Comparison of Markov processes via infinitesimal generators

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December 2, 2010

Summary: We derive comparison results for Markov processes with respect to stochastic orderings induced by function classes. Our main result states that stochastic monotonicity of one process and comparability of the infinitesimal generators implies ordering of the processes. Unlike in previous work no boundedness assumptions on the function classes are needed anymore. We also present an integral version of the comparison result which does not need the local comparability assumption of the generators. The method of proof is also used to derive comparison results for time-discrete Markov processes.

1 Introduction

Ordering conditions for Markov processes in terms of conditions on the infinitesimal generators have been given in several papers in the literature. Massey (1987), Herbst and Pitt (1991), Chen and Wang (1993) and Chen (2004) describe stochastic ordering for discrete state spaces for diffusions and diffusions with jumps in terms of their infinitesimal generators. For bounded generators and in the case of discrete state spaces Daduna and Szekli (2006) give a comparison result for the stochastic ordering of Markov processes in terms of comparison of their generators. Rüschendorf (2008) (abbreviated in the sequel as [Ru]) established a comparison result for Markov processes on polish spaces using boundedness conditions on the order defining function classes. These boundedness conditions arise from the method of proof used in that paper which makes essential use of an idea in Liggetts characterization result for association of Markov processes (see Liggett (1985)). A similiar idea was also used in a paper of Cox et al. (1996) and Greven et al. (2002) for the special case of directionally convex ordering of a system of interacting diffusions. For Lévy processes some general ordering results were derived in Bergenthum and Rüschendorf (2007) (abbreviated in the sequel as [BeRu]) and for the case of supermodular ordering in Bäuerle et al. (2008). The comparison results in [BeRu, 2007] go beyond the frame of Markov processes to semimartingales and are based on stochastic analysis (Itô's formula and generalized Kolmogorov backward equation).

In this paper we extend the approach in [Ru, 2008] based on strongly continuous semigroups and their infinitesimal generators. In Section 2 we recollect the necessary notation and results on strongly continuous semigroups on Banach spaces $\mathbb{B}$ and their generators. This generality allows to omit the restrictive boundedness conditions in previous papers which prevent applications to interesting orderings defined by nonbounded function classes as e.g. convex orderings. As consequence we obtain general ordering results for Markov Processes in Section 3 by the same simple method as in [Ru, 2008]. Furthermore we give a variant of this comparison result where the conditions appear in an integrated form. This

AMS 2000 subject classification: 60E15, 60J25, 90B15
Key words and phrases: Stochastic ordering, stochastic monotonicity, Markov processes, infinitesimal generators,

integral version of the comparison result can be useful in those cases where a local comparability of the generators is difficult to achieve. We discuss several applications that can be dealt with the generalized approach in this paper. Moreover the simple method of proof in continuous time which is used in Section 3 is adapted also to the discrete time case. This in fact gives a new proof to a classical comparison result in for discrete time processes.

2 Strongly continuous semigroups and their infinitesimal generators

In this section we recollect some important notions and results from semigroup theory on general Banach spaces. Our main references are Engel and Nagel (2000) and Pazy (1983).

Let $T = (T_t)_{t\in\mathbb{R}_+}$ be a semigroup of bounded linear operators on a Banach space $(\mathbb{B}, ||\cdot||)$ i.e.

$$\begin{aligned} T_0 &= \mathrm{id}, \\ T_{t+s} &= T_t T_s \text{ for } s,t \geq 0. \end{aligned} \tag{2.1}$$

T is called *uniformly continuous*, if

$$\lim_{t\downarrow 0} ||T_t - \mathrm{id}\,|| = 0. \tag{2.2}$$

T is called a *contraction*, if

$$||T_t f|| \leq ||f||,\ \forall t \geq 0,\ f \in \mathbb{B}. \tag{2.3}$$

T is called a *strongly continuous semigroup* on $\mathbb{B}$ also called C_0-semigroup, if

$$\lim_{t\downarrow 0} T_t f = f,\ \forall f \in \mathbb{B}. \tag{2.4}$$

If T is strongly continuous then

$$\lim_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} T_s f ds = T_t f,\ f \in \mathbb{B}. \tag{2.5}$$

The *infinitesimal generator* $A : \mathcal{D}_A \subset \mathbb{B} \to \mathbb{B}$ of a strongly continuous semigroup is defined as

$$Af := \lim_{t\downarrow 0} \frac{1}{t}\left(T_t f - f\right) \tag{2.6}$$

for its domain

$$\mathcal{D}_A := \left\{ f \in \mathbb{B} : \lim_{t\downarrow 0} \frac{1}{t}\left(T_t f - f\right) \text{ exists} \right\}.$$

We list two classical examples of semigroups.

Examples 2.1 *(1)* For the Banach space $\mathbb{B} = C_b([0,\infty))$ of bounded continuous functions on $[0,\infty)$ with sup-norm the translation semigroup T is defined by

$$\left(T_t f\right)(x) := f(x+t). \tag{2.7}$$

T is a C_0-contraction semigroup on $C_b([0,\infty))$ with infinitesimal generator

$$Af = \frac{d}{dx} f = f' \tag{2.8}$$

and with domain $\mathcal{D}_A$, the set of all $f \in C_b$ with $f' \in C_b$. Obviously A is not a bounded operator.

(2) Consider the diffusion semigroup T on $\mathbb{B} = \mathscr{L}^p(\mathbb{R}^d)$, $1 \le p < \infty$, defined by

$$T_t f(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{2.9}$$

T is a strongly continuous semigroup on $\mathscr{L}^p(\mathbb{R}^d)$, the class of measurable, p-times integrable functions on $\mathbb{R}^d$ w.r.t. Lebesgue measure. The infinitesimal generator Δ of T is the closure of the *Laplace operator*

$$\Delta f(x) = \sum_{i=1}^d \partial_{ii}^2 f(x_1, \dots, x_d). \tag{2.10}$$

A is defined for all f in the *Schwarz space*

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d);\ f \text{ rapidly decreasing} \right\}.$$

The following result is basic (see Engel and Nagel (2000, Lemma 1.3 in Chapter II)).

Lemma 2.2 *Let $(A, \mathcal{D}_A)$ be the infinitesimal generator of a strongly continuous semigroup T. Then it holds:*

(1) *$A : \mathcal{D}_A \subset \mathbb{B} \to \mathbb{B}$ is a closed, densely defined linear operator, i.e. $\mathcal{D}_A \subset \mathbb{B}$ is dense.*

(2) *If $f \in \mathcal{D}_A$, then $T_t f \in \mathcal{D}_A$ and*

$$\frac{d}{dt} T_t f = T_t A f = A T_t f,\ f \in \mathbb{B},\ t \ge 0. \tag{2.11}$$

(3) *For $f \in \mathbb{B}$, $t \ge 0$ holds*

$$\int_0^t T_s f ds \in \mathcal{D}_A. \tag{2.12}$$

(4) *For $t \ge 0$ holds*

$$T_t f - f = A \int_0^t T_s f ds,\ \text{if } f \in \mathbb{B} \tag{2.13}$$

$$= \int_0^t T_s A f ds,\ \text{if } f \in \mathcal{D}_A. \tag{2.14}$$

For a time homogeneous Markov process $X = (X_t)_{t \ge 0}$ on some measure space $(\mathbb{E}, \mathscr{E})$ let P_t denote the transition kernel

$$P_t(x, B) = P(X_t \in B | X_0 = x),\ x \in \mathbb{E},\ B \in \mathscr{E} \tag{2.15}$$

and $T = (T_t)$ the corresponding transition semigroup defined by

$$\begin{aligned} T_t f(x) &= \int_{\mathbb{E}} P_t(x, dy) f(y) \\ &= E\left(f(X_t) | X_0 = x\right) \end{aligned} \tag{2.16}$$

for f in a suitable Banach space $\mathbb{B}$ of functions on $\mathbb{E}$. Then this puts Markov processes in the framework of semigroups.

Examples 2.3 (1) Let $B_t = (B_t^1, \dots, B_t^d)$ be a Brownian motion on $\mathbb{R}^d$, then

$$P_t(x, V) = (2\pi t)^{-d/2} \int_V e^{-\frac{|y-x|^2}{2t}} dy \tag{2.17}$$

for $0 < t,\ V \in \mathscr{B}(\mathbb{R}^d)$. The corresponding semigroup T can be considered e.g. on C_0 or on C_b.

(2) Let $X = (X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with semigroup $T = (T_t)$ and infinitesimal generator A, then $C_0^2 \subset \mathcal{D}_A$ and

$$\begin{aligned} Af(x) = &\sum_{i=1}^{d} b^i \partial_i f(x) + \frac{1}{2} \sum_{i,j=1}^{d} c^{i,j} \partial_{ij}^2 f(x) \\ &+ \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \sum_{i=1}^{d} \partial_i f(x) y^i \mathbb{1}_{\{y^i : |y^i| \leq 1\}} \right) F(dy) \end{aligned} \tag{2.18}$$

for $f \in C_0^2$, where (b, c, F) is the local characteristic of X with drift vector $b = (b^i)_{1\leq i\leq d}$, $c = (c^{i,j})_{i,j\leq d}$ the diffusion characteristic, a symmetric, positive semidefinite matrix and F the Lévy measure of X i.e. $F(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) F(dx) < \infty$ characterizing its jumps.

(3) For countable state spaces $\mathbb{E}$ we denote by

$$p_t(x, y) := P_t(x, \{y\}),\ x, y \in \mathbb{E},\ t \geq 0 \tag{2.19}$$

the transition function,

$$\mathbf{P}_t := (p_t(x, y))_{x,y\in\mathbb{E}} \text{ the } \textit{transition matrix}$$

and

$$\mathbf{Q} := (q(x, y))_{x,y\in\mathbb{E}} \text{ the } \textit{infinitesimal generator}$$

or *intensity matrix*, where

$$q(x, y) = \lim_{t\downarrow 0} \frac{1}{t} p_t(x, y) \text{ for } x \neq y \tag{2.20}$$

$$q(x, x) = \lim_{t\downarrow 0} \frac{1}{t} \left(p_t(x, x) - 1\right) \tag{2.21}$$

assuming existence and finiteness of the limits. For many applications of ordering results to queuing networks the theory of continuous time Markov chains with transition function p_t respectively induced semigroup $T = (T_t)_{t\geq 0}$ is essential.

The following proposition from Sato (1999, E.34.10) gives an example for the usefulness of the generalized frame of semigroups on Banach spaces different from C_0 or C_b.

Proposition 2.4 *Let $(X_t)_{t\geq 0}$ be a time homogeneous translation invariant Markov process with $X_0 = x$ and transition function P_t. Then the corresponding semigroup $T = (T_t)_{t\geq 0}$*

$$T_t f(x) = \int_{\mathbb{R}^d} P_t(x, dy) f(y) = \int_{\mathbb{R}^d} P_t(dy) f(x + y),\ f \in \mathscr{L}^p \tag{2.22}$$

is a C_0-contraction semigroup on $\mathscr{L}^p = \mathscr{L}^p(\mathbb{R}^d),\ 1 \leq p < \infty$.

For a strongly continuous semigroup $T = (T_t)_{t\geq 0}$ with infinitesimal generator A define $F(t) := T_t f,\ t \geq 0$. Then by (2.11) F solves the *homogeneous Cauchy problem*

$$F'(t) = AF(t),\ F(0) = 0 \text{ for } f \in \mathcal{D}_A. \tag{2.23}$$

The link to the inhomogeneous Cauchy problem and the characterization of its solution in the following theorem is the fundamental tool for the comparison result in the following section. We adapt the proof of Liggett (1985, Chapter I, Theorem 2.15), to the general class of strongly continuous semigroups on a Banach space as considered in this section. Some related results can be found in Pazy (1983) or in Yan (1987) as well.

Theorem 2.5 (Inhomogeneous Cauchy problem) *Let $T = (T_t)_{t\geq 0}$ be a strongly continuous semigroup on $\mathbb{B}$ with infinitesimal generator A and let $F, G : [0,\infty) \to \mathbb{B}$ be functions such that*

(1) $F(t) \in \mathcal{D}_A$*, for all* $t \geq 0$.

(2) $\int_0^t T_{t-s}G(s)ds$ *exists for all* $t \geq 0$.

(3) F *solves the inhomogeneous Cauchy problem, i.e.*

$$F'(t) = AF(t) + G(t) \text{ for } t \geq 0,$$

then

$$F(t) = T_t F(0) + \int_0^t T_{t-s}G(s)ds. \tag{2.24}$$

Proof: We consider the $\mathbb{B}$ valued function $v(s) = T_{t-s}F(s)$ which is well-defined, since T is a C_0-semigroup on $\mathbb{B}$. Furthermore we have

$$\begin{aligned}\frac{T_{t-s-h}F(s+h) - T_{t-s}F(s)}{h} &= T_{t-s}\left(\frac{F(s+h)-F(s)}{h}\right) \\ &+ \left(\frac{T_{t-s-h} - T_{t-s}}{h}\right)F(s) \\ &+ (T_{t-s-h} - T_{t-s})F'(s) \\ &+ (T_{t-s-h} - T_{t-s})\left(\frac{F(s+h)-F(s)}{h} - F'(s)\right).\end{aligned}$$

The first term converges to $T_{t-s}F'(s)$ as h tends to zero, since T is a bounded linear operator. To see that the second term converges to $-T_{t-s}AF(s)$ as h tends to zero one has to use the first assumption and (2.11). The third expression converges to zero as h tends to zero, since T is a strongly continuous semigroup that is it fulfills (2.4). Treating the last term in the same way and using additionally the linearity of T we see that it disappears as h tends to zero. As a result $v(s)$ is differentiable for $0 < s < t$ and using the third assumption we obtain

$$\frac{d}{ds}v(s) = T_{t-s}F'(s) - T_{t-s}AF(s) = T_{t-s}G(s).$$

Moreover $T_{t-s}G(s)$ is integrable by assumption. Integrating this equality yields

$$F(t) - T_t F(0) = \int_0^t T_{t-s}G(s)ds \text{ for all } t \geq 0,$$

which proves the claim. □

Remark 2.6 Assumption 2.5-(2) holds in particular if G is continuous on $[0,\infty)$. If G is not continuous but in $\mathscr{L}^1([0,\tau],\mathbb{B})$, then 2.5-(2) does hold as well, since we have $||T_{t-\cdot}G||_{\mathscr{L}^1([0,\tau],\mathbb{B})} \leq c||G||_{\mathscr{L}^1([0,\tau],\mathbb{B})}$ for $t \in [0,\tau)$ and $c > 0$.

3 Comparison of Markov processes

3.1 Continuous time Markov processes

We assume that X and Y are two time homogeneous Markov processes with values in $(\mathbb{E},\mathscr{E})$. Let $S = (S_t)$ and $T = (T_t)$ denote their semigroups which we assume are strongly continuous semigroups on some Banach function space $\mathbb{B}$ on $\mathbb{E}$ like $C_0(\mathbb{E})$, $C_b(\mathbb{E})$,

$\mathscr{L}^p(\mathbb{E}, \mu)$ where $\mathbb{E}$ allows to define these structures as Banach spaces. Denote by A and B the corresponding infinitesimal generator of S and T respectively. Let $\mathcal{F} \subset \mathbb{B}$ be a set of real functions on $\mathbb{E}$ and let $\leq_{\mathcal{F}}$ denote the corresponding *stochastic order* on $M^1(\mathbb{E})$, the set of probability measures on $\mathbb{E}$ defined by

$$\mu \leq_{\mathcal{F}} \nu \text{ if } \int f d\mu \leq \int f d\nu, \text{ for all } f \in \mathcal{F}. \tag{3.25}$$

We assume that

$$\mathcal{F} \subset \mathcal{D}_A \cap \mathcal{D}_B. \tag{3.26}$$

Theorem 3.1 (Conditional comparison result) *Assume that*

(1) *X is stochastically monotone w.r.t. $\mathcal{F}$ i.e. $f \in \mathcal{F}$ implies $S_t f \in \mathcal{F}$ for all $t \geq 0$, and*

(2)
$$Af \leq Bf \quad [P^{X_0}], \textit{ for all } f \in \mathcal{F}. \tag{3.27}$$

Then

$$S_t f \leq T_t f \quad [P^{X_0}], \, f \in \mathcal{F}. \tag{3.28}$$

Proof: Define for $f \in \mathcal{F}$ the function $F : [0, \infty) \to \mathbb{B}$ by $F(t) := T_t f - S_t f$. Then F satisfies the differential equation

$$\begin{aligned} F'(t) &= BT_t f - AS_t f \\ &= B(T_t f - S_t f) + (B - A)(S_t f). \end{aligned}$$

By the stochastic monotonicity assumption holds $S_t f \in \mathcal{F}$ and thus $H(t) := (B-A)(S_t f)$ is well-defined in $\mathbb{B}$ and $H(t) \geq 0$ by (3.27). Thus F solves the nonhomogeneous Cauchy problem that is

$$F'(t) = BF(t) + H(t), \; F(0) = 0. \tag{3.29}$$

Since $H(t) \geq 0$ it follows that $\int_0^t T_{t-s} H(s) ds$ exists and is finite. Further by Lemma 2.2-2 and by the stochastic monotonicity assumption we obtain

$$F(t) = T_t f - S_t f \in \mathcal{D}_B.$$

Thus the assumptions of Theorem 2.5 are satisfied and imply that the solution $F(t)$ has an integral representation of the form

$$\begin{aligned} F(t) &= T_t F(0) + \int_0^t T_{t-s} H(s) ds \\ &= \int_0^t T_{t-s} H(s) ds \quad \text{as } F(0) = 0. \end{aligned}$$

Since $H(s) \geq 0$ it follows that $F(t) \geq 0$ for all $t > 0$ and thus the statement of the theorem. □

For diffusion processes with jumps a related comparison theorem is given by Zhang (2006). He applies this comparison result to discuss the existence and uniqueness of invariant probability measures for uniformly elliptic diffusion processes with jumps. Several examples where the local comparison condition for the infinitesimal generators is easy to verify were given by [Ru, 2008].

Example 3.2 For pure diffusion processes X and Y in $\mathbb{R}^d$ with respective diffusion matrices $a := (a_{ij}(x))$ and $b := (b_{ij}(x))$, the infinitesimal generators are given by

$$Af(x) = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \text{ and } Bf(x) = \frac{1}{2}\sum_{ij} b_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{3.30}$$

Thus for convex ordering the comparison condition (3.27) is implied by

$$Af(x) \leq Bf(x),\ f \in \mathcal{F}_{\text{cx}} \cap C^2$$

which is equivalent to the positive semidefiniteness of the matrix $b - a$. Note that since the choice $\mathscr{L}^1(\mathbb{E})$ as the considered Banach space is included, an application of the comparison result to convex ordering in the nonbounded case is justified by our extension of the comparison results.

Define the componentwise ordering of processes X, Y by

$$(X) \leq_{\mathcal{F}} (Y) \text{ if } Eh(X_{t_1}, \ldots, X_{t_k}) \leq Eh(Y_{t_1}, \ldots, Y_{t_k}) \tag{3.31}$$

for all $0 \leq t_1 < \cdots < t_k$ and for all functions h that are componentwise in $\mathcal{F}$ and are integrable. Thus componentwise ordering is an ordering of the finite dimensional distributions. As in [Ru, 2008, Corollary 2.4] we obtain the following comparison result as consequence of the conditional ordering theorem (Theorem 3.1) and the *separation theorem* (see [BeRu, 2007, Proposition 3.1]) for the ordering of Markov processes.

Corollary 3.3 (Componentwise ordering result) *Assume that the conditions of Theorem 3.1 hold true and that $X_0 \leq_{\mathcal{F}} Y_0$. Then the componentwise ordering $(X) \leq_{\mathcal{F}} (Y)$ of the processes X and Y holds.*

For some applications the pointwise ordering conditions on the infinitesimal generators as in (3.27) are too strong. For instance in the pure diffusion case with generator as in equation (3.30) with coefficients given by

$$a_{ij}(x) = -1 \text{ and } b_{ij}(x) = 2 \cdot \left(\mathbb{1}_{\mathbb{R}^d_+}(x) - \mathbb{1}_{\mathbb{R}^d_-}(x)\right) \text{ for all } i, j \in \{1, \ldots, d\} \tag{3.32}$$

the pointwise ordering condition is not true for the class of directionally convex functions. It turns out that a weaker integral condition allows to obtain an integral comparison result. In order to describe the development of stochastic dependence between two processes X and Y we establish the integral comparison result for more general function classes. For example we would like to have conditions that imply that over the time t the positive dependence in process Y between Y_0 and Y_t is getting stronger than in process X between X_0 and X_t. That is

$$Eh(X_0, X_t) \leq Eh(Y_0, Y_t) \text{ for all } t > 0 \tag{3.33}$$

and for all integrable dependence functions $h \in \mathcal{F}_{[2]}$. For several results on dependence ordering of this type we refer to Daduna and Szekli (2006) and Daduna et al. (2006). In order to compare w.r.t. such functions h we need the information about the type of comparison which is contained in a tupel of function classes $(\mathcal{F}, \mathcal{F}^{(2)})$. Therefore let $\mathcal{F}^{(2)}$ be a function class on $\mathbb{E}^2$ and let $\mathcal{F}$ be a function on $\mathbb{E}$ and define

$$\mathcal{F}_{[2]} := \{f \in \mathcal{F}^{(2)} |\ f(x, \cdot),\ f(\cdot, x) \in \mathcal{F}\}.$$

Typical examples for $\mathcal{F}^{(2)}$ are $\mathcal{F}_{\text{cx}}(\mathbb{R}^{2d}), d \in \mathbb{N}$, the class of convex functions on $\mathbb{R}^{2d}$ and $\mathcal{F}_{\text{sm}}(\mathbb{R}^{2d}), d \in \mathbb{N}$ the class of supermodular functions on $\mathbb{R}^{2d}$ and for $\mathcal{F}$ are $\mathcal{F}_{\text{cx}}(\mathbb{R}^d)$ resp. $\mathcal{F}_{\text{sm}}(\mathbb{R}^d)$ (for definitions and properties, see Müller and Stoyan (2002)). Then $\mathcal{F}_{[2]} = \mathcal{F}_{\text{cx}}(\mathbb{R}^{2d})$ resp. $\mathcal{F}_{[2]} = \mathcal{F}_{\text{sm}}(\mathbb{R}^{2d})$, since the functions in $\mathcal{F}^{(2)}$ are componentwise in $\mathcal{F}_{\text{cx}}(\mathbb{R}^d)$ and $\mathcal{F}_{\text{sm}}(\mathbb{R}^d)$ respectively

Let X and Y be two time homogeneous Markov processes as introduced at the beginning of this section. Additionally we assume that for every $f \in \mathcal{F}_{[2]}$ the existence of the integrals $Ef(X_0, X_t), Ef(Y_0, Y_t)$ and the domain condition (3.26).

Theorem 3.4 (Integral comparison result w.r.t. $\mathcal{F}_{[2]}$) *Assume that*

(1) *X is stochastically monotone w.r.t. $\mathcal{F}_{[2]}$ i.e. $f \in \mathcal{F}_{[2]}$ implies $(S_t f(x,\cdot)) \in \mathcal{F}$, for $x \in \mathbb{E}$ and $t \geq 0$,*

(2) *for $\tau \in \mathbb{R}_+$ the map $s \mapsto (T_{\tau-s}Bf(x,\cdot))$ is integrable on $[0,\tau)$ for all $f \in \mathcal{F}_{[2]}$, $x \in \mathbb{E}$ and*

(3) *for all $f \in \mathcal{F}_{[2]}$ and all $t \geq s$ it holds that*

$$\begin{aligned}\int_{\mathbb{E}}\int_{\mathbb{E}} &(Af(x,\cdot))(y)P^Y_{t-s}(x,dy)P^{X_0}(dx)\\ &\leq \int_{\mathbb{E}}\int_{\mathbb{E}} (Bf(x,\cdot))(y)P^Y_{t-s}(x,dy)dP^{X_0}(dx). \end{aligned} \tag{3.34}$$

Then

$$\int_{\mathbb{E}} (S_t f(x,\cdot)))(x)P^{X_0}(dx) \leq \int_{\mathbb{E}} (T_t f(x,\cdot))(x)P^{X_0}(dx) \text{ for all } f \in \mathcal{F}_{[2]} \text{ and } t \geq 0. \tag{3.35}$$

Proof: Let $x \in \mathbb{E}$ and define for $f \in \mathcal{F}_{[2]}$ the function $F_x : [0,\infty) \to \mathbb{B}$ by $F_x(t)(\cdot) := (T_t f(x,\cdot) - S_t f(x,\cdot))(\cdot)$. Then F satisfies the differential equation

$$\begin{aligned}F'_x(t) &= \frac{d}{dt}(T_t f(x,\cdot)) - \frac{d}{dt}(S_t f(x,\cdot))\\ &= B(T_t f(x,\cdot)) - A(S_t f(x,\cdot))\\ &= B(T_t f(x,\cdot) - S_t f(x,\cdot))) + ((B-A)(S_t f(x,\cdot))\end{aligned}$$

By the stochastic monotonicity assumption holds $S_t f(x,\cdot) \in \mathcal{F}$ and thus $H_x(t)(\cdot) := ((B-A)(S_t f(x,\cdot))(\cdot)$ is well-defined in $\mathbb{B}$. Thus F_x solves the nonhomogeneous Cauchy problem, i.e.

$$F'_x(t) = BF_x(t) + H_x(t) \text{ and } F_x(0) = 0.$$

Further we obtain

$$\begin{aligned}\int_0^t \|T_{t-s}H_x(s)\|ds &= \int_0^t \|(T_{t-s}(BS_s f(x,\cdot)) - T_{t-s}(AS_s f(x,\cdot)))\|ds\\ &\leq \int_0^t \|T_{t-s}(BS_s f(x,\cdot)))\|ds + \int_0^t \|T_{t-s}(AS_s f(x,\cdot))\|ds\\ &< \infty.\end{aligned}$$

This estimate follows from the stochastic monotonicity assumption, the integrability of $T_{t-\cdot}Bf(x,\cdot)$ and property (2.14) from Lemma 2.2. Hence $\int_0^t T_{t-s}H_x(s)ds$ exists and is finite. Moreover by Lemma 2.2-2 and the stochastic monotonicity assumption we have

$$F_x(t) = T_t f(x,\cdot) - S_t f(x,\cdot) \in \mathcal{D}_B.$$

Thus the assumptions of Theorem 2.5 are fulfilled and imply that the solution $F_x(t)$ has an integral representation has a respecta

$$\begin{aligned}F_x(t) &= T_t F_x(0) + \int_0^t T_{t-s}H_x(s)ds\\ &= \int_0^t T_{t-s}H_x(s)ds \quad \text{as } F_x(0) = 0.\end{aligned}$$

By assumption we can integrate both sides to obtain

$$\begin{aligned}\int_{\mathbb{E}} F_x(t)P^{X_0}(dx) &= \int_{\mathbb{E}}\int_0^t T_{t-s}H_x(s)dsP^{X_0}(dx) \\ &= \int_0^t \int_{\mathbb{E}} T_{t-s}H_x(s)P^{X_0}(dx)ds,\end{aligned}$$

where we apply the Theorem of Fubini. But since the inner integral has a representation of the form

$$\begin{aligned}&\int_{\mathbb{E}}\int_{\mathbb{E}} H_x(s)(y)P^Y_{t-s}(x,dy)P^{X_0}(dx) \\ &\quad = \int_{\mathbb{E}}\int_{\mathbb{E}} \left((B-A)(S_sf(x,\cdot)\right)(y)P^Y_{t-s}(x,dy)P^{X_0}(dx),\end{aligned}$$

condition (3.34) delivers the positivity of $\int_{\mathbb{E}} F_x(t)P^{X_0}(dx)$ which proves the statement of the theorem. □

In the case of equal initial distributions we obtain from (3.35) for all $t > 0$

$$\begin{aligned}Ef(X_0,X_t) &= \int E(f(x,X_t)|X_0 = x)P^{X_0}(dx) \\ &= \int \Big(\left(S_tf(x,\cdot)\right)(z)\Big)_{\big|z=x} P^{X_0}(dx) \\ &\leq \int \Big(\left(T_tf(x,\cdot)\right)(z)\Big)_{\big|z=x} P^{X_0}(dx) \\ &= \int E(f(x,Y_t)|Y_0 = x)P^{Y_0}(dx) \\ &= Ef(Y_0,Y_t).\end{aligned}$$

The latter theorem is also true for functions in $\mathcal{F}$, i.e. for function classes on $\mathbb{E}$. Thus in the case of equal initial distribution we obtain the following corollary.

Corollary 3.5 (Integral comparison result w.r.t. $\mathcal{F}$) *Assume that $P^{X_0} = P^{Y_0}$ and*

(1) X is stochastically monotone w.r.t. $\mathcal{F}$,

(2) for $\tau \in \mathbb{R}_+$ the map $s \mapsto (T_{\tau-s}Bf)$ is integrable on $[0,\tau)$ for all $f \in \mathcal{F}$ and

(3) for all $f \in \mathcal{F}$ and all $t \geq s$ it holds that

$$\int_{\mathbb{E}}\int_{\mathbb{E}} Af(y)P^Y_{t-s}(x,dy)P^{X_0}(dx) \leq \int_{\mathbb{E}}\int_{\mathbb{E}} Bf(y)P^Y_{t-s}(x,dy)dP^{X_0}(dx). \tag{3.36}$$

Then

$$X_t \leq_{\mathcal{F}} Y_t \text{ for all } t \geq 0. \tag{3.37}$$

Remark 3.6 (a) In the particular case when Y is a stationary homogeneous Markov process with invariant distribution π and X is a homogeneous Markov process with initial distribution π, then condition (3.36) becomes

$$\int_{\mathbb{E}} Af(x)\pi(dx) \leq 0, \tag{3.38}$$

since it holds that $\int P^Y_t(x,dy)\pi(dx) = \pi(dy)$ for all $t > 0$. In situations like (3.32) the local comparability of the generators is not true, so Theorem 3.1 is not applicable, however condition (3.38) is satisfied. Thus we obtain $X_t \leq_{\mathcal{F}} Y_t$ for all $t \geq 0$ from Corollary 3.5.

(b) One further application of Corollary 3.5 is the comparison of bivariate distribution functions of Markov processes. As it is easily seen from the proof of Theorem 3.4 condition (3.34) can be replaced by

$$\int_V \int_{\mathbb{E}} (Af(y))\, P^Y_{t-s}(x, dy) P^{X_0}(dx) \leq \int_V \int_{\mathbb{E}} (Bf(y))\, P^Y_{t-s}(x, dy) dP^{X_0}(dx), \tag{3.39}$$

where $V \in \mathscr{E}$ in order to obtain

$$\int_V (S_t f(x))) P^{X_0}(dx) \leq \int_V (T_t f(x))\, P^{X_0}(dx)$$

for all $f \in \mathcal{F}$ and $t \geq 0$, if both expressions exist.

Now consider $\mathbb{E} = \mathbb{R}$ and $V := (-\infty, u]$. From Corollary 3.5 we have a comparison of the two dimensional distribution functions of (X_0, X_t) and (Y_0, Y_t) for all $s, u \in \mathbb{R}$ and $f := \mathbb{1}_{(-\infty,s]}$:

$$\begin{aligned} P(X_0 \leq u, X_t \leq s) &= \int_{-\infty}^{u} S_t f(x) P^{X_0}(dx) \\ &\leq \int_{-\infty}^{u} T_t f(x) P^{X_0}(dx) = P(Y_0 \leq u, Y_t \leq s) \end{aligned} \tag{3.40}$$

provided that the assumptions of Corollary 3.5 are fulfilled. If X and Y additionally have the same marginals then (3.40) implies the concordance order.

In several cases condition (3.34) is not verifiable. In general one is interested in comparing Markov processes without knowing much about their transition kernels. The integral comparison theorem implies however that a bivariate comparison of X and Y is possible if a suitable local comparison condition on the infinitesimal generators holds true. The next result implies in particular the integral comparison result in (3.35).

Corollary 3.7 (Bivariate comparison result) *Assume that $P^{X_0} = P^{Y_0}$ and*

(1) *X is stochastically monotone w.r.t. $\mathcal{F}_{[2]}$ i.e. $f \in \mathcal{F}_{[2]}$ implies $(S_t f(x, \cdot)) \in \mathcal{F}$, for $x \in \mathbb{E}$ and $t \geq 0$,*

(2) *for $\tau \in \mathbb{R}_+$ the map $s \mapsto (T_{\tau-s} Bf(x, \cdot))$ is integrable on $[0, \tau)$ for all $f \in \mathcal{F}_{[2]}$, $x \in \mathbb{E}$ and*

(3) *for all $f \in \mathcal{F}_{[2]}$ and all $x \in \mathbb{E}$ it holds that*

$$Af(x, \cdot) \leq Bf(x, \cdot). \tag{3.41}$$

Then

$$Ef(X_0, X_t) \leq Ef(Y_0, Y_t) \text{ for all } f \in \mathcal{F}_{[2]} \text{ and } t \geq 0. \tag{3.42}$$

Note that in general it is not enough to assume condition (3.27) in order to get (3.42).

In networks for example telecommunication networks or company supply networks where different velocities in evolution are extant, it is valuable to understand their internal dependencies in order to respond reasonably to changes in the evolution of the network. Is it possible to transfer the knowledge about the dependencies in the faster evolving network to the slower evolving network and vice versa? It is of interest to know how the dependence of a network driven by a Markov process X alters when changing the speed of development. An intuitive approach is to study speeding-down versions of a Markov process. Several results of this type have been obtained in Bäuerle and Rolski (1998) and in Kulik and Wichelhaus (2007).

For $f \in \mathcal{F}_{[2]}$, $x \in \mathbb{E}$ and $c \in (0,1)$ consider the speeding-down Markov process $\hat{X}^c$ induced by the following infinitesimal generator:

$$Bf(x,\cdot) = \begin{cases} c \cdot Af(x,\cdot), & \text{for } y \in \mathbb{E} \text{ s.t. } \left(Af(x,\cdot)\right)(y) \geq 0, \\ 0 & \text{for } y \in \mathbb{E} \text{ s.t. } \left(Af(x,\cdot)\right)(y) < 0. \end{cases} \tag{3.43}$$

Thus in general both inequalities

$$\begin{aligned} &Af(x,\cdot) < Bf(x,\cdot) \text{ on } \{Af(x,\cdot) < 0\} \text{ and} \\ &Af(x,\cdot) \geq cAf(x,\cdot) = Bf(x,\cdot) \text{ on } \{Af(x,\cdot) \geq 0\} \end{aligned}$$

hold true. In consequence Corollary 3.7 is not applicable since condition (3.41) does not hold for the infinitesimal generators of the Markov processes X and $\hat{X}^c$.

The following result generalizes the speeding-down property in (3.43). Assuming that the infinitesimal generators of the corresponding Markov processes fulfill condition (3.34) allows to apply the integral comparison result Theorem 3.4.

Corollary 3.8 *Let X be a homogeneous Markov process. Assume that X is stochastically monotone w.r.t. $\mathcal{F}_{[2]}$. Moreover for $x \in \mathbb{E}$ and $\tau \in \mathbb{R}_+$ let the map $s \mapsto T_{\tau-s}Af(x,\cdot)$ be integrable on $[0,\tau)$ for all $f \in \mathcal{F}_{[2]}$. If for some $c = (c_1, c_2) \in [0,1)^2$ and all $f \in \mathcal{F}_{[2]}$ it holds that*

$$\begin{aligned} &\int \int_{\{Af(x,\cdot)<0\}} (1-c_2) \cdot \left(Af(x,\cdot)\right)(y) P^X_{t-s}(x,dy) P^{X_0}(dx) \\ &\quad \leq \int \int_{\{Af(x,\cdot)\geq 0\}} (c_1 - 1) \cdot \left(Af(x,\cdot)\right)(y) P^X_{t-s}(x,dy) P^{X_0}(dx), \end{aligned} \tag{3.44}$$

then

$$Ef(X_0, X_t) \leq Ef(\hat{X}^c_0, \hat{X}^c_t), \text{ for all } f \in \mathcal{F}_{[2]}$$

where $\hat{X}^c$ is the speeding-down version of X induced by the infinitesimal generator

$$Bf(x,\cdot) = \begin{cases} c_1 \cdot Af(x,\cdot)(y), & \text{for } y \in \mathbb{E} \text{ s.t. } \left(Af(x,\cdot)\right)(y) \geq 0, \\ c_2 \cdot Af(x,\cdot)(y), & \text{for } y \in \mathbb{E} \text{ s.t. } \left(Af(x,\cdot)\right)(y) < 0. \end{cases} \tag{3.45}$$

Proof: For proving this statement we use Theorem 3.4. Due to the construction in (3.45) the properties of X are transferred to $\hat{X}^c$. Thus the only thing to do is to check condition (3.34). But this follows from (3.44). □

3.2 Discrete time Markov processes

Now we transfer the method of proof of the comparison result for the continuous time case to the discrete time case. Let $\mathcal{F} \subset \mathbb{B}$ be a set of real functions on $\mathbb{E}$ and let $X = (X_n)_{n\in\mathbb{N}_0}$ and $Y = (Y_n)_{n\in\mathbb{N}_0}$ be real-valued, discrete-time homogeneous Markov processes. Denote the one-step transition kernels for X and Y by $\mathbf{K}^X : \mathbb{E} \times \mathscr{E} \to [0,1]$ and $\mathbf{K}^Y : \mathbb{E} \times \mathscr{E} \to [0,1]$. Also define as usual for a kernel $\mathbf{K}$

$$\mathbf{K}f(x) = \int_{\mathbb{E}} f(y)\mathbf{K}(x,dy).$$

By a modification of the method of proof of Theorem 3.1 we obtain a discrete time version of the *conditional comparison result*. This in fact gives a new proof to a classical result (see e.g. Müller and Stoyan (2002, Theorem 5.2.11)).

Proposition 3.9 (Discrete time conditional comparison result; w.r.t. $\mathcal{F}$)

(a) *Assume that*

(1) $\mathbf{K}^Y$ *is stochastically monotone, i.e.* $f \in \mathcal{F}$ *implies* $\mathbf{K}^Y f \in \mathcal{F}$*, and*

(2)
$$\mathbf{K}^X(x,\cdot) \leq_{\mathcal{F}} \mathbf{K}^Y(x,\cdot) \text{ for all } x \in \mathbb{E}. \tag{3.46}$$

Then we have

$$\int f(x)\mathbf{K}_n^X(y,dx) \leq \int f(x)\mathbf{K}_n^Y(y,dx), \text{ for all } f \in \mathcal{F},\ n \in \mathbb{N}_0,$$

where $\mathbf{K}_n$ *denotes the* n*-step transition kernel* $(\mathbf{K})^n$*.*

(b) If in addition X *and* Y *possess the same initial distribution, then it holds that*

$$X_n \leq_{\mathcal{F}} Y_n, \text{ for all } n \in \mathbb{N}_0.$$

Proof: (a) As in the proof of the *conditional comparison result* in Theorem 3.1 define for $f \in \mathcal{F}$ the function $F : \mathbb{N}_0 \to \mathbb{B}$ by $F(n) := \mathbf{K}_n^Y f - \mathbf{K}_n^X f$. Then we obtain similar to the case of continuous time a recursive equation for F:

$$\begin{aligned} F(n+1) - F(n) &= (\mathbf{K}^Y - \mathrm{id})(\mathbf{K}_n^Y f) - (\mathbf{K}^X - \mathrm{id})(\mathbf{K}_n^X f) \\ &= (\mathbf{K}^X - \mathrm{id})(\mathbf{K}_n^Y f - \mathbf{K}_n^X f) + (\mathbf{K}^Y - \mathbf{K}^X)(\mathbf{K}_n^Y f) \\ &= (\mathbf{K}^X - \mathrm{id})F(n) + (\mathbf{K}^Y - \mathbf{K}^X)(\mathbf{K}_n^Y f). \end{aligned}$$

This yields

$$F(n+1) = \mathbf{K}^X F(n) + (\mathbf{K}^Y - \mathbf{K}^X)(\mathbf{K}_n^Y f) \tag{3.47}$$
$$F(0) = 0. \tag{3.48}$$

For $n = 0$ we have $F(1) = (\mathbf{K}^Y - \mathbf{K}^X)f \geq 0$, by condition (3.46).

For $n = 1$ this implies that $F(2) = \mathbf{K}^X F(1) + (\mathbf{K}^Y - \mathbf{K}^X)(\mathbf{K}^Y f) \geq 0$ by using the $\leq_{\mathcal{F}}$-monotonicity of $\mathbf{K}^Y$ and the positivity preserving property of $\mathbf{K}^X$. Proceeding inductively we get for all $n \in \mathbb{N}_0$ that

$$0 \leq F(n) = \mathbf{K}_n^Y f - \mathbf{K}_n^X f,$$

i.e. we have

$$\int f(x)\mathbf{K}_n^X(y,dx) \leq \int f(x)\mathbf{K}_n^Y(y,dx).$$

(b) Taking expectations, bearing in mind that X and Y have the same initial distribution, we get $X_n \leq_{\mathcal{F}} Y_n$ for all $n \in \mathbb{N}_0$. □

Observe that equation (3.47) can be rewritten in

$$F(n+1) = \sum_{k=0}^{n} \left(\mathbf{K}_k^X \left(\mathbf{K}^Y - \mathbf{K}^X \right) \left(\mathbf{K}_{n-k}^Y f \right) \right) \text{ for all } n \in \mathbb{N}_0, \tag{3.49}$$

using the recursive definition of $F(n)$ and that fact that $F(0) = 0$. This enables us to establish a comparison result for $f \in \mathcal{F}$, where the conditions on the transition kernel appear in integrated form.

Proposition 3.10 (Discrete integral comparison result w.r.t. $\mathcal{F}$) *Let* $X = (X_n)_{n\in\mathbb{N}_0}$ *and* $Y = (Y_n)_{n\in\mathbb{N}_0}$ *be discrete-time homogeneous Markov processes. Assume that* $P^{X_0} = P^{Y_0} =: \pi$*, and*

(1) $\mathbf{K}^Y$ *is stochastically monotone w.r.t.* $\mathcal{F}$*, and*

(2) *for all* $f \in \mathcal{F}$ *it holds that*

$$0 \leq \int_{\mathbb{E}} \Big(\mathbf{K}_n^X \left(\mathbf{K}^Y - \mathbf{K}^X\right) f\Big)(x)\pi(dx) \text{ for all } n \in \mathbb{N}_0 \tag{3.50}$$

Then

$$X_n \leq_{\mathcal{F}} Y_n \text{ for all } n \in \mathbb{N}_0.$$

Proof: As in the proof of our integral comparison result Theorem 3.4 define for $f \in \mathcal{F}$ the function $F : \mathbb{N}_0 \to \mathbb{B}$ by $F(n)(\cdot) := \left(\mathbf{K}_n^Y - \mathbf{K}_n^X f\right)(\cdot)$. Then by integrating both sides with respect to π and using (3.49) we obtain

$$\int F(n+1)(x)\pi(dx) = \int \sum_{k=0}^{n} \Big(\mathbf{K}_k^X \left(\mathbf{K}^Y - \mathbf{K}^X\right) \left(\mathbf{K}_{n-k}^Y f\right)\Big)(x)\pi(dx),$$

which is non-negative due to assumption (3.50) and the stochastic monotonicity of Y. Moreover for all $f \in \mathcal{F}$ it holds that

$$\begin{aligned} Ef(Y_{n+1}) - Ef(X_{n+1}) &= \int E(f(Y_{n+1})|Y_0 = x) - E(f(X_{n+1})|X_0 = x)\pi(dx) \\ &= \int \Big(\mathbf{K}_{n+1}^Y f(x) - \mathbf{K}_{n+1}^X f(x)\Big)\pi(dx) \\ &= \int F(n+1)(x)\pi(dx), \end{aligned}$$

and the statement of the propostion follows. □

Note that condition (3.50) reduces to $Ef(X_0) \leq Ef(Y_1)$ for all $f \in \mathcal{F}$ if X is stationary with invariant distribution π. Then in this case $X_n \leq_{\mathcal{F}} Y_n$ is obvious, since Y is stochastically monotone w.r.t. $\mathcal{F}$.

The recursive formula in (3.49) is also true for $f \in \mathcal{F}_{[2]}$ if for $x \in \mathbb{E}$ we define the function $F_x : \mathbb{N} \to \mathbb{B}$ by $F_x(n)(\cdot) := \left(\mathbf{K}^Y f(x,\cdot) - \mathbf{K}^X f(x,\cdot)\right)(\cdot)$. Then we obtain similarly the following corollary.

Corollary 3.11 (Discrete integral comparison result w.r.t. $\mathcal{F}_{[2]}$) *Let* $X = (X_n)_{n\in\mathbb{N}_0}$ *and* $Y = (Y_n)_{n\in\mathbb{N}_0}$ *be discrete-time, homogeneous Markov processes. Assume that* $P^{X_0} = P^{Y_0} =: \pi$*, and*

(1) $\mathbf{K}^Y$ *is stochastically monotone w.r.t.* $\mathcal{F}_{[2]}$*, i.e.* $f \in \mathcal{F}_{[2]}$ *implies* $\mathbf{K}^Y f(x,\cdot) \in \mathcal{F}$*, for all* $x \in \mathbb{E}$*, and*

(2) *for all* $f \in \mathcal{F}_{[2]}$ *and all* $x \in \mathbb{E}$ *it holds that*

$$0 \leq \int_{\mathbb{E}} \Big(\mathbf{K}_n^X \left(\mathbf{K}^Y - \mathbf{K}^X\right) f(x,\cdot)\Big)(x)\pi(dx) \text{ for all } n \in \mathbb{N}_0. \tag{3.51}$$

Then

$$Ef(X_0, X_n) \leq Ef(Y_0, Y_n) \text{ for all } f \in \mathcal{F}_{[2]},\ n \in \mathbb{N}_0.$$

Acknowledgements. We are thankful to an anonymous referee for carefully reading the manuscript and posing the question whether an extension of the continuous time results to the discrete time case is possible.

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