After the introduction of some basic notions from convex analysis in Chapter 2 we derive in Chapter 3 the basic characterization of optimal total risk minimizing allocations and give a link to Pareto-optimal allocations. Due to the multivariate structure the proof of this characterization needs a new element in the analysis. In Chapter 4 we specialize to law invariant convex risk measures ϱ_i. A characterization of their subgradients leads to a close connection between optimal allocations and worst case portfolio vectors. More precisely it is shown that optimal allocations are

[1]Kiesel and Rüschendorf is abbreviated within this paper with [KR], Rüschendorf with [R].

comonotone w.r.t. a common worst case scenario measure. Further they exhibit a worst case dependence structure w.r.t. some specific max-correlation risk measure. In Chapter 5 we derive a new general existence criterion for optimal allocations and finally discuss some examples in Chapter 6.

2 Some notions from convex analysis

Throughout this paper we consider convex, lower semicontinuous (lsc) proper risk functions ϱ on L_d^p called in the following risk functionals. Generally, for a convex proper function on a locally convex space E paired with its dual space E^* by $(E, E^*, \langle\cdot,\cdot\rangle)$ we denote by

$$f^* : E^* \to \overline{\mathbb{R}}, \quad f^*(x^*) = \sup_{x\in E}(\langle x^*, x\rangle - f(x)), \quad x^* \in E^*, \tag{2.1}$$

the convex conjugate and by

$$f^{**} : E \to \overline{\mathbb{R}}, \quad f^{**}(x^*) = \sup_{x^*\in E^*}(\langle x, x^*\rangle - f^*(x^*)), \quad x \in E, \tag{2.2}$$

the bi-conjugate of f. Let further

$$\partial f(x) = \{x^* \in E^* \mid f(y) - f(x) \geq \langle x^*, y - x\rangle, \forall y \in E\} \tag{2.3}$$

denote the set of subgradients of f in x. Then

$$\partial f(x) = \{x^* \in E^* \mid \forall y \in E, \langle x^*, y\rangle \leq D(f; x)(y)\}, \tag{2.4}$$

where $D(f, x)(y)$ is the right directional derivative of f in x in direction y. This connection is useful in the applications in order to calculate the subgradient.

For a proper convex function f with $\partial f(x) \neq \emptyset$ holds

$$\begin{aligned} x^* \in \partial f(x) &\Leftrightarrow \langle x^*, x\rangle = f^*(x^*) + f(x) && (2.5)\\ &\Leftrightarrow \langle x^*, x\rangle - f(x) = \sup_{y\in E}(\langle x^*, y\rangle - f(y)). && (2.6) \end{aligned}$$

Thus x is a minimizer of f if and only if

$$0 \in \partial f(x) \qquad \text{(Fermat's rule).} \tag{2.7}$$

If f is furthermore lower-semicontinuous, then we get by the Fenchel–Moreau-Theorem the equivalence:

$$0 \in \partial f(x) \Leftrightarrow x \in \partial f^*(0). \tag{2.8}$$

$\partial f^*(0)$ is the set of all minimizers of f.

For convex, lsc proper risk functionals ϱ_i it follows that ϱ_i^* are proper and

$$(\bigwedge \varrho_i)^* = \sum_{i=1}^n \varrho_i^* \tag{2.9}$$

(see Barbu and Precupanu (1986, Chapter 2)). In consequence we obtain

$$\bigcap_{i=1}^n \operatorname{dom} \varrho_i^* \neq \emptyset \Rightarrow \operatorname{dom}(\wedge \varrho_i) \neq \emptyset \tag{2.10}$$

(see [KR] (2008, Proposition 2.1)). For all results on convex duality we refer to Rockafellar (1974) and Barbu and Precupanu (1986).

In this paper we deal with the dual pair $(L_d^p, L_d^q, \langle \cdot, \cdot \rangle_d)$ where $\langle \cdot, \cdot \rangle_d$ denotes the canonical scalar product on the product spaces

$$\langle Z, X \rangle_d := \sum_{j=1}^d E Z^j X^j \tag{2.11}$$

for $X = (X^1, \dots, X^d) \in L_d^p$, $Y = (Y^1, \dots, Y^d) \in L_d^q$, where q is the conjugate index to p, $\frac{1}{p} + \frac{1}{q} = 1$. In the case $p = 1$, $q = \infty$ the dual space is the set ba_d^q of d-tuples of finitely additive normal P-continuous measures integrating $|x|^q$. To avoid cumbersome notation we still use L_d^q in this case. For law invariant risk functionals as used in the second part of the paper in fact we can reduce to the class of probability measures and thus to L_d^q.

The portfolio vectors $(\xi_i)_{1 \le i \le n}$ are contained in the corresponding productspaces defining the dual pair $((L_d^p)^n, (L_d^q)^n, \langle \cdot, \cdot \rangle_d^n)$ where for $X = (X_1, \dots, X_n) \in (L_d^p)$, $Z = (Z_1, \dots, Z_n) \in (L_d^q)^n$ the scalar product is given by

$$\langle Z, X \rangle_d^n := \sum_{i=1}^n \langle Z_i, X_i \rangle_d. \tag{2.12}$$

We will use the notation $\langle Z, X \rangle = \langle Z, X \rangle_d^n$ when omitting the indices does not lead to confusion.

3 Optimal allocations of portfolios

To characterize Pareto-optimal allocations we describe at first allocations which minimize the total risk, i.e. solutions of the infimal convolution

$$\bigwedge \varrho_i(X) = \inf \left\{ \sum_{i=1}^n \varrho_i(\xi_i) \mid (\xi_i) \in \mathcal{A}(X) \right\}. \tag{3.1}$$

The inf-convolution problem is a restricted optimization problem. It can be transformed into an unrestricted global minimization problem

$$\bigwedge \varrho_i(X) = \inf\left\{\overline{\varrho}(\xi) + 1_{\mathcal{A}(X)}(\xi) \mid \xi \in (L_d^p)^n\right\} \tag{3.2}$$

where $\overline{\varrho}(\xi) := \sum_{i=1}^n \varrho_i(\xi_i)$ and for a convex set A, $\mathbb{1}_A$ denotes the convex indicator

$$\mathbb{1}_A(x) = \begin{cases} 0, & x \in A, \\ \infty, & x \notin A. \end{cases} \tag{3.3}$$

We generally assume that there exists at least one n-allocation $\xi \in \mathcal{A}^n(X)$, where $\overline{\varrho}$ is continuous and finite. For $f : E \to \mathbb{R} \cup \{\infty\}$ the *domain of continuity* is denoted by

$$\operatorname{domc}(f) := \{x \in E \mid f \text{ is finite and continuous in } x\}.$$

The inf-convolution problem resp. the minimal total risk problem is called *well posed* for a given portfolio X if

$$\operatorname{domc}(\overline{\varrho}) \cap \mathcal{A}(X) \neq \emptyset. \tag{3.4}$$

The following is the basic characterization of minimal total risk allocations which extends the developments for real risks to the portfolio case. For the ample literature to this theorem see the references as mentioned in the introduction.

Theorem 3.1 (Characterization of minimal total risk) *Let ϱ_i be risk functionals on L_d^p, $1 \le p \le \infty$. Let $X \in L_d^p$ be a risk portfolio such that the minimal total risk problem is well posed and let $(\eta_i) \in \mathcal{A}^n(X)$ be a risk allocation. Then the following statements are equivalent:*

1) (η_i) has minimal total risk (w.r.t. $\varrho_1, \dots, \varrho_n$ and X)

2) $\exists V \in L_d^q : V \in \partial\varrho_i(\eta_i), \quad 1 \le i \le n$ (3.5)

3) $\exists V \in L_d^q : \eta_i \in \partial\varrho_i^*(V), \quad 1 \le i \le n$ (3.6)

Proof: The equivalence of 2) and 3) is a well known result in convex analysis (see e.g. Barbu and Precupanu (1986), Aubin (1993)). The proof of the equivalence of 1) and 2) needs in the multivariate case some additional arguments compared to the corresponding proof in the one-dimensional case as in Jouini et al. (2007), Acciaio (2007), and [KR] (2008).

Let $\overline{\xi} = (\xi_1, \dots, \xi_n) \in \mathcal{A}^n(X)$ be an allocation with total minimal risk, i.e.

$$\bigwedge \varrho_i(X) = \sum_{i=1}^n \varrho_i(\xi_i).$$

Due to Fermat's rule the representation in (3.2) implies

$$0 \in \partial(\overline{\varrho} + \mathbb{1}_{\mathcal{A}^n(X)})(\overline{\xi}). \tag{3.7}$$

The infimal convolution is well posed for X. In consequence the subdifferential sum formula (see Barbu and Precupanu (1986, Chapter 3, Theorem 2.6)) is applicable to the right-hand side of (3.7) and yields

$$0 \in \partial\overline{\varrho}(\overline{\xi}) + \partial\mathbb{1}_{\mathcal{A}^n(X)}(\overline{\xi}). \tag{3.8}$$

Thus there exists an element $\Lambda \in (L_d^p)^n$ with

$$\Lambda \in \partial\overline{\varrho}(\overline{\xi}) \quad \text{and} \quad -\Lambda \in \partial\mathbb{1}_{\mathcal{A}^n(X)}(\overline{\xi}). \tag{3.9}$$

This leads to the equation

$$\overline{\varrho}(\overline{\xi}) + \overline{\varrho}^*(\Lambda) = \langle \Lambda, \overline{\xi} \rangle. \tag{3.10}$$

In the next step we show that (3.10) implies the existence of some $V \in L_d^q$ such that

$$\varrho_i(\xi_i) + \varrho_i^*(V) = \langle V, \xi_i \rangle, \quad \forall i, \tag{3.11}$$

i.e. all components Λ_i of Λ are identical to $V \in L_d^q$. This results from the following proposition.

Proposition 3.2 *For all $X \in L_d^p$ and $\overline{\xi} \in \mathcal{A}^n(X)$ holds*

$$\partial\mathbb{1}_{\mathcal{A}^n(X)}(\overline{\xi}) = \Big\{ \overline{Z} \in (L_p^q)^n \mid \overline{Z} = \sum_{i=1}^n Z e_i, Z \in L_d^q \Big\}, \tag{3.12}$$

where e_i is the i-th unit vector of the n-fold product space $(L_d^q)^n$. Thus the product Ze_i is understood as the element of $(L_d^q)^n$ which has Z as its i-th component and $0 \in L_d^q$ otherwise.

Proof of Proposition 3.2: By definition of the subdifferential we have

$$\partial\mathbb{1}_{\mathcal{A}^n(X)}(\overline{\xi}) = \{ \overline{Z} \in (L_d^q)^n \mid \langle \overline{\eta}, \overline{Z} \rangle \le \langle \overline{\xi}, \overline{Z} \rangle, \quad \forall \overline{\eta} \in \mathcal{A}^n(X) \}$$

If $\overline{Z} = \sum_{i=1}^n Z e_i$, with $Z \in L_d^q$ and $\overline{\eta} \in \mathcal{A}^n(X)$, then

$$\begin{aligned} \langle \overline{\eta}, \overline{Z} \rangle &= \sum_{i=1}^n \langle \eta_i, Z \rangle \\ &= \Big\langle \sum_{i=1}^n \eta_i, Z \Big\rangle = \Big\langle \sum_{i=1}^n \xi_i, Z \Big\rangle = \langle \overline{\xi}, \overline{Z} \rangle. \end{aligned}$$

Thus $\overline{Z} \in \partial \mathbb{1}_{\mathcal{A}^n(X)}$.

Conversely for $\overline{Z} \in \partial \mathbb{1}_{\mathcal{A}^n(X)}(\xi)$ and $\overline{\eta} \in \mathcal{A}^n(X)$ holds

$$\langle \overline{\eta}, \overline{Z} \rangle \leq \langle \overline{\xi}, \overline{Z} \rangle. \tag{3.13}$$

Choosing $\overline{\eta}$ of the form $\overline{\eta} = \overline{\xi} + \eta e_k - \eta e_\ell$ with $k, \ell \in \{1, \dots, n\}$ and $\eta \in L_d^p$ we obtain from (3.13)

$$\langle \eta, \overline{Z}_k \rangle \leq \langle \eta, \overline{Z}_\ell \rangle.$$

Reverting the roles of k, ℓ we obtain the opposite inequality and consequently $Z_k = Z$, $1 \leq k \leq n$ for some $Z \in L_d^q$. Thus $\overline{Z} = \sum_{i=1}^n Z e_i$ is of the form as stated in (3.12). □

Continuation of the proof of Theorem 3.1: From the definition of the convex conjugate it follows that

$$\overline{\varrho}^*(\overline{Z}) = \sum_{i=1}^n \varrho_i^*(\overline{Z}_i), \quad \overline{Z} = (\overline{Z}_1, \dots, \overline{Z}_n) \in (L_d^q)^n.$$

From Proposition 3.2 we obtain for Λ as in (3.9) $\Lambda = \sum_{i=1}^n V e_i$ with $V \in L_d^q$. (3.10) then implies

$$\sum_{i=1}^n (\varrho_i(\xi_i) + \varrho_i^*(V)) = \sum_{i=1}^n \langle \xi_i, V \rangle. \tag{3.14}$$

Since by the Fenchel inequalities

$$\varrho_i(\xi_i) + \varrho_i^*(V) \geq \langle \xi_i, V \rangle, \quad \forall i \tag{3.15}$$

(3.14) implies equality in (3.15) and in consequence

$$V \in \partial \varrho_i(\xi_i), \quad \forall i.$$

Thus 1) implies 2). The above given proof can be reverted to yield also the opposite direction. □

To obtain a connection of minimizing the total risk to Pareto-optimality we introduce as in [KR] (2008) a condition called non-saturation property. We say that ϱ has the *non-saturation property* if

$$\text{(NS)} \qquad \inf_{X \in L_d^p} \varrho(X) \text{ is not attained} \tag{3.16}$$

The non-saturation property is a weak property of risk measures. It is implied in particular by the cash invariance property. Under the (NS) condition Pareto-optimality is related to the problem of minimizing the total weighted risk. This is described by the *weighted minimal convolution* $(\bigwedge \varrho_i)_\gamma(X)$ defined for weight vectors $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ by

$$(\bigwedge \varrho_i)_\gamma(X) := \inf \left\{ \sum_{i=1}^n \gamma_i \varrho_i(\xi_i) \mid (\xi_1, \dots, \xi_n) \in \mathcal{A}^n(X) \right\}. \tag{3.17}$$

The connection between Pareto-optimality and minimizing total weighted risk goes back in more special cases to the early papers in insurance (see e.g. Gerber (1979)).

Theorem 3.3 (Characterization of Pareto-optimal allocations) *Let ϱ_i, $1 \le i \le n$, be risk functionals on L_d^p satisfying the non-saturation conditions (NS). Then for $X \in L_d^p$ and $(\xi_1, \dots, \xi_n) \in \mathcal{A}^n(X)$ the following are equivalent*

1) $(\xi_1, \dots, \xi_n)$ is a Pareto-optimal allocation of X w.r.t. $\varrho_1, \dots, \varrho_n$

2) $\exists \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n_{>0}$ such that

$$(\bigwedge \gamma_i)_\gamma(X) = \sum_{i=1}^n \gamma_i \varrho_i(\xi_i) \tag{3.18}$$

3) $\exists \gamma = (\gamma_i) \in \mathbb{R}^n_{>0}$ and $\exists V \in L_d^q$ such that

$$V \in \gamma_i \partial \varrho_i(\xi_i), \quad \forall i \tag{3.19}$$

4) $\exists \gamma = (\gamma_i) \in \mathbb{R}^n_{>0}$ and $\exists V \in L_d^q$ such that

$$\xi_i \in \partial(\gamma_i \varrho_i)^*(V), \quad \forall i \tag{3.20}$$

Proof: The proof of Theorem 3.3 follows by a similar line of arguments as in [KR] (2008) in the one-dimensional case. □

The intersection condition (3.19) can also be described by saying that

$$V \in \partial(\bigwedge \varrho_i)_\gamma(X). \tag{3.21}$$

This is a consequence of the following proposition.

Proposition 3.4 *If $(\xi_i) \in \mathcal{A}^n(X)$ minimizes the total risk w.r.t. $\varrho_1, \dots, \varrho_n$, then*

$$\partial(\bigwedge \varrho_i)(X) = \bigcap_{i=1}^n \partial \varrho_i(\xi_i). \tag{3.22}$$

Proof: (3.22) is a consequence of the definition of subgradients of ϱ_i and $\bigwedge \varrho_i$ using the Fenchel inequality. The details are as in the one-dimensional case (see [KR] (2008, Prop. 3.2)). □

4 Law invariant risk measures and comonotonicity

For the specialization to law invariant risk measures ϱ_i on L_d^p optimal allocations take a more specific form and are connected with multivariate comonotonicity. By the classical comonotone improvement theorem of Landsberger and Meilijson (1994), see also Dana and Meilijson (2003) and Ludkovski and Rüschendorf (2008) for some extensions, any allocation $(\xi_i) \in \mathcal{A}^n(X)$ in $d = 1$ can be improved by a comonotone allocation uniformly w.r.t. all convex law invariant risk measures ϱ_i. A uniform improvement result can not be expected in the multivariate case. There is no notion of comonotonicity in $d \geq 1$ which is applicable to all law invariant risk measures. There is however a senseful notion of *μ-comonotonicity* introduced in Ekeland et al. (2009) which allows to construct an μ-comonotone improvement of an allocation with however a more restricted range of improvement. See the recent paper of Carlier et al. (2009). In this section we establish that Pareto-optimal allocations w.r.t. law invariant risk measures are μ-comonotone for certain scenario measures μ.

For $1 \leq p \leq \infty$ we consider finite, law invariant convex risk measures ϱ on L_d^p. In fact we take the insurance version $\Psi(X) = \varrho(-X)$ which has a simpler monotonicity property, $\Psi : L_d^p \to \mathbb{R}^1$. Let $Y \in D_d^q$, where

$$D_d^q = \{(Y_1, \dots, Y_d) : Y_i \geq 0, Y_i \in L^q, E_P Y_i = 1, 1 \leq i \leq d\} \subset L_d^q \tag{4.1}$$

is the set of d-tuples of P densities and let $\mu = P^Y$ denote the distribution of Y. The maximal correlation risk measure in direction Y resp. μ is defined by

$$\begin{aligned} \widehat{\Psi}_Y(X) &= \sup_{\widetilde{X} \sim X} E\widetilde{X} \cdot Y \\ &= \sup_{\widetilde{Y} \sim \mu} EX \cdot \widetilde{Y} = \Psi_\mu(X) \end{aligned} \tag{4.2}$$

(see [R] (2006)). $\Psi_\mu = \widehat{\Psi}_Y$ is a law invariant coherent risk measure on L_d^p. For the representation in (4.2) μ can be identified with the d-tuple of its marginals $(\mu_1, \dots, \mu_d) \in M_d^1$. With this identification any finite law invariant, convex risk measure Ψ on L_d^p has a robust representation of the form

$$\Psi(X) = \max_{\mu \in Q} (\Psi_\mu(X) - \alpha(\mu)), \tag{4.3}$$

where Q is a weakly closed subset of $Q_{d,p} = \{Q \in M_d^1 \mid \frac{dQ_i}{dP} \in L^q(P)\}$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha(\cdot)$ is some law invariant penalty function (see [R] (2006)). We choose in the following α as the minimal penalty function corresponding to the Fenchel conjugate ϱ^* of ϱ. Equivalently we can write (4.3) in the form

$$\Psi(X) = \max_{\mu \in A} (\Psi_\mu(X) - \alpha(\mu)), \tag{4.4}$$

where $A \subset \{\mu \in M^q(\mathbb{R}^d_+, \mathcal{B}^d_d) \mid \exists Y \in D^q_d \text{ such that } Y \sim \mu\}$ is a weakly closed subset of distributions of density vectors with q-integrable components.

For $X \in L^p_d$ and $Y \in L^q_d$ define that X, Y are *optimally coupled*, $X \sim_{\mathrm{oc}} Y$ if

$$\widehat{\Psi}_Y(X) = \sup_{\widetilde{Y} \sim Y} EX \cdot \widetilde{Y} = EX \cdot Y, \tag{4.5}$$

where $\widetilde{Y} \sim Y$ means equality in distribution. For any $Y \in L^q_d$, $Y \sim \mu$ holds a symmetry relation

$$\Psi_\mu(X) = \sup_{\widetilde{X} \sim X} E\widetilde{X} \cdot Y. \tag{4.6}$$

We next give a basic characterization of subgradients of convex law invariant risk measures Ψ on L^p_d with representation as in (4.3) with $\alpha = \Psi^*$. Define the risk functional

$$F(\mu) := F_X(\mu) := \Psi_\mu(X) - \Psi^*(\mu). \tag{4.7}$$

$\mu_0 \in A$ is called a *worst case scenario measure* of Ψ for X if

$$F_X(\mu_0) = \max_{\mu \in A} F_X(\mu) = \Psi(X). \tag{4.8}$$

Theorem 4.1 *For a finite, convex, law invariant risk measure Ψ on L^q_d and for $X \in L^p_d$, $Y_0 \in L^q_d$ with $\mu_0 = P^{Y_0} \in A$ the following statements are equivalent:*

1) $Y_0 \in \partial\Psi(X)$

2) a) μ_0 is a worst case scenario of Ψ for X

b) $X \sim_{\mathrm{oc}} Y_0$

Proof: 1) $\Rightarrow$ 2) For $Y_0 \in \partial\Psi(X)$ we have for all $Z \in L^p_d$:

$$\Psi(X) - \Psi(Z) \le EY_0 \cdot (X - Z). \tag{4.9}$$

Thus we obtain from law invariance of Ψ

$$0 = \Psi(X) - \Psi(\widetilde{X}) \le EY_0 \cdot (X - \widetilde{X}), \quad \forall \widetilde{X} \sim X$$

or equivalently

$$EY_0 \cdot \widetilde{X} \le EY_0 \cdot X, \quad \forall \widetilde{X} \sim X.$$

This however is equivalent to

$$\Psi_{\mu_0}(X) = \sup_{\widetilde{X} \sim X} EY_0 \cdot \widetilde{X} \sim X = EY_0 \cdot X$$

and thus $X \sim_{\mathrm{oc}} Y_0$, i.e. condition a) holds.

For the proof of b) note that $Y_0 \in \partial\Psi(X)$ implies also

$$\Psi(X) = EX \cdot Y_0 - \Psi^*(Y_0). \tag{4.10}$$

In consequence we obtain

$$\begin{aligned} F_X(\mu_0) &= \sup_{\widetilde{Y} \sim Y_0} E\widetilde{Y} \cdot X - \Psi^*(Y_0) \\ &= \max_{\mu \in A} \sup_{\widetilde{Y} \sim Y} (E\widetilde{Y} \cdot X - \Psi^*(\mu)) = \max_{\mu \in A} F_X(\mu), \end{aligned}$$

i.e. μ_0 is a worst case scenario measure.

2) $\Rightarrow$ 1) Let now $Y_0 \in L_d^q$ with $\mu_0 = P^{Y_0} \in A$ fulfill that $Y_0 \sim_{\mathrm{oc}} X$ and that μ_0 is a worst case scenario measure. Then we obtain

$$\begin{aligned} \Psi(X) &= \max_{\mu \in A} (\Psi_\mu(X) - \Psi^*(\mu)) \\ &= \max_{\mu \in A} F_X(\mu) = F_X(\mu_0) \\ &= \Psi_{\mu_0}(X) - \Psi^*(\mu_0) \\ &= EY_0 \cdot X - \Psi^*(Y_0) \end{aligned}$$

using that Ψ^* is also law invariant. Thus $Y_0 \in \partial\Psi(X)$ is a subgradient of Ψ in X. $\square$

Theorem 4.1 combined with the subgradient intersection condition in the characterization theorem of minimal risk allocations in Theorem 3.1 implies that optimal risk allocations have a specific μ-comonotonicity property where μ satisfies the intersection condition. There is also a close connection to the notion of worst case portfolios which concerns a worst case dependence structure for fixed marginal distributions. For a risk measure Ψ on L_d^p a portfolio $X = (X_1, \dots, X_n)$ is called a *worst case portfolio* with respect to Ψ if

$$\Psi\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \sup_{\widetilde{X}_i \sim X_i} \Psi\left(\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i\right) \tag{4.11}$$

(see [R] (2009)). Combining Theorems 3.1, 4.1 with the characterization of worst case portfolios in [R] (2009) we obtain the following result connecting the notion of optimal allocations with μ-comonotonicity and with the notion of worst case portfolios.

Theorem 4.2 (Optimal allocations for law invariant risk measures) *Let $\Psi_1, \dots, \Psi_n$ be a finite, lsc convex law invariant risk measures on L_d^p with scenario sets A_i. Let $X \in L_d^p$ be a risk vector such that the minimal risk allocation problem for X is well posed. Let $F_i(\mu) = F_{i,\xi_i}(\mu) = \Psi_\mu(\xi_i) - \Psi_i^*(\mu)$ denote the risk functional of ξ_i w.r.t. Ψ_i.*

For an allocation $(\xi_i) \in \mathcal{A}^n(X)$ the following statements are equivalent:

1) (ξ_i) *has minimal the total risk (w.r.t.* $\Psi_1, \dots, \Psi_n$ *and* X*)*

2)
$$\exists V \in L_d^q : V \in \partial \Psi_i(\xi_i), \quad 1 \le i \le n \tag{4.12}$$

3) a) $\exists$ *joint worst case scenario measure* $\mu_0 \in \mathcal{M}_d^q$ *for all* ξ_i *w.r.t.* Ψ_i*, i.e.*

$$F_i(\mu_0) = \sup_{\mu \in A_i} F_i(\mu) = \Psi_i(\xi_i), \quad 1 \le i \le n \tag{4.13}$$

b) $\xi_1, \dots, \xi_n$ *are* μ_0 *comonotone.*

4) a) $\exists$ *joint worst case scenario measure* μ_0 *for all* ξ_i *w.r.t.* Ψ_i*.*

b) $\xi_1, \dots, \xi_n$ *is a worst case dependence structure for the max correlation risk measure* Ψ_{μ_0}*.* (4.14)

Remark 4.3 *From the characterization of risk minimizing allocations in* (4.13) *we obtain for an optimal allocation* $(\xi_i) \in \mathcal{A}(X)$

$$\begin{aligned}\sum_{i=1}^n \Psi_i(\xi_i) &= \sum_{i=1}^n F_i(\mu_0) = \sum_{i=1}^n (\Psi_{\mu_0}(\xi_i) - \Psi_i^*(\mu_0)) \\ &= \Psi_{\mu_0}\left(\sum_{i=1}^n \xi_i\right) - \sum_{i=1}^n \Psi_i^*(\mu_0) \\ &= \Psi_{\mu_0}(X) - \sum_{i=1}^n \Psi_i^*(\mu_0).\end{aligned} \tag{4.15}$$

In case all Ψ_i *are coherent risk measures* (4.15) *implies that*

$$\wedge \Psi_i(X) = \sum_{i=1}^n \Psi_i(\xi_i) = \Psi_{\mu_0}(X). \tag{4.16}$$

5 Existence of minimal risk allocations

As main result in this section we derive a characterization of the existence of risk minimizing allocations as well as give several sufficient conditions. In the one-dimensional case existence results for optimal allocations have been based on the monotone improvement theorem (see Jouini et al. (2007) and Acciaio (2007)) which allows to restrict to allocations $\xi_i = f_i(X)$ with some monotone functions f_i, which allows to apply Dini's theorem. Alternatively a strong intersection condition (SIS) from convex analysis (see Barbu and Precupanu (1986)) has been used in [KR] (2008) and Filipović and Svindland (2007).

In this section we shall make use of the *subdifferential sum formula* for functions f, g:

$$(\mathrm{SD}(x)) \qquad \partial(f+g)(x) = \partial f(x) + \partial g(x) \tag{5.1}$$

which is used in convex analysis for dealing with existence of the convolution (see Barbu and Precupanu (1986)). There is a close link with the following *epigraph condition* for the conjugates f^*, g^*:

$$(\mathrm{EC}) \qquad \mathrm{epi}(f+g)^* = \mathrm{epi}(f^*) + \mathrm{epi}(g^*). \tag{5.2}$$

The following theorem of Burachik and Jeyakumar (2005, Theorem 3.1) extends previous results and states that the EC-condition implies the subdifferential sum formula.

Theorem 5.1 *Let f, $g : X \to (-\infty, \infty]$ be proper, lsc convex functions on a Banach space X such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. If f, g fulfill the epigraph condition (EC), then they satisfy the subdifferential sum formula SD(x) for all $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$.*

Subdifferentiability of $f \wedge g$ at a point x and the subdifferential sum formula SD(x^*) for the conjugates f^*, g^* implies existence of a minimizer of $f \wedge g$ at x. For f, g as in Theorem 5.1 the following theorem is essentially a reformulation of Theorem 2.3 in [KR] (2008).

Theorem 5.2 (Local existence) *Assume that $f \wedge g$ is subdifferentiable at x and assume that the subdifferential sum formula w.r.t. f^* and g^* holds for some $x^* \in \partial(f \wedge g)(x)$*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*). \tag{5.3}$$

Then there exists an allocation $(\xi_1, \xi_2) \in \mathcal{A}^2(x)$ which minimizes the total risk,

$$f \wedge g(x) = f(\xi_1) + g(\xi_2). \tag{5.4}$$

Proof: The proof follows as in [KR] (2008, Theorem 2.3). In that paper the strong intersection property (SIS) was postulated and used to imply the subdifferentiability of $f \wedge g$. $\square$

Next we establish that the conditions in Theorem 5.2 are in fact equivalent to the existence of a minimizer. The infimal convolution $f \wedge g$ is called *exact in x* if the inf is attained at x as in (5.4); it is called *exact* if this holds for all $x \in X$. Let f, g be functions as in Theorem 5.1.

Proposition 5.3 *The following statements are equivalent*

1) $f \wedge g$ is exact

2) $f \wedge g$ *is subdifferentiable at all* $x \in X$ *and* $\forall x^* \in \partial(f \wedge g)(x)$ *the subdifferential sum formula holds for* f^*, g^*, *i.e.*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*). \tag{5.5}$$

Proof: The direction 2) $\Rightarrow$ 1) follows from Theorem 5.1.

For the converse direction there exists to $x \in X$ an allocation $(\xi_1, \xi_2) \in \mathcal{A}^2(x)$ with minimal total risk. Then by the characterization result in Theorem 3.1 and using Proposition 3.4 there exists an element

$$x^* \in \partial f(\xi_1) \cap \partial g(\xi_2) = \partial(f \wedge g)(x). \tag{5.6}$$

Thus $f \wedge g$ is subdifferentiable in x.

To establish the subdifferential sum formula (5.5) let $x^* \in \partial(f \wedge g)(x)$. Then we obtain

$$x \in \partial(f \wedge g)^*(x^*) = \partial(f^* + g^*)(x^*) \tag{5.7}$$

(see (Barbu and Precupanu, 1986, Corollary 1.4, Chapter 2)). On the other hand

$$x^* \in \partial f(\xi_1) \cap \partial f(\xi_2) \tag{5.8}$$

for any solution (ξ_1, ξ_2) of $(f \wedge g)(x)$ by Theorem 3.1. This implies

$$\xi_1 \in \partial f^*(x^*) \text{ and } \xi_2 \in \partial g^*(x^*).$$

Thus we obtain

$$x = \xi_1 + \xi_2 \in \partial f^*(x^*) + \partial g^*(x^*). \tag{5.9}$$

(5.9) implies the inclusion

$$\partial(f^* + g^*)(x^*) \subset \partial f^*(x^*) + \partial g^*(x^*).$$

Therefore, equality holds since the opposite inclusion holds generally true. □

By Theorem 5.1 and Proposition 5.3 the epigraph condition (EC) holding true for f^*, g^* together with subdifferentiability of $f \wedge g$ implies existence of optimal allocations. In the following we improve this statement and establish that the subdifferentiability condition can be skipped. Our proof is based essentially on the following proposition which is a restatement of Proposition 2.2 of Boţ and Wanka (2006) for the conjugates f^*, g^* of f, g.

Proposition 5.4 *Assume that* $\operatorname{dom}(f^*) \cap \operatorname{dom}(g^*) \neq \emptyset$. *Then the following statements are equivalent:*

1) The epigraph condition (EC) holds for f^*, g^*, *i.e.*

$$\operatorname{epi}(f^* + g^*)^* = \operatorname{epi}(f) + \operatorname{epi}(g). \tag{5.10}$$

2)
$$(f^* + g^*)^* = f \wedge g \text{ and } f \wedge g \text{ is exact.} \tag{5.11}$$

Based on the equivalence in Proposition (5.3) we next see that the first condition in 2) of Proposition 5.4 of Boţ and Wanka (2006) can be omitted.

Proposition 5.5 *If the equivalent conditions of Proposition 5.3 hold true, then*

$$(f^* + g^*)^* = f \wedge g. \tag{5.12}$$

Proof: For $x \in X$ the subdifferentiability of $f \wedge g$ and the subdifferential sum formula (5.5) imply the existence of some $x^* \in \partial(f \wedge g)(x)$ such that

$$x \in \partial(f \wedge g)^*(x^*) = \partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*). \tag{5.13}$$

The exactness of $f \wedge g$ and the characterization of minimal allocations in Theorem 3.1 imply the existence of $(\xi_1, \xi_2) \in \mathcal{A}^2(x)$ such that

$$\xi_1 \in \partial f^*(x^*) \text{ and } \xi_2 \in \partial g^*(x^*). \tag{5.14}$$

In consequence we obtain from (5.13)

$$\langle x^*, x \rangle = (f^* + g^*)(x^*) + (f^* + g^*)^*(x). \tag{5.15}$$

From (5.14) we conclude

$$\langle x^*, \xi_1 \rangle = f^*(x^*) + f(\xi_1), \tag{5.16}$$

as well as

$$\langle x^*, \xi_2 \rangle = g^*(x^*) + g(\xi_2). \tag{5.17}$$

Summing up (5.16) and (5.17) and comparing this to (5.14) we conclude

$$(f^* + g^*)^*(x) = f(\xi_1) + g(\xi_2) = (f \wedge g)(x), \tag{5.18}$$

and thus (5.12) holds true. $\square$

As consequence of Theorem 5.1, Propositions 5.3–5.4 we now obtain equivalence of exactness of $f \wedge g$ to the epigraph condition for f^*, g^*. This is our main existence result for optimal allocations and improves in particular Theorem 5.2.

Theorem 5.6 (Existence of optimal allocations) *Let f, g be proper lsc convex functions from a Banach space X to $(-\infty, \infty]$ such that* $\operatorname{dom}(f^*) \cap \operatorname{dom}(g^*) \neq \emptyset$.

Then the following statements are equivalent:

1) $f \wedge g$ *is exact.*

2) The epigraph condition (EC) holds for f^*, g^*, *i.e.*

$$\operatorname{epi}((f^* + g^*)^*) = \operatorname{epi} f + \operatorname{epi} g. \tag{5.19}$$

3) $f \wedge g$ *is subdifferentiable at all* $x \in X$ *and for all* $x^* \in \partial(f \wedge g)(x)$ *holds the subdifferential sum formula*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

For sublinear functions it is even possible to omit the subdifferentiability property of $f \wedge g$ in 3). Since Corollary 3.1 of Burachik and Jeyakumar (2005) provides following equivalence.

Corollary 5.7 *Let* f, g *be funcitons like in Theorem 5.6 with the additional condition of positive homegenity. Then the following conditions are equivalent:*

1) The epigraph condition (EC) holds for f^*, g^*, *i.e.*

$$\operatorname{epi}((f^* + g^*)^*) = \operatorname{epi} f + \operatorname{epi} g.$$

2) For all $x^* \in \operatorname{dom} f^* \cap \operatorname{dom} g^*$ *holds the subdifferential sum formula*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

Thus Theorem 5.6 becomes

Proposition 5.8 *Let* f, g *be funcitons like in Theorem 5.6 with the additional condition of positive homegenity. Then the following conditions are equivalent:*

1) $f \wedge g$ *is exact.*

2) The epigraph condition (EC) holds for f^*, g^*, *i.e.*

$$\operatorname{epi}((f^* + g^*)^*) = \operatorname{epi} f + \operatorname{epi} g.$$

3) For all $x^* \in \operatorname{dom} f^* \wedge \operatorname{dom} g^*$ *holds the subdifferential formula*

$$\partial(f^* + g^*)(x^*) = \partial f^*(x^*) + \partial g^*(x^*).$$

In general the epigraph condition (EC) for f^*, g^* in (5.19) is not easy to check. In the following proposition we restate some sufficient conditions for the epigraph condition (EC) in (5.19) which by Theorem 5.6 implies the existence of optimal allocations. All these sufficient conditions can be found in Boţ and Wanka (2006). The strong intersection condition (SIS) was also used in [KR] (2008).

We need some notation. For a subset $D \subset X$ denote by

$$\operatorname{core}(D) := \{d \in D \mid \forall x \in X, \exists \varepsilon > 0, \forall \lambda \in (-\varepsilon, \varepsilon), d + \lambda x \in D\} \tag{5.20}$$

the *core* of D. Further let $\operatorname{icr}(D)$ denote the *intrinsic core* of D relative to the *affine hull* $\operatorname{aff}(D)$ of D. Further for D convex define the *strong quasi-relative interior* of D by

$$\operatorname{sqri}(D) := \{x \in D \mid \operatorname{cone}(D - x) \text{ is a closed subspace}\}. \tag{5.21}$$

Proposition 5.9 (Interior point conditions) *Any of the following interior point conditions implies the epigraph condition* (5.19).

$$\operatorname{dom} f^* \cap \operatorname{int} \operatorname{dom} g^* \neq \emptyset \tag{5.22}$$

$$0 \in \operatorname{core}(\operatorname{dom} g^* - \operatorname{dom} f^*) \tag{5.23}$$

$$0 \in \operatorname{sqri}(\operatorname{dom} g^* - \operatorname{dom} f^*) \tag{5.24}$$

$$\begin{aligned} &0 \in \operatorname{icr}(\operatorname{dom} g^* - \operatorname{dom} f^*) \ \textit{and} \\ &\operatorname{aff}(\operatorname{dom} g^* - \operatorname{dom} f^*) \ \textit{is a closed subspace.} \end{aligned} \tag{5.25}$$

The statements in Proposition 5.9 are given in Boţ and Wanka (2006) where also further relations between these conditions are discussed. For more sufficient conditions on (5.19) we refer to the references therein.

In the following we extend the results of this section to more than two functions. It is clear how the infimal convolution, the subdifferential sum formula and the epigraph condition are formulated for n functions. All preceding statements can be carried to this setup straight forward except that in Proposition 5.9. Each interior point condition has to be stated as a system of $n-1$ conditions, to imply the epigraph condition

$$\operatorname{epi}\left(\left(\sum_{i=1}^n g_i^*\right)^*\right) = \sum_{i=1}^n \operatorname{epi} g_i. \tag{5.26}$$

Proposition 5.10 *For lower semicontinuous functions $g_1, \dots, g_n$ any of the following interior point conditions implies the epigraph condition* (5.26).

$$\text{(SIS)} \qquad \bigcap_{i=1}^{n-1} \operatorname{int} \operatorname{dom} g_i^* \cap \operatorname{dom} g_k^* \neq \emptyset, \quad k \in \{2, \dots, n\} \tag{5.27}$$

$$0 \in \operatorname{core}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right), \quad k \in \{2, \dots, n\} \tag{5.28}$$

$$0 \in \operatorname{sqri}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right), \quad k \in \{2, \dots, n\} \tag{5.29}$$

$$\begin{aligned} &0 \in \operatorname{icr}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right) \textit{ and} \\ &\operatorname{aff}\left(\bigcap_{i=1}^{k-1} \operatorname{dom} g_i^* - \operatorname{dom} g_k^*\right) \textit{ is a closed subspace}, \quad k \in \{2, \dots, n\}. \end{aligned} \tag{5.30}$$

Proof: At first we observe that the strong intersection condition

$$\text{(SIS)} \qquad \bigcap_{i=1}^{n-1} \operatorname{int} \operatorname{dom} g_i^* \cap \operatorname{dom} g_n^* \neq \emptyset$$

is equivalent to the following system of interior point conditions

$$\bigcap_{i=1}^{k-1} \operatorname{int} \operatorname{dom} g_i^* \cap \operatorname{dom} g_k^* \neq \emptyset, \quad k = 2, \dots, n. \tag{5.31}$$

The proof of Proposition 5.10 is based on the fact that the infimal convolution of functions $g_1, \dots, g_n$ can be solved iteratively, i.e.

$$\bigwedge_{i=1}^{n} g_i = \left(\bigwedge_{i=1}^{n-1} g_i\right) \wedge g_n. \tag{5.32}$$

Assume that any of the interior point conditions of Proposition 5.9 holds for the functions $f_{n-1}^* := \left(\bigwedge_{i=1}^{n-1} g_i\right)^* = \sum_{i=1}^{n-1} g_i^*$ and g_n^*. This means with the equivalence of (SIS) and (5.31) that in Proposition 5.10 one of the conditions hold for $k = n$, where $\operatorname{dom}\left(\sum_{i=1}^{n-1} g_i^*\right) = \bigcap_{i=1}^{n-1} \operatorname{dom} g_i^*$ resp. $\operatorname{int}\operatorname{dom}\left(\sum_{i=1}^{n-1} g_i^*\right) = \bigcap_{i=1}^{n-1} \operatorname{int}\operatorname{dom} g_i$. Then we get as consequence of Proposition 5.9

$$\operatorname{epi}((f_{n-1}^* + g_n^*)^*) = \operatorname{epi}(g_n) + \operatorname{epi}\left(\left(\sum_{i=1}^{n-1} g_i^*\right)^*\right). \tag{5.33}$$

Note that f_{n-1} is not necessarily lower semicontinuous. If we assume that any of the interior point conditions of Proposition 5.9 holds additionally for the functions $f_{n-2}^* := \left(\bigwedge_{i=1}^{n-2} g_i\right)^* = \sum_{i=1}^{n-2} g_i^*$ and g_{n-1}^* (which corresponds to $k = n - 1$ in Proposition 5.10) we get again from Proposition 5.9

$$\operatorname{epi}((f_{n-2}^* + g_{n-1}^* + g_n^*)^*) = \operatorname{epi}(g_n) + \operatorname{epi}(g_{n-1}) + \operatorname{epi}\left(\left(\sum_{i=1}^{n-2} g_i^*\right)^*\right). \tag{5.34}$$

Proceeding further the same way and using the facts that $\operatorname{dom}\left(\sum_{i=1}^{k-1} g_i^*\right) = \bigcap_{i=1}^{k-1} \operatorname{dom} g_i^*$ resp. $\operatorname{int}\operatorname{dom}\left(\sum_{i=1}^{n-1} g_i^*\right) = \bigcap_{i=1}^{k-1} \operatorname{int}\operatorname{dom} g_i^*$ holds for any $k \in \{2, \dots, n\}$ we see that any system of conditions of Proposition 5.10 implies (5.26). □

Obviously it is sufficient for the application of Proposition 5.10 that for $k \in \{2, \dots, n\}$ any of the interior point conditions hold.

6 Uniqueness

Uniqueness of optimal allocations is a consequence of strict convexity.

Proposition 6.1 *Let Ψ_i, $i \in \{1, \dots, n-1\}$ be strictly convex risk functionals on L_d^p in the following sense*

$$\Psi_i(\lambda X + (1 + \lambda)Y) < \lambda \Psi_i(X) + (1 - \lambda)\Psi_i(Y) \text{ for all } \lambda \in (0, 1)$$

for all X, $Y \in \operatorname{dom} \Psi_i$. Then any optimal allocation of $X \in L_d^p$ with $\wedge \Psi_i(X) < \infty$ is unique.

Proof: We suppose that for $X \in L_d^p$ with $\bigwedge_{i=1}^n \Psi_i(X) < \infty$ there exist two minimizer $(X_1, \dots, X_n) \in (L_d^p)^n$ and $(Y_1, \dots, Y_n) \in (L_d^p)^n$ of the total risk. Then the allocation $Z_i := \lambda X_i + (1 - \lambda)Y_i$ with $\lambda \in (0, 1)$ defines an allocation of X with

$$\sum \Psi_i(Z_i) < \lambda \sum \Psi_i(X_i) + (1 - \lambda) \sum \Psi_i(Y_i) = \bigwedge_{i=1}^{n} \Psi_i(X).$$

This contradicts the optimality of $(X_1, \dots, X_n)$. □

Remark 6.2 *It is obvious that in Proposition 6.1 it is necessary to postulate the strict convexity for at least $n-1$ risk functionals. If there were less than $n-1$ strict convex risk functionals then as consequence one could not exclude the existence of a rearrangement $(\overline{X}_1, \dots, \overline{X}_n) \in (L_d^p)^n$ of an optimal allocation $(X_1, \dots, X_n) \in (L_d^p)^n$ which is optimal, too.*

If for example Ψ_{n-1} and Ψ_n are constant on L_d^p, then any rearrangement $(\overline{X}_1, \dots, \overline{X}_n) \in (L_d^p)^n$ of $(X_1, \dots, X_n) \in (L_d^p)^n$ defined by $\overline{X}_i = X_i$, $i \in \{1, \dots, n-2\}$ $\overline{X}_{n-1} = X_{n-1} + Y$, $\overline{X}_n = Y_n - Y$ for any $Y \in L_d^p$ is optimal, too.

Some further uniqueness results can be found in [KR] (2008). There it is shown, e.g., that strict convexity of the risk functional Ψ_i implies the uniqueness of the risk contribution $X_i \in L_d^p$ of an optimal allocation $(X_1, \dots, X_n) \in (L_d^p)^n$. Further a uniqueness result is proved for weighted versions of the allocation problem which implies uniqueness of Pareto-optimal allocations for cash invariant risk measures, which are strictly convex on L_0^∞, the class of risks $X \in L^\infty$ with $EX = 0$.

These results can be easily adapted to the multivariate case as considered in this paper.

7 Examples and Remarks

7.1 Dilated risk functionals

Let ϱ be a convex risk functional on L_d^p. The class of dilated risk functionals ϱ_λ is defined for $\lambda > 0$ by

$$\varrho_\lambda(X) = \lambda\varrho\Big(\frac{1}{\lambda}X\Big), \tag{7.1}$$

where multiplication is componentwise. Then the following rules hold true:

$$\begin{aligned} \varrho_\lambda^* = \lambda\varrho^*, \quad \partial\varrho_\lambda(X) &= \partial\varrho\Big(\frac{1}{\lambda}X\Big) \\ \text{and} \quad \partial\varrho_\lambda^*(V) &= \lambda\partial\varrho^*(V). \end{aligned} \tag{7.2}$$

As consequence we obtain from the characterization of minimal risk allocations in Theorem 3.1 a simple optimal allocation rule.

Proposition 7.1 (Dilated risk functional) *Let ϱ be a convex risk functional on L_d^p, $1 \le p \le \infty$. Let $X \in L_d^p$ be a risk portfolio such that the total risk problem is well-posed for the dilated risk measures $\varrho_i = \varrho_{\lambda_i}$, $1 \le i \le n$, $\lambda_i > 0$ and let $\Lambda := \sum_{i=1}^n \lambda_i$ and assume that $\frac{1}{\Lambda}X \in \operatorname{int}(\operatorname{dom}\varrho)$. Then the proportional allocation $\xi_i := \frac{\lambda_i}{\Lambda}X$, $1 \le i \le n$, defines a total risk minimizing allocation of X.*

Proof: For the proof we check the intersection condition (3.5) of Theorem 3.1. This holds true by definition of the proportional allocation:

$$\begin{aligned} \bigcap_{i=1}^n \partial\varrho_i(\xi_i) &= \bigcap_{i=1}^n \partial\varrho\Big(\frac{1}{\lambda_i}\xi_i\Big) = \bigcap_{i=1}^n \partial\varrho\Big(\frac{1}{\Lambda}X\Big) \\ &= \partial\varrho\Big(\frac{1}{\Lambda}X\Big) \neq \emptyset. \end{aligned} \tag{7.3}$$

□

In the particular case when $\varrho_i = \varrho$, $1 \le i \le n$, the allocation $\xi_i = \frac{1}{n}X$, $1 \le i \le n$, is risk minimizing.

7.2 Multivariate expected risk function

Let $r : \mathbb{R}^d \to \mathbb{R}$ be strictly convex, continuously differentiable and satisfy the growth condition

$$|r(x)| \leq C(1 + \|x\|^p) \text{ for some } C \in \mathbb{R}, \quad p > 1. \tag{7.4}$$

Then r induces the corresponding risk functional

$$\varrho_r : L_d^p \to \mathbb{R}, \quad \varrho_r(X) := Er(X), \quad X \in L_d^p. \tag{7.5}$$

By the growth condition ϱ_r is a finite risk functional on L_d^p. To determine the subdifferential of ϱ_r on L_d^p we next prove that $\nabla r(X) \in L_d^q$.

Lemma 7.2 *Let $\frac{1}{p} + \frac{1}{q} = 1$ and let r satisfy the growth condition* (7.4)*. Then for $X \in L_d^p$ holds*

$$\nabla r(X) \in L_d^q. \tag{7.6}$$

Proof: Define the linear operator $F : L_d^p \to \overline{\mathbb{R}}$ by

$$F(Y) := E\langle \nabla r(X), Y\rangle \tag{7.7}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product. We establish that F is well defined and norm bounded. Note that convexity of r implies

$$\begin{aligned}
\|F\| &= \sup_{\|Y\|_{L_d^p} \leq 1} E\langle \nabla r(X), Y\rangle \\
&= \sup_{\|Y\|_{L_d^p} \leq 1} (E(r(X+Y) - r(X)) \\
&\leq C_1 + \sup_{\|Y\|_{L_d^p} \leq 1} Er(X+Y) \\
&\leq C_1 + \sup_{\|Y\|_{L_d^p} \leq 1} E\left(\frac{1}{2} r(2X) + \frac{1}{2} r(2Y)\right) \\
&\leq C_2 + \frac{1}{2} \sup_{\|Y\|_{L_d^p} \leq 1} Er(2Y) \\
&\leq C_2 + \frac{1}{2} CE(1 + 2\|Y\|^p) \leq C_3.
\end{aligned}$$

By the Riesz representation theorem there exists a unique $Z \in L_d^q$ such that

$$\int \langle \nabla r(X), Y\rangle dP = F(Y) = \int \langle Z, Y\rangle dP \quad \forall Y \in L_d^p. \tag{7.8}$$

In consequence we obtain $\nabla r(X) = Z \in L_d^q$. □

Since $\alpha \to \frac{1}{\alpha}(r(X+\alpha Y)-r(X))$ is monotone increasing for $X,\, Y \in L_d^p$, we obtain by the monotone convergence theorem for the directional derivative of ϱ_r in X in direction Y

$$\begin{aligned} D_{\varrho_r}(X,Y) &= \lim_{\alpha\downarrow 0} \frac{\varrho_r(X+\alpha Y)-\varrho_r(X)}{\alpha} \\ &= \lim_{\alpha\downarrow 0} E\frac{r(X+\alpha Y)-r(X)}{\alpha} \\ &= E\nabla r(X)\cdot Y. \end{aligned} \tag{7.9}$$

Thus ϱ_r is Gateaux differentiable with subgradient

$$\partial\varrho_r(X) = \{\nabla r(X)\} \in L_d^q. \tag{7.10}$$

For a family $r_1,\dots,r_n$ of convex functions as above with corresponding expected risk functionals $\varrho_{r_1},\dots,\varrho_{r_n}$ on L_d^p we consider the optimal risk allocation problem for $X \in L_d^p$. By Theorem 3.1 an optimal allocation $(\xi_i) \in \mathcal{A}^n(X)$ with minimal total risk is characterized by the optimality equation

$$\nabla r_i(\xi_i) = \nabla r_j(\xi_j), \quad 1 \le i \le n. \tag{7.11}$$

This is a multivariate extension of the classical Borch theorem to $d \ge 1$. For strictly convex r_i, ∇r_i is one-to-one and as a consequence we obtain

$$\xi_i = (\nabla r_i)^{-1}\cdot \nabla r_1(\xi_1), \quad 2 \le i \le n. \tag{7.12}$$

The critical allocation condition then becomes

$$\xi_1 + \sum_{i=2}^n (\nabla r_i)^{-1}\cdot\nabla r_1(\xi_1) = X. \tag{7.13}$$

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