Conditional limit theorems for random excursions

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Abstract

The main result of this paper establishes under some general assumptions a conditional limit theorem for an iid sequence of random excursions to the conditional excursion measure of the (concatenated) Itô point process of excursions in the Lévy case. This result extends a corresponding limit theorem for the rescaled discrete height process of sequences of critical Galton–Watson trees in the finite variance case in Le Gall (2010) as well as for random walk excursions in an early paper of Doney (1985). As consequence we obtain several known and new results in terms of conditioned convergence of random walk excursions and Galton–Watson trees.

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1. Introduction

In a number of publications in the past 10-15 years, the fundamental role of Itô's excursion theory to the asymptotic properties of large random trees has been uncovered and developed in particular in connection with the asymptotics of Galton–Watson trees conditioned to be large in some sense. For detailed descriptions of these developments and for surveys of this field see Duquesne and Le Gall (2002), Le Gall (2005) and Le Gall (2010).

The basic idea of this development is the following. When studying scaling limits for a sequence of independent Galton–Watson trees one obtains (in the finite variance case) that the rescaled contour process or a variant of it, the height process, under the condition that the Galton–Watson tree is large can be obtained as first tree in the sequence satisfying the constraint. Equivalently the tree corresponds to the first excursion away from zero that satisfies this property. Under rescaling this excursion converges to the first excursion of a reflected Brownian motion satisfying this property. And as result from Itô's excursion theory one can identify the limit with a conditioned version of the Itô excursion measure, i.e., the intensity measure of Itô's point process of excursions.

This program yields, in the case of Galton–Watson trees conditioned to have a large progeny, in the limit a normalized Brownian excursion which serves as coding of the continuum random tree (CRT) and thus implies as consequence the original Aldous result of convergence to the CRT w.r.t. Gromov–Hausdorff distance (see Le Gall (2010) for a survey on this approach). For several particular conditionings like size or height in degenerate or non-degenerate form, limit theorems for conditioned Galton–Watson trees have been obtained in this way in the finite variance case, for example in Drmota and Gittenberger (1997) who treated the profile process in order to obtain the limiting distribution of the width of the tree conditioned to its size. A general conditional limit theorem for the rescaled height process of a sequence of iid Galton–Watson trees has been given in Le Gall (2010) in the case of finite reproduction variance, when conditioning under a regular excursion set A.

The Lévy height process was introduced in Le Gall and Le Jan (1998) allowing to define an analogon to the excursion process of Brownian motion as possible limit object of discrete contour or height processes in the infinite variance case. Related convergence results for trees and forests were then established by Duquesne and Le Gall (2002) in the infinite variance case and further conditional limit theorems for the discrete height, the contour and the random

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walk process to the Lévy excursion resp. Lévy height excursion process have been found in Duquesne and Le Gall (2002), Duquesne (2003) and Kortchemski (2012), but also in a paper of Kersting (1998).

Conditional functional limit theorems have also been investigated for the study of recurrent random walks and Markov processes. A general conditional limit theorem for fixed conditioning sets has been given in an early work of Doney (1985) for the case of asymptotically stable random walk excursions.

The main aim of our paper, which is based on the dissertation of Kühn (2013), is to establish a general conditional limit theorem of normalized excursion processes to the conditional excursion measure of the concatenated reflected Lévy process in the limit. We state this result under conditioning on a general sequence of regular sets assuming joint convergence of the scaled concatenated excursion process together with the scaled discrete local time process. In the application to conditional Galton–Watson trees we obtain in particular an extension of Le Gall's (2010) general convergence result to the infinite variance case. After an introduction to the basic notions and constructions in Section 2 we state our main limit theorem for conditional excursions in Section 3. In Section 4 we obtain as consequence extensions of known conditional limit theorems for random walk excursions. As applications we obtain several new results for the conditional waiting time process in queuing theory. In Section 5 we re-derive and extend several known conditional limit theorems for the infinite variance case of Galton–Watson trees for various related processes like contour, random walk and height process to some general forms of conditionings. We obtain also new conditional limit theorems for the case of conditioning under the total path length and under the width. Finally in Section 6 we collect proofs of some lemmas and propositions.

2. Excursion space, excursion measures, and some notation

We start by defining the state space of excursions, which are functions in the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of càdlàg trajectories with compact support. It is defined as

$$\mathcal{E} := \{f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \mid f(t) > 0 \;\forall\; 0 < t < \zeta(f) < \infty\},$$

where $\zeta(f) := \sup\{t > 0 \mid f(t) > 0\}$ denotes the length or duration of the excursion. The space $\mathcal{E}$ is equipped with the topology which is generated by the distance

$$d_{\mathcal{E}}(f, g) := \delta(f, g) + |\zeta(f) - \zeta(g)|,$$

where δ stands for the Skorokhod (Prokhorov) distance, generating the Skorokhod J_1-topology, and the corresponding Borel-σ-field. Note that the mappings $y \mapsto \bar{y}$, where $\bar{y} := \sup_{0\le s\le\zeta(y)} y_s$, and $y \mapsto \zeta(y)$ are continuous for the topology induced by $d_{\mathcal{E}}$.

Let X be a Lévy process and let $(-\infty, 0]$ be regular for X. We denote by $I_t := \inf_{0\le s\le t} X_s$ the running infimum of X and write $Y := X - I$ for the Lévy process reflected at its infimum. Y is a strong Markov process with respect to $\mathcal{F}$, the canonical augmented filtration generated by X. Let $L = L(Y)$ be the local time at zero of the Markov process Y, namely a continuous, adapted, increasing process such that:

1. The support of the Stieltjes measure dL is included in the zero-set $\mathcal{Z} := \{t \ge 0 \mid Y_t = 0\}$ of Y.
2. For every stopping time T with $Y_T = 0$ a.s. on $\{T < \infty\}$, the shifted process $(Y_{T+t}, L_{T+t} - L_T)_{t\ge0}$ is independent of $\mathcal{F}_T$ under $P(\cdot \mid T < \infty)$ and distributed as (Y, L) under P.

The local time L is specified by the above conditions up to a constant factor. In the special case when X is spectrally positive, we may choose $L = -I$. The inverse local time $(\tau_l)_{l\ge0}$, which is often referred to as the ladder time process, is defined as

$$\tau_l := \inf\{t \ge 0 \mid L_t > l\}$$

with the convention that $\inf \emptyset = \infty$. It is an increasing and right-continuous process, adapted to the filtration $(\mathcal{F}_{\tau_l})_{l\ge0}$. Moreover, for every $l \ge 0$ with $\tau_l < \infty$, we have $\tau_l \in \mathcal{Z}$, meaning that the range of τ corresponds to the set of real times, at which X reaches new infima. In other words, each element l of the discontinuity set

$$D := \{l \ge 0 \mid \tau_{l-} < \tau_l\}$$

corresponds to the excursion of X over its infimum straddling the interval (τ_{l-}, τ_l). This fact motivates the following definition of the excursion process $(y_l)_{l\geq 0}$ taking values in the excursion space $\mathcal{E} \cup \{\Upsilon\}$, extended by an element Υ as placeholder for the process value at local times, at which no excursion is taking place, and which is defined by: L.R.:Υ ok?

$$\begin{aligned} &\text{If } l \in D, \text{ set } y_l(t) := \begin{cases} Y_{\tau_l+t} & 0 \leq t \leq \tau_l - \tau_{l-} \\ 0 & t > \tau_l - \tau_{l-}. \end{cases} \\ &\text{If } l \notin D, \text{ set } y_l \quad := \Upsilon \end{aligned} \tag{1}$$

By an important result due to Itô (1972), the excursion process $(y_l)_{l\geq 0}$ of the reflected process Y is a $(\mathcal{F}_{\tau_l})_{l\geq 0}$ *Poisson point process* taking values in the excursion space $\mathcal{E} \cup \{\Upsilon\}$. Its intensity measure N, which is called the *excursion measure* of the reflected process Y, is a σ-finite measure on the excursion space $\mathcal{E}$. Let $A \in \mathcal{E}$ be a measurable set with $0 < N(A) < \infty$ and denote on $T_A := \inf\{l \geq 0 \mid y_l \in A\}$ the first entrance time of the excursion process (y_l) in A. Then T_A is an $(\mathcal{F}_{\tau_l})_{l\geq 0}$ stopping time and exponentially distributed with parameter $N(A)$. The first excursion y_{T_A} that falls in the set A is distributed according to the conditioned excursion measure

$$N(\,\cdot \mid A) = \frac{N(A \cap \,\cdot\,)}{N(A)}.$$

Of particular interest is the case of α-stable Lévy processes since their excursions are the only possible limits for random walk excursions. Assuming that $\alpha \in (1, 2]$ such that we have $\rho := P(X_t > 0) \in (0, 1)$ where the positivity parameter ρ does not depend on $t > 0$ and that $|X|$ is no subordinator, then, according to Bertoin (1996, Chap. VIII.4), the distribution of the length of a generic excursion of Y is given by:

$$N(\zeta > s) = \frac{s^{\rho-1}}{\Gamma(\rho)}, \quad \frac{dN^{\zeta}}{d\lambda}(s) = \mathbb{1}_{[0,\infty)}(s)\frac{1-\rho}{\Gamma(\rho)}s^{\rho-2}. \tag{2}$$

In the α-stable case the excursion process N and the local time process have the following scaling properties.

Let X be α-stable with $\alpha \in (1, 2]$ and define for $\lambda > 0$ the scaling transformation $\Phi_\lambda : \mathbb{D} \to \mathbb{D}$ by $\Phi_\lambda : (y_t)_{t\geq 0} \mapsto (\lambda^{-1/\alpha} y_{\lambda t})_{t\geq 0}$, then it holds that

$$L_t(Y) \overset{(d)}{=} L_t\left(\Phi_\lambda(Y)\right) = \lambda^{-1/\alpha} L_{\lambda t}\left(Y\right), \tag{3}$$

$$N(A) = \lambda^{-1/\alpha} N\left(\Phi_\lambda(A)\right). \tag{4}$$

3. Conditional limit theorem for scaled excursions

Let $F : \mathcal{E} \to \mathcal{E}$ be a measurable mapping on the excursion space which is length-preserving, meaning that $\zeta(y) = \zeta(F(y))$, $y \in \mathcal{E}$. Defining $F(\Upsilon) := \Upsilon$, it can be checked easily that

$$(F(y_l))_{l\geq 0}$$

is an $(\mathcal{F}_{\tau_l})_{l\geq 0}$ *Poisson point process* whose intensity measure Δ is the image of N under the mapping F, $\Delta = N^F$. For any $t > 0$, we denote y^t the excursion of Y straddling the timepoint t. Further we define

$$g_t := \sup\{s \leq t \mid Y_s = 0\} \text{ and } d_t := \inf\{s \geq t \mid Y_s = 0\}$$

as the left and right endpoints of the excursion straddling time t and by $\zeta_t = d_t - g_t$ its length. Then we have $y^t = (y^t_s)_{0\leq s\leq \zeta_t} = \left(Y_{g_t+s}\right)_{0\leq s\leq \zeta_t}$. We write $E^{F(Y)}$ for the process which results by concatenating the excursions of y^F. It is defined by

$$E_t^{F(Y)} := \begin{cases} 0, & t \in \mathcal{Z}, \\ F(y^t)_{t-g_t}, & t \notin \mathcal{Z}. \end{cases} \tag{5}$$

Although there is no information about whether $E^{F(Y)}$ possesses Markovian or even Lévy properties, it is clear, that it takes values in the space of càdlàg functions. Defining $(L(Y)_t)_{t\geq 0}$ to be the local time of $E^{F(Y)}$ at zero makes sense since $E_t^{F(Y)} = 0$ iff $Y_t = 0$.

A random excursion is a random variable taking values in the space $\mathcal{E}$. Given a sequence $(e^n)_{n\in\mathbb{N}}$ of random excursions, we define the concatenated process $(E_t)_{t\geq 0}$ taking values in the space $\mathbb{D}$ by setting

$$E_t = e^n_{t-\sum_{k=1}^n \zeta(e^k)} \Leftrightarrow t \in \left[\sum_{k=1}^{n} \zeta(e^k), \sum_{k=1}^{n+1} \zeta(e^k)\right). \tag{6}$$

We then define the local time at zero of E via

$$L_t := \#\{0 \leq s \leq t \mid E_t = 0\}.$$

In the following theorem we derive under some general conditions weak convergence of conditioned random excursions of Lévy processes.

Let $(e^n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random excursions and denote by $(E_t)_{t\geq 0}$ the concatenated process. For a given sequence $a_n \nearrow \infty$ and for $n \in \mathbb{N}$ we denote by

$$\left(e^{(n)}_t\right)_{0\leq t\leq \zeta(e^n)/n} = \left(\frac{1}{a_n} e_{nt}\right)_{0\leq t\leq \zeta(e^n)/n}$$

the scaled excursions, and by

$$\left(E^{(n)}\right)_{t\geq 0} = \left(\frac{1}{a_n} E_{nt}\right)_{t\geq 0}$$

the scaled concatenated process. Finally let

$$\left(L^{(n)}_t\right)_{t\geq 0} := \left(\frac{1}{a_n} L_{nt}\right)_{t\geq 0}$$

be the scaled discrete local time of E.

A measurable set $A \subseteq \mathcal{E}$ is called Δ-*regular*, if

$$\lim_{\varepsilon \searrow 0} \Delta(A_\varepsilon) = \Delta(A),$$

where $A_\varepsilon := \{e \in \mathcal{E} \mid d_{\mathcal{E}}(A, e) < \varepsilon\}$ denotes its ε-enlargement. For a sequence of measurable sets $(A_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ we write $A_{(n)} := \{(\frac{1}{a_n} e_{nt})_{t\geq 0} \mid (e_t)_{t\geq 0} \in A_n\}$ for the corresponding scaled set. The notation $\xrightarrow{d_{\mathcal{H}}}$ means convergence with respect to the Hausdorff topology based on the metric $d_{\mathcal{E}}$.

Theorem 3.1 (Conditional limit theorem for random excursions)**.**
Let $(e^n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random excursions and let X be a Lévy process such that $(-\infty, 0)$ is regular for X. Let $F : \mathcal{E} \to \mathcal{E}$ be measurable and length-preserving such that for a suitable sequence $a_n \nearrow \infty$ it holds that

$$\left(E^{(n)}, L^{(n)}\right) \xrightarrow[n\to\infty]{(d)} \left(E^{F(Y)}, L^{F(Y)}\right) \tag{7}$$

in $\mathbb{D}^2$. Let $A \in \mathcal{B}(\mathcal{E})$ be Δ-regular and open and let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{E})$ with $P(e^n \in A_n) > 0$ for all $n \in \mathbb{N}$, such that the following assumptions hold:

$$A_{(n)} \xrightarrow{d_{\mathcal{H}}} A. \tag{A1}$$

There exists an $\varepsilon_0 > 0$ such that

$$\forall 0 < \varepsilon < \varepsilon_0 : \frac{\Delta(A_{(n)\varepsilon} \cap A_\varepsilon)}{\Delta(A_\varepsilon)} = 1 - g^\varepsilon_n\,, \quad \frac{\Delta(A_{(n)\varepsilon} \cap A_\varepsilon)}{\Delta(A_{(n)\varepsilon})} = 1 - h^\varepsilon_n, \tag{A2}$$

for nonnegative sequences g^ε_n and h^ε_n with $\sum_{n=1}^\infty (g^\varepsilon_n + h^\varepsilon_n) < \infty$. Then the scaled excursion $e^{(n)}$ converges, conditionally on $A_{(n)}$, to the conditional excursion measure, i.e.

$$\mathcal{L}\left(e^{(n)} \;\middle|\; e^{(n)} \in A_{(n)}\right) \xrightarrow{(d)} \Delta(\,\cdot\,|A).$$

Remark 3.1. The assumption that $(-\infty, 0)$ is regular for X has to be stated to ensure the existence of a continuous version of the local time at zero. In the case that e^n are discrete random walk excursions, only α-stable Lévy processes can appear as possible limits in Theorem 3.1 and for $\alpha \in (1, 2]$, this assumption holds true since X is of unbounded variation,

Proof. Since $0 < P(e^n \in A_n) < \infty$, $n \in \mathbb{N}$, we have $\inf\{k \in \mathbb{N} \mid e^{(n)} \in A_{(n)}\} < \infty$ almost surely. Therefore we can consider the first excursion of $E^{(n)}$ that falls in the set $A_{(n)}$, denoting it $e^{(n),A_{(n)}}$. Since the law of $e^{(n),A_{(n)}}$ is the conditioned measure

$$\mathcal{L}\left(e^{(n),A_{(n)}}\right) = \mathcal{L}\left(e^{(n)} \;\middle|\; e^{(n)} \in A_{(n)}\right),$$

and the law of e_{T_A} is given by

$$\mathcal{L}\left(e_{T_A}\right) = \mathcal{L}\left(F(y_{T_{F^{-1}(A)}})\right) = \frac{N(F^{-1}(\cdot \cap A))}{N(F^{-1}(A))} = \Delta\left(\cdot \mid A\right),$$

we need to show that $e^{(n),A_{(n)}} \xrightarrow{(d)} e_{T_A}$.

By the Skorokhod representation theorem, we may assume that (7) holds almost surely; thus being left with showing that $e^{(n),A_{(n)}} \xrightarrow{\text{a.s.}} e_{T_A}$.

Let $(l_A, r_A) := (\tau_{T_A-}, \tau_{T_A})$ be the excursion interval of e_{T_A} and let m_A be any point in its interior. We have $E^{F(Y)}_{m_A}|_{(m_A-\varepsilon, m_A+\varepsilon)} > 0$ for some $\varepsilon > 0$ since $\zeta(e_{T_A}) > 0$ almost surely. Therefore, the convergence $E^{(n)} \xrightarrow{a.s.} E^{F(Y)}$ with respect to the J_1- topology ensures that $E^{(n)}_{m_A} > 0$ for n large enough, as well as the existence of an excursion $\bar{e}^{(n),A}$ of $E^{(n)}$ which straddles m_A. Let $(l_{(n),A}, r_{(n),A})$ be its excursion interval. We first show that $\bar{e}^{(n),A} \xrightarrow{a.s.} e_{T_A}$ with respect to $d_{\mathcal{E}}$ and afterwards, we identify $\bar{e}^{(n),A}$ and $e^{(n),A_{(n)}}$.

For the former convergence with respect to $d_{\mathcal{E}}$ we need to establish J_1-convergence and convergence of the excursion length. We start by showing

$$l_{(n),A} \xrightarrow{a.s.} l_A \quad \text{and} \quad r_{(n),A} \xrightarrow{a.s.} r_A. \tag{8}$$

which implies $\zeta(\bar{e}^{(n),A}) \xrightarrow{n\to\infty} \zeta(e_{T_A})$. The J_1-convergence of $E^{(n)}$ implies that for all $0 < \varepsilon < \zeta(e_{T_A})$ it holds that $E^{(n)}_{l_A+\varepsilon} > 0$ and $E^{(n)}_{r_A-\varepsilon} > 0$ for n large enough:

There exists a sequence of timeshifts λ_n, converging to the identity locally uniformly, such that $E^{(n)}_{\lambda_n(l_A+\delta)} \xrightarrow{a.s.} E^{F(Y)}_{l_A+\delta} > 0$. We can choose n large enough such that $|\lambda_n(l_A+\delta) - l_A| < \varepsilon$ and $E^{(n)}_{\lambda_n(l_A+\delta)} > 0$. The same argument applies to the right excursion endpoint r_A. This gives

$$\liminf_{n\to\infty} r_{(n),A} \geq r_A \text{ and } \limsup_{n\to\infty} l_{(n),A} \leq l_A. \tag{9}$$

By the definition of the local time at zero of a Markov process, the definition of L and $E^{F(Y)}$, we have $\operatorname{supp}(dL) = \{s \,| E^{F(Y)}_s = 0\}$. Since $E^{F(Y)}_{l_A} = E^{F(Y)}_{r_A} = 0$, it holds that for all $\varepsilon > 0$:

$$L^{F(Y)}_{l_A+\varepsilon} - L^{F(Y)}_{l_A} > 0 \text{ and } L^{F(Y)}_{r_A} - L^{F(Y)}_{r_A-\varepsilon} > 0. \tag{10}$$

Since $L^{(n)} \xrightarrow{a.s.} L^{F(Y)}$ in the J_1-topology, we can once again choose n large enough and $\delta > 0$ such that $|\lambda_n(l_A+\delta) - l_A| < \varepsilon$ and $L^{(n)}_{\lambda_n(l_A)+\delta} - L^{(n)}_{l_A} > 0$ by (10). This gives $L^{(n)}_{l_A+\varepsilon} - L^{(n)}_{l_A} > 0$ for n large enough and the same way, it holds that $L^{(n)}_{r_A} - L^{(n)}_{r_A-\varepsilon} > 0$ for n large enough. On the other hand, by definition, $L^{(n)}_t = \frac{1}{a_n}\#\{0 \leq s \leq t \mid E^{(n)}_s = 0\}$. Therefore, for all $\varepsilon > 0$ there exist some $s, t \in [0, \varepsilon)$ such that $E^{(n)}_{l_A+t} = E^{(n)}_{r_A+s} = 0$ for n large enough. This implies

$$\liminf_{n\to\infty} l_{(n),A} \geq l_A \text{ and } \limsup_{n\to\infty} r_{(n),A} \leq r_A,$$

and together with (9), we arrive at (8). Together with $E^{(n)} \xrightarrow{a.s.} E^{F(Y)}$ it follows, that

$$\bar{e}^{(n),A} \xrightarrow{a.s.} e_{T_A} \tag{11}$$

with respect to $d_{\mathcal{E}}$.

It only remains to be shown that for n large enough $\bar{e}^{(n),A}$ coincides with $e^{(n),A_{(n)}}$. In other words we need to show that the excursion of $E^{(n)}$ which straddles m_A is the first one to be in the set $A_{(n)}$ for n large enough.

We first claim that almost surely for ε small enough and n large enough it holds that $e_{T_{A_{(n),\varepsilon}}} = e_{T_A}$. To be precise, we want to establish

$$P(\liminf_{k\to\infty} \liminf_{n\to\infty} \{e_{T_{A_{(n),\varepsilon_k}}} = e_{T_A}\}) = 1$$

for a suitable sequence $\varepsilon_k \searrow 0$. This can be shown by proving the following two statements:

$$P\left(\liminf_{k\to\infty} \{T_{A_{\varepsilon_k}} = T_A\}\right) = 1, \tag{12}$$

$$P\left(\liminf_{k\to\infty} \liminf_{n\to\infty} \{T_{A_{(n),\varepsilon_k}} = T_{A_{\varepsilon_k}}\}\right) = 1. \tag{13}$$

Both statements follow from the regularity of the set A and the assumptions (A1) and (A2). As A is regular, we have that $\lim_{\varepsilon\searrow 0} \Delta(A_\varepsilon) = \Delta(A)$. By choosing a sequence $(\varepsilon_k)_{k\geq 1}$ such that $1 - \frac{\Delta(A)}{\Delta(A_{\varepsilon_k})} \leq 2^{-k}$, it follows that

$$P(T_{A_{\varepsilon_k}} \neq T_A) = P(e_{T_{A_{\varepsilon_k}}} \notin A) = 1 - \Delta(A \mid A_{\varepsilon_k}) \leq 2^{-k}.$$

By Borel–Cantelli we have $P(\limsup_{k\to\infty} T_{A_{\varepsilon_k}} \neq T_A) = 0$ which shows (12). To verify Equation (13) we show that

$$\forall\, 0 < \varepsilon < \varepsilon_0 : \; P(\limsup_{n\to\infty} \{T_{A_{(n)\varepsilon}} \neq T_{A,\varepsilon}\}) = 0.$$

This follows since

$$\begin{aligned} P(T_{A_{(n),\varepsilon}} \neq T_{A,\varepsilon}) &\leq P(e_{T_{A_\varepsilon}} \notin A_{(n),\varepsilon}) + P(e_{T_{A_{(n),\varepsilon}}} \notin A_\varepsilon) \\ &= 2 - \frac{\Delta(A_{(n),\varepsilon} \cap A_\varepsilon)}{\Delta(A)} - \frac{\Delta(A_{(n),\varepsilon} \cap A_\varepsilon)}{\Delta(A_{(n),\varepsilon})} \\ &= h_n^\varepsilon + g_n^\varepsilon \end{aligned}$$

is summable according to Assumption (A2).

We now show convergence of the zero-set before time T_A similar as already done before. The almost sure convergence $L^{(n)} \to L$ together with the fact that

$$\mathcal{Z} := \operatorname{supp}(dL) = \{t \geq 0 \mid E^{F(Y)} = 0\} \tag{14}$$

$$\mathcal{Z}^{(n)} := \operatorname{supp}(dL^{(n)}) = \{t \geq 0 \mid E^{(n)} = 0\} \tag{15}$$

implies the convergence of the zero-sets. Identity (14) implies that for any $\alpha \in \mathcal{Z}^{(n)}$ it holds that

$$\text{for all } \varepsilon > 0 \text{ and } n \in \mathbb{N} : \; L^{(n)}_{\alpha+\varepsilon} - L^{(n)}_{\alpha-\varepsilon} > 0 \text{ a.s.} \tag{16}$$

Let $0 < K < \infty$. The convergence $L^{(n)} \xrightarrow{a.s.} L^{F(Y)}$ for the Skorokhod topology ensures the existence of a sequence of time shifts $\lambda_n : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lambda_n(0) = 0$, $\lambda_n(t) \nearrow \infty$ for $t \nearrow \infty$, such that

$$\sup_{t\geq 0} |\lambda_n(t) - t| \xrightarrow{n\to\infty} 0, \text{ and } \sup_{t\leq K} \left| L^{(n)}_{\lambda_n(t)} - L_t \right| \xrightarrow{n\to\infty} 0. \tag{17}$$

The first part in (17) gives, that for any fixed $\varepsilon > 0$ we can choose n large enough such that $\lambda_n(\alpha+\varepsilon) > \alpha$ and $\lambda_n(\alpha-\varepsilon) < \alpha$ for all $\alpha \in \mathcal{Z}^{(n)}$ and, thanks to (16), $L^{(n)}_{\lambda_n(\alpha+\varepsilon)} - L^{(n)}_{\lambda_n(\alpha-\varepsilon)} > 0$ for $\alpha \leq K$. Since the second part of (17) gives, that $L^{(n)}_{\lambda_n(t+\varepsilon)} - L^{(n)}_{\lambda_n(t-\varepsilon)} \xrightarrow{n\to\infty} L_{t+\varepsilon} - L_{t-\varepsilon}$ uniformly on compact sets, we have $L_{\alpha+\varepsilon} - L_{\alpha-\varepsilon} > 0$ for n large enough and all $\alpha \in \mathcal{Z}^{(n)}$ with $\alpha \leq K$.

By (14), for any $\varepsilon > 0$, we can choose n large enough, such that for all $\alpha \in \mathcal{Z}^{(n)},\ \alpha \leq K$, there exist zeros of $E^{F(Y)}$ in the intervals $(\alpha + \varepsilon, \alpha - \varepsilon)$, resulting in $\mathcal{Z}^{(n)} \cap [0, K] \subseteq B_\varepsilon(\mathcal{Z}) \cap [0, K]$.

In same way, one can show that almost surely, for all $\varepsilon > 0$ we have $B_\varepsilon(\mathcal{Z}) \cap [0, K] \subseteq \mathcal{Z}^{(n)} \cap [0, K]$ for n large enough. In total, we get almost sure Hausdorff convergence of the zero-sets:

$$\mathcal{Z}^{(n)} \cap [0, K] \xrightarrow{a.s.} \mathcal{Z} \cap [0, K], \quad 0 < K < \infty. \tag{18}$$

This fact, together with $E^{(n)} \to E^{F(Y)}$ implies the following. For any $\varepsilon_0 > 0$ and n large enough, and for any excursion $e^{(n)}$ of $E^{(n)}$ before T_A, there exists an excursion e of $E^{F(Y)}$ before T_A, such that $d_{\mathcal{E}}(e^{(n)}, e) < \varepsilon_0$. But since $e_{T_{A_{(n),\varepsilon}}} = e_{T_A}$ we have $e \notin A_{(n),\varepsilon_0}$ almost surely and therefore, $e^{(n)} \notin A_{(n)}$ for n large enough, almost surely.

It remains to be shown, that $\bar{e}^{(n),A} \in A_{(n)}$ for n large enough, almost surely. As A is open and $\bar{e}^{(n),A} \xrightarrow{n\to\infty} e_{T_A} \in A$, we have that for n large enough, $d_{\mathcal{E}}(\partial A, \bar{e}^{(n),A}) > \delta$ for some $\delta > 0$. On the other hand, for n large enough, the Hausdorff-convergence $A_n \xrightarrow{d_{\mathcal{H}}} A$ ensures, that $A \subseteq A_{(n),\delta}$. It follows, that $B_\delta(\bar{e}^{(n),A}) \subseteq A \subseteq A_{(n),\delta}$ and therefore $\bar{e}^{(n),A} \in A_{(n)}$. We now have $\bar{e}^{(n),A} = e^{(n),A_{(n)}}$ as desired. □

Remark 3.2. Denote by $\mathcal{S}$ the space of compact subsets of $\mathbb{R}_+$. From the proof of Theorem 3.1 we see that the Hausdorff-convergence of the zero sets in (18) is essential. In consequence assumption (7) can be replaced by the condition

$$(E^{(n)}, \mathcal{Z}^{(n)}) \xrightarrow{(d)} (E^{F(Y)}, \mathcal{Z}) \text{ in the product space } \mathbb{D} \times \mathcal{S}, \tag{19}$$

where in the second coordinate, the zero-sets $\mathcal{Z}^{(n)}$ converge as random closed sets, cf. Molchanov (1993), which is the same as requesting convergence of the random measures $dL^{(n)}$, see Molchanov (1962, Prop. 8.16). The topology is taken to be the product topology of the Skorokhod topology on $\mathbb{D}$ and the topology induced by $d_{\mathcal{H}}$ on $\mathcal{S}$. The latter is sometimes referred to as the *hit-or-miss-topology*, cf. Molchanov (1993, Chap. 1).

We next discuss the conditions of Theorem 3.1. In order to meet the convergence condition of Theorem 3.1, it is crucial to establish joint convergence of the concatenated process and its local time at zero. A useful result to that purpose due to Chaumont and Doney (2009) gives a connection between convergence of a random walk and joint convergence with its local time at the supremum.

Theorem 3.2 (Chaumont and Doney (2009, Thrm. 2, Cor. 1))**.** *Let X be any Lévy process, such that* 0 *is regular for* $(0, \infty)$ *and assume that there exists a sequence of random walks $S^{(n)}$, such that $S^{(n)} \xrightarrow{a.s.} X$. Then it holds that*

$$\left(S^{(n)}_{\lfloor nt \rfloor}, \frac{1}{a_n}\widehat{L}^{(n)}_{\lfloor nt \rfloor}\right)_{t\geq 0} \xrightarrow{(d)} (X_t, \widehat{L}_t)_{t\geq 0}. \tag{20}$$

Here, $S^{(n)}$ is already considered as a normed sequence and $\widehat{L}^{(n)}$ and $\widehat{L}$ refer to the local time at the supremum of $S^{(n)}$ and X respectively, meaning $\widehat{L}_k := \#\{j \in \{1, \ldots, k\} \mid S_{j-1} < S_j, S_j = \max_{i\leq j} S_i\}$ and $\widehat{L}_t := L^0((\sup_{0\leq s\leq t} X_s - X_t)_{t\geq 0})$.

As a consequence of Theorem 3.2 we get that convergence of the random walk reflected at its supremum to the Lévy process reflected at its supremum holds jointly with convergence of the respective local times. By reflection, the above limit theorem also holds for Lévy processes reflected at their infimum, given that 0 is regular for $(-\infty, 0)$. Several further related results have been established in the article, for example an extension of the above theorem, where the sequence of random walks $S^{(n)}$ is replaced by a sequence of scaled Lévy processes $X^{(n)}$.

The following Lemma allows us to apply a continuous mapping theorem together with Theorem 3.2 in order to verify the assumptions of Theorem 3.1 in different cases. Several examples will be discussed subsequently.

Lemma 3.3. *Let $F : \mathcal{E} \to \mathcal{E}$ be continuous and length preserving. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg functions in $\mathbb{D}$ with $h_n \xrightarrow{n\to\infty} h$ for some $h \in \mathbb{D}$ with respect to the Skorokhod topology. Denote*

$$\mathcal{Z}^K = \{t \leq K \mid h(t) = 0\} \text{ and } \mathcal{Z}^K_n = \{t \leq K \mid h_n(t) = 0\}$$

the zero-sets of h_n and h respectively. Further assume, that for all $0 < K < \infty$, it holds that $\mathcal{Z}^K_n \xrightarrow{n\to\infty} \mathcal{Z}^K$ with respect to $d_{\mathcal{H}}$. Then it holds that

$$h^F_n \xrightarrow{n\to\infty} h^F$$

with respect to the Skorokhod topology.

4. Random walk excursions, applications to queuing theory

Based on the Chaumont and Doney (2009) Theorem (Theorem 3.1) we apply in this section the conditional limit theorem to random walk excursions. Our result is an extension of an early result of Doney (1985) who considers the case of sationary conditioning sets. As a motivation we discuss some applications to queuing theory. The following example shows the relevance of reflected random walks and is excursions in the simplest case of single server queues.

Example 4.1 (Typical busy periods in single server queues). In a single server queue with one serving line (Lindley model) the waiting time for customer k between his arrival in the queue until the time being served can be written as

$$W_1 = W_0 = 0, \quad W_k = \max\{0, W_{k-1} + M_k\}, \ k \geq 2, \tag{21}$$

where $M_k = A_k - B_k$ is the difference between the serving time A_k that is needed to serve customer $k-1$ and B_k is the time between the arrivals of customer $k-1$ and k. We assume that the driving sequence $M = (M_k)_{k\geq 1}$ of the waiting line W is an iid sequence. Then, the waiting time W_k can be rewritten as

$$W_k = S_k - \inf_{0\leq j\leq k} S_j, \ \text{where } S_k := \sum_{j=1}^{k} M_j, \ S_0 = 0, \tag{22}$$

and thus yields a representation as a reflected random walk whose excursions will be referred to as *runs*. Each excursion stands for a *busy period* in the queue. Let $T_1 := 0$ and recursively define

$$\begin{aligned} U_k &:= \sup\{j \geq T_k \mid W_{T_k} = W_{T_k+1} = \cdots = W_{j-1} = 0, \ W_j > 0\}, \ k \geq 1 \\ T_{k+1} &:= \inf\{j > U_k \mid W_j = 0\}, \ k \geq 1 \end{aligned} \tag{23}$$

to get a sequence of runs respectively busy periods:

$$\left(w_t^k\right)_{0\leq t\leq T_{k+1}-T_k} := \left(W_{T_k+t}\right)_{0\leq t\leq T_{k+1}-T_k}, \quad k \in \mathbb{N}. \tag{24}$$

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Let $S_0 = 0, S_n := \sum_{k=1}^{n} M_k$ a random walk with iid increments M_k in the domain of attraction of a stable law with index $\alpha \in (1,2]$ and with norming sequence $(a_n)_{n\in\mathbb{N}}$. Then by a well known limit theorem of Skorokhod (1957)

$$\left(\frac{1}{a_n} S_{\lfloor nt\rfloor}\right)_{t\geq 0} \xrightarrow{(d)} (X_t)_{t\geq 0} \text{ in } in\mathbb{D},$$

where X is an α-stable Lévy process. Denote as in (22) by

$$W_k = S_k - \inf_{0\leq j\leq k} S_j$$

the reflected random walk. Since X has no fixed discontinuities Proposition 2.4 in Jacod and Shiryaev (2002, Chap. VI) and the continuous mapping theorem, cf. Billingsley (1968, Thrm. 5.1), imply

$$\left(\frac{1}{a_n} W_{\lfloor nt\rfloor}\right)_{t\geq 0} \xrightarrow{(d)} (Y_t)_{t\geq 0}, \tag{25}$$

where $Y = X - I$ is the reflected Lévy process.

Thus Theorem 3.2 implies with help of the Skorokhod representation theorem joint convergence

$$\left(\frac{1}{a_n} W_{\lfloor nt\rfloor}, \frac{1}{a_n} \bar{L}_{\lfloor nt\rfloor}\right)_{t\geq 0} \xrightarrow{(d)} (Y_t, L_t)_{t\geq 0}, \tag{26}$$

where $\bar{L}_k$ refers to the discrete local time at zero of W.

For the reflected random walk W define the sequence of empty queues (U_k), of waiting times T_k and of runs (i.e. excursions) (w^k) as in (23) and (24).

Due to the strong Markov property of the reflected random walk, the sequence of random walk excursions $\left(w_t^k\right)_{0\le t\le T_{k+1}-T_k}$, $k\in\mathbb{N}$, forms an independent sequence of identically distributed excursions, taking values in the excursion space $\mathcal{E}$ and their concatenation forms the walk W. As consequence of (26) Theorem 3.1 is applicable to random walk excursions. It gives an extension of the main theorem of Doney (1985) and allows to deduce conditional limit distributions for typical busy periods.

In the following corollary we consider the typical busy period when the length is known to be large or when the maximum weighting time is known to be large. Denote by

$$\left(w_t^{(n)}\right)_{0\le t\le\zeta(w^{(n)})} = \left(\frac{1}{a_n}w_{\lfloor nt\rfloor}\right)_{0\le t\le\zeta(w)/n}$$

the scaled random walk excursion, i.e. the scaled typical busy period. For $y\in\mathcal{E}$ we denote by $\bar{y} := \sup_{0\le t\le\zeta(y)} y_t$ the height of the excursion.

Theorem 4.1 (Conditioning of random walk excursions on length and height)**.** *Let $(t_n)_{n\in\mathbb{N}}$ be a sequence nonegative real numbers with $\lim_{n\to\infty} t_n = t < \infty$ such that $\sum_{n=1}^{\infty}|t-t_n| < \infty$. Then the scaled random walk excursions converge to the conditional excursion measures, i.e.*

$$\mathcal{L}(w^{(n)} \mid \zeta(w) > n\cdot t_n) \xrightarrow{(d)} N(\,\cdot\,|\zeta(y) > t\,) \tag{27}$$

$$\mathcal{L}(w^{(n)} \mid \bar{w} > a_n\cdot t_n) \xrightarrow{(d)} N(\,\cdot\,|\bar{y} > t\,) \tag{28}$$

where N is the excursion measure of the reflected α-stable Lévy process Y.

In the above statement, F was chosen to be the identity. The verification of the convergence condition (A1) and the regularity condition (A2) of the assumptions of Theorem 3.1 is similar as in Examples 5.1 and 5.2 in Section 5 and therefore is omitted.

In the following application we consider the cumulated waiting time process c as an excursion which keeps track of the total waiting time of all customers up to the specific time point. Define the cumulative waiting times

$$c_k := \sum_{j=0}^{k} w_j,\quad k = 0,\ldots,\zeta(w)$$

and denote the scaled waiting time process by

$$c_t^{(n)} := \frac{1}{a_n}c_{\lfloor nt\rfloor} = \int_0^t w_s^{(n)}\,ds,\quad 0\le t\le\zeta(w^{(n)}). \tag{29}$$

In order to apply Theorem 3.1, we need to verify that the map

$$I:\mathcal{E}\to\mathcal{E},\ f\mapsto\left(\int_0^t f(s)\,ds\right)_{0\le t\le\zeta(f)} \tag{30}$$

is measurable with respect to $d_{\mathcal{E}}$. Clearly, I maps to $\mathcal{E}$ and is length preserving. So measurability with respect to the Skorokhod topology remains to be shown. We show that it is even continuous.

Lemma 4.2. *The mapping*

$$I:\mathcal{E}\to\mathcal{E},\ f\mapsto\left(\int_0^t f(s)\,ds\right)_{0\le t\le\zeta(f)} \tag{31}$$

is continuous with respect to $d_{\mathcal{E}}$.

Since we have $c^{(n)} = I(w^{(n)})$, we obtain with help of Remark 3.3, the convergence in (25) and the continuous mapping theorem, that

$$W^{(n),I} \xrightarrow{(d)} Y^I.$$

The constant conditioning sequence of sets $A_n = A = \{\zeta > t\}$ obviously satisfies conditions (A1) and (A2) and is regular thanks to (2), remembering $\zeta(w) = \zeta(I(w))$. Thus an application of Theorem 3.1 results in the following corollary, which states, that given a busy period is long, its cumulative waiting times converge in distribution to the integral function of a Lévy excursion, given its duration is long:

Corollary 4.3 (Limit theorem for the cumulated waiting time conditioned on the length)**.** *Let $(c_t)_{0\leq t\leq\zeta(c)}$ denote the cumulated waiting time process as defined in* (29)*. Then*

$$\mathcal{L}\left(\left(c_t^{(n)}\right)_{0\leq t\leq\zeta(c)/n}\,\middle|\,\zeta(w)>n\right)\xrightarrow{(d)} N\left(\left(\int_0^t y_s\,ds\right)_{0\leq t\leq\zeta(y)}\,\middle|\,\zeta>1\right)$$

where N is the excursion measure related to the reflected α-stable Lévy process Y.

It seems possible to derive the results in this section also for several of the classical extensions of the single server queues.

5. Limit theorems for conditional Galton–Watson trees

Central applications of the conditional limit theorem for random excursions concern critical Galton–Watson trees where the mapping F refers to mapping excursions of the reflected process Y to excursions of the height process H. In order to make the paper selfcontained readable we start by introducing some basic notions for trees and introduce to the various processes coding a random tree.

A discrete, finite, rooted tree is a connected, acyclic, plane graph with a distinguished vertex ρ called the root. For a given tree τ, we can order its vertices according to depth-first search order $\prec$ and indicate this ordering with help of indices: $v_0 \prec \cdots \prec v_{|\tau|-1}$ where we write $|\tau|$ for the number of vertices and $|v|$ of the depth for a vertex v in the tree. We briefly introduce two of the various *coding processes* of a discrete tree. The codings shall be understood as mappings or walks on the real line, inheriting some or all of the genealogical information of a given tree.

Profile process: The profile process $(Z_j)_{j\geq 0}$, coding the generations of a tree τ keeps track of the generation sizes as follows:

$$Z_j = \#\{v\in\tau \mid |v| = j\},\ \ j\geq 0. \tag{32}$$

Random walk: For every $v\in\tau$, $s(v) := \#\{n\in\mathbb{N} \mid v.n\in\tau\}$ denotes by the number of children of v in τ. The random walk $(S_j)_{0\leq j\leq|\tau|}$ coding a tree τ is defined as

$$\begin{aligned}
S_0 &:= 0\\
S_j &:= \sum_{i=1}^{j}\left(s(v_{j-1})-1\right),\ 1\leq j\leq|\tau|.
\end{aligned} \tag{33}$$

The random walk, also known as *Harris walk* or *Lukasiewicz path*, keeps track of the overall excession of individuals in each generation. This means, that if a vertex v has $s(v)$ children, the next generation grows by $s(v)-1$ vertices compared to the generation of v. The property

$$\begin{aligned}
&S_0 = 0,\\
&S_j \geq 0,\ \ j=1,\dots,|\tau|-1,\\
&S_{|\tau|} = -1,
\end{aligned} \tag{34}$$

holds and we have $\min\{0 \leq j \leq |\tau| \mid S_j = -1\} = |\tau|$. Therefore, the random walk of each finite tree forms an excursion starting at 0 and ending at $|\tau|$ in -1.

Height process: The height process $(H_j)_{0\leq j\leq|\tau|-1}$ coding a tree τ keeps track of the depth of vertices, visited in depth-first search order:

$$H_j := |v_j|, \ \ 0 \leq j \leq |\tau| - 1.$$

For $j \geq 0$ let $(\widehat{S}^{(j)}_k)_{0\leq k\leq j}$ be the reflected random walk defined by $\widehat{S}^{(j)}_k = S_j - S_{j-k}$. In this context, reflected means, that the trajectories of $\widehat{S}^{(t)}$ are the trajectories of S reflected at the point $(\frac{j}{2}, \frac{S_j}{2})$. Then it holds that

$$H_j = \#\{0 \leq k \leq j-1 \mid S_k = \min_{k\leq i\leq j} S_i\} \tag{35}$$

$$= \#\{1 \leq k \leq j \mid \widehat{S}^{(j)}_k = \max_{0\leq i\leq k} \widehat{S}^{(j)}_i\}. \tag{36}$$

Contour process: We describe the contour process in an informal way by assuming the tree being embedded in the plane such that all edges have length one. We start at time zero and visit each vertex of the tree in depth-first search order by walking along the edges with speed 1, always taking the shortest (unique) path from v_i to v_{i+1} and assigning to each time point j by $\kappa(j)$ the vertex which is visited at that time. The contour process $(C_j)_{0\leq j\leq(2|\tau|-1)}$ coding a tree τ is then defined by

$$C_j := |\kappa(j)|, \ 0 \leq j \leq 2(|\tau| - 1).$$

The coding functions S, H and C entail the entire information of the tree, meaning that there is a unique way to reconstruct the tree when given one of the coding functions. They entail specific information on the tree as functionals. Thus for example the height and width of the tree, the number of leaves and relationships like ancestral lines, latest common ancestors or distances between vertices are functionals of the coding functions.

The profile process Z in contrast just keeps track of the successive generation sizes of the tree and therefore entails much less information about the tree structure.

Let $(Z_k)_{k\geq 0}$ be a *Galton–Watson process* with reproduction law $(\mu_j)_{j\geq 0}$, started at $Z_0 = 1$, meaning that $Z_{k+1} := \sum_{i=1}^{Z_k} X^k_i$ for $(X^k_i, i \geq 1, k \geq 1)$ a family of independent, identically distributed and integer valued random variables with $\mu_j := P(X^k_i = j)$ and denote by $\tau = \tau^Z$ the corresponding (random) Galton–Watson tree. We assume that the reproduction law μ is critical, $\mathrm{E}[Z_1] = 1$, with $\mu_0 + \mu_1 < 1$ and aperiodic, $\gcd\{i \in \mathbb{N} \mid \mu_i > 0\} = 1$ and further, that the jump law ν where $\nu_j := \mu_{j-1}, \ j \geq -1$ is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$.

A sequence of iid Galton–Watson trees $(\tau^i)_{i\in\mathbb{N}}$ will be called a *Galton–Watson forest* and the coding of the forest is obtained by the concatenation of the coding functions:

$$\bar{S}_k := S^{j+1}_{k-\sum_{i=1}^{j}|\tau^i|} - j \qquad \text{for } k \in \left[\sum_{i=1}^{j}|\tau^i|, \sum_{i=1}^{j+1}|\tau^i|\right), \tag{37}$$

$$\bar{H}_k := H^{j+1}_{k-\sum_{i=1}^{j}|\tau^i|} \qquad \text{for } k \in \left[\sum_{i=1}^{j}|\tau^i|, \sum_{i=1}^{j+1}|\tau^i|\right), \tag{38}$$

$$\bar{C}_k := C^{j+1}_{k-2\sum_{i=1}^{j}|\tau^i|-j} \qquad \text{for } k \in \left[2\sum_{i=1}^{j}|\tau^i| - j, 2\sum_{i=1}^{j+1}|\tau^i| - j - 1\right). \tag{39}$$

The Lévy height process was introduced in Le Gall and Le Jan (1998) and Duquesne and Le Gall (2002). Throughout this section, we assume that X is a spectrally positive Lévy process, does not drift to $+\infty$ and that its Laplace exponent ψ satisfies $\int_1^\infty \frac{1}{\psi(\lambda)}\, d\lambda < \infty$.

For each $t \geq 0$ we set

$$\widehat{X}^{(t)}_s := X_t - X_{(t-s)-} \ 0 \leq s < t, \tag{40}$$
$$\widehat{X}^{(t)}_t := X_t.$$

Then a duality property holds and there is equality in distribution:

$$(X_s)_{0\le s\le t} \overset{(d)}{=} (\widehat{X}^{(t)}_s)_{0\le s\le t}. \tag{41}$$

For $s \le t$, we set $\widehat{S}^{(t)}_r := \sup_{0\le s\le r} X^{(t)}_s$ and note that the number of right-minima of X equals the number of records of $\widehat{X}^{(t)}$:

$$\{0 \le s \le t \mid X_s = \inf_{s\le r\le t} X_r\} = \{0 \le s \le t \mid \widehat{X}^{(t)}_s = \sup_{0\le r\le s} \widehat{X}^{(t)}_r\}. \tag{42}$$

This leads to the definition of the *Lévy height process* at time t as the number of right minima of the reflected random walk up to time t (in analogy to (35)):

$$H_t := L_t\left(\widehat{X}^{(t)} - \widehat{S}^{(t)}\right), \tag{43}$$

where $L_t(\widehat{X}^{(t)} - \widehat{S}^{(t)})$ is the local time at time t of the process $\widehat{X}^{(t)} - \widehat{S}^{(t)}$. Note that due to (41) and Bertoin (1996, Chap. VI.1, Prop.1), the process $(\widehat{X}^{(t)} - \widehat{S}^{(t)})_{0\le s\le t}$ possesses the strong Markov property and therefore, the existence of a process $(H_t)_{t\ge0}$ defined as above follows.

Remark 5.1 (The height process in the Brownian case). We note that $\widehat{X}^{(t)}$ possesses no positive jumps only in the Brownian case and therefore $\widehat{S}^{(t)}$ is a local time for $\widehat{X}^{(t)} - \widehat{S}^{(t)}$. Writing

$$\widehat{S}^{(t)}_t = \sup_{0\le r\le t} \widehat{X}^{(t)}_r = \sup_{0\le r\le t} X_t - X_{(t-r)} = X_t - I_t,$$

we see that in the Brownian case it holds that $(H_t)_{t\ge0} = (X_t - I_t)_{t\ge0}$ and the height process is the reflected Brownian motion.

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The normalization of H_t is given by the representation

$$H_t = \lim_{\epsilon\searrow0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{\{X_s < I^t_s + \epsilon\}} \; ds$$

according to Le Gall and Le Jan (1998, Sec. 4.3) and Duquesne and Le Gall (2002, Lem. 1.2.1.), where the limit holds as convergence in distribution. Moreover there exists a continuous modification of $(H_t)_{t\ge0}$.

Remark 5.2. Remember that g_t denotes the left endpoint of the excursion y^t of $X - I$ which straddles t. As pointed out in Duquesne and Le Gall (2002, Chap. 1), we can verify that H_t does only depend on the excursion of $Y = X - I$ straddling t and that we can write

$$h^t = F(y^t)$$

as a measurable mapping of the corresponding excursion of Y. Moreover it holds that

$$H_t = 0 \Leftrightarrow Y_t = 0$$

and it is therefore natural to define

$$L_t(H) := L_t(Y).$$

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Denote by $\bar{S}, \bar{H}, \bar{C}$ and $\bar{Z}$ the coding processes of a μ-Galton–Watson forest. Let $(h_k)_{k\ge0}$, $(c_k)_{k\ge0}$ and $(s_k)_{k\ge0}$ be the height-, contour- and random-walk coding excursions of the μ-Galton–Watson tree (see (37)–(39)) and denote by

$$h^{(n)}_t := \frac{a_n}{n} h_{\lfloor nt\rfloor},\ s^{(n)}_t := \frac{1}{a_n} s_{\lfloor nt\rfloor},\ c^{(n)}_t := \frac{a_n}{n} c_{\lfloor 2nt\rfloor},\ t \ge 0.$$

the rescaled excursion processes. Let Δ be the excursion measure of the Lévy height process H and let N be the excursion measure of $Y = X - I$. The following conditional convergence result states, that Theorem 3.1 may be applied to a sequence of rescaled height- respectively contour-excursions of a Galton–Watson tree, in the general setting where the offspring distribution of the reproduction law may be of infinite variance and are in the domain of attraction of a stable law with index $\alpha \in (1, 2]$.

By Duquesne and Le Gall (2002) the basic necessity of convergence of the rescaled height process as well as the rescaled contour process has already been established. Also note the overview in the preface of this thesis to see, that in Duquesne and Le Gall (2002, Prop. 2.5.2), conditioned convergence of the Galton–Watson coding processes, given the height of the tree, and in Duquesne (2003, Thrm. 3.1) conditioned convergence given the total population was established. These specific results, respectively the non-degenerate versions, are given in the examples following Theorem 5.1.

Theorem 5.1 (Conditional convergence of the rescaled coding processes)**.** *For a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{E})$ and an open set $A \in \mathcal{B}(\mathcal{E})$ such that the Assumptions* (A1) *and* (A2) *of Theorem 3.1 are satisfied, the following conditional convergence result holds:*

$$\mathcal{L}(h^{(n)} \mid h^{(n)} \in A_{(n)}) \xrightarrow{(d)} \Delta(\,\cdot\,|A), \tag{44}$$

$$\mathcal{L}(c^{(n)} \mid h^{(n)} \in A_{(n)}) \xrightarrow{(d)} \Delta(\,\cdot\,|A), \tag{45}$$

$$\mathcal{L}(s^{(n)} \mid s^{(n)} \in A_{(n)}) \xrightarrow{(d)} N(\,\cdot\,|A). \tag{46}$$

Proof. The theorem is a consequence of Theorem 3.1 and of the fact that

$$\left(\frac{1}{a_n}\bar{S}_{\lfloor nt\rfloor}, \frac{a_n}{n}\bar{H}_{\lfloor nt\rfloor}, \frac{a_n}{n}\bar{C}_{\lfloor 2nt\rfloor}\right)_{t\geq 0} \xrightarrow{(d)} (X_t, H_t, H_t)_{t\geq 0}, \tag{47}$$

where $(H_t)_{t\geq 0}$ denotes the Lévy height process of the spectrally positive, α-stable limiting Lévy process X. Convergence in (47) holds in $\mathbb{D}(\mathbb{R}^3)$ as is proven in Duquesne and Le Gall (2002). We make use of the joint convergence $(\frac{1}{a_n}\bar{S}_{\lfloor nt\rfloor}, \frac{a_n}{n}\bar{H}_{\lfloor nt\rfloor})_{t\geq 0} \xrightarrow{(d)} (X_t, H_t)_{t\geq 0}$, and note that the local time equals the negative infimum of the spectrally positive Lévy process $L(H) = L(Y) = -I$. In terms of the coding processes, due to (37) and (38), it holds that the local time satisfies the same relation $\bar{L}(H)_k = -\inf_{0\leq j\leq k}\bar{S}_j$. With help of a continuous mapping theorem, we get the joint convergence

$$\left(\frac{a_n}{n}\bar{H}_{\lfloor nt\rfloor}, \frac{1}{a_n}\bar{L}(H)_{\lfloor nt\rfloor}\right)_{t\geq 0} \to (H_t, L_t(H))_{t\geq 0} \tag{48}$$

In consequence the assumptions of Theorem 3.1 are fulfilled and we get (44) and (46) by choosing $F = \mathrm{id}$ respectively F as in Remark 5.2. Then (45) follows from (44) in the same manner as in Duquesne and Le Gall (2002, Thrm. 2.4). $\square$

As previously remarked we give two specific applications of Theorem 5.1 to conditioning to the height or size of a critical Galton–Watson tree. These conditionings have already been treated in Duquesne and Le Gall (2002) and Duquesne (2003), except that one may vary the size respectively height conditions with n as long as the sets meet the assumptions of Theorem 5.1.

Example 5.1 (Conditioning on the length). Let τ be a Galton–Watson tree whose reproduction law satisfies the stated assumptions in this section. We condition its law on its size being larger than $n \cdot t_n$, where $(t_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative reals, such that $t_n \xrightarrow{n\to\infty} t$ and in addition, $\sum_{n=1}^{\infty}|t_n - t| < \infty$.

We apply Theorem 5.1, (44) to the height process and the sets $A_n = \{\zeta > n \cdot t_n\}$ respectively the scaled sets $A_{(n)} = \{\zeta > t_n\}$ and the limit set $A = \{\zeta > t\}$. Alternatively, one could use (46) since the lengths of excursions of the discrete random walk and the discrete height process both coincide with the size of the tree. We can also directly apply (45) to the contour process of the tree, hereby noting that due to the scaling, we have $2|\tau| = \zeta$ and so use $A_n = \{\zeta > 2 \cdot n \cdot t_n\}$ instead.

Condition (A2) is easily verified with help of (2) which specifies the law of ζ under N respectively Δ. We have $A_{(n),\varepsilon} = \{\zeta > t_n - \varepsilon\}$ and $A_\varepsilon = \{\zeta > t - \varepsilon\}$ and this gives $A_{(n),\varepsilon} \cap A_\varepsilon = \{\zeta > (t_n \vee t) - \varepsilon\}$.

Since $\zeta(h) = \zeta(y)$ and $\rho = 1 - 1/\alpha$ since X is spectrally positive, we have from (2)

$$\frac{\Delta(A_{(n),\varepsilon} \cap A_\varepsilon)}{\Delta(A_\varepsilon)} = \left(\frac{(t_n \vee t) - \varepsilon}{t - \varepsilon}\right)^{-\frac{1}{\alpha}},$$

and

$$g_n^\varepsilon = 1 - \left(\frac{t - \varepsilon}{(t_n \vee t) - \varepsilon}\right)^{\frac{1}{\alpha}} \overset{\alpha \in (1,2]}{\leq} \frac{(t_n \vee t) - t}{(t_n \vee t) - \varepsilon} \leq \frac{(t_n \vee t) - t}{t - \varepsilon} = \frac{1}{t - \varepsilon}(t_n - t)^+.$$

The same way, we find $h_n^\varepsilon \leq \frac{1}{t-\varepsilon}(t - t_n)^+$. The assumption, that $\sum |t_n - t| = \sum (t - t_n)^+ + (t - t_n)^+ < \infty$ ensures that the sequences g_n^ε and h_n^ε are summable.

Obviously A is regular, measurable and open. Thus Condition (A1) holds since we have $d_{\mathcal{H}}(A_{(n)}, A) = |t_n - t| \xrightarrow{n \to \infty} 0$. Altogether, we obtain

Corollary 5.2 (Conditioning on the length)**.** *The scaled coding excursions $h^{(n)}$, $c^{(n)}$, $s^{(n)}$ of the height-, the contour and the random walk processes of the Galton–Watson tree conditioned under the length converge to the conditional measure Δ of the Lévy height process H resp. the conditioned excursion of the reflected Lévy process $Y = X - I$,*

$$\mathcal{L}(h^{(n)} \mid |\tau| > nt_n) \xrightarrow{(d)} \Delta(\,\cdot \mid \zeta > t),$$
$$\mathcal{L}(c^{(n)} \mid |\tau| > nt_n) \xrightarrow{(d)} \Delta(\,\cdot \mid \zeta > t),$$
$$\mathcal{L}(s^{(n)} \mid |\tau| > nt_n) \xrightarrow{(d)} N(\,\cdot \mid \zeta > t).$$

┘

The following lemma states some useful scaling properties of the excursion measure Δ and the Lévy height local time L which can be applied to verify the regularity condition in this example but also in the following examples.

Lemma 5.3. *Let H be the Lévy height process, resulting from an α-stable Lévy process X with $\alpha \in (1, 2)$. For any $\lambda > 0$ consider the scaling operator $\Phi_\lambda^* : \mathbb{D} \to \mathbb{D}$ given by $\Phi_\lambda^*(x)_t := \lambda^{\frac{1}{\alpha}-1} x_{\lambda t}$, then we have:*

$$L_t(H) \overset{(d)}{=} L_t\left(\Phi_\lambda^*(H)\right) = \lambda^{-1/\alpha} L_{\lambda t}(H), \tag{49}$$
$$\Delta(A) = \lambda^{-1/\alpha} \Delta\left(\Phi_\lambda^*(A)\right), \quad A \in \mathcal{B}(\mathcal{E}). \tag{50}$$

Example 5.2 (Conditioning on the height). Let τ be a Galton–Watson tree whose reproduction law satisfies the assumptions in this section. We condition its law on its height being larger than $\frac{n}{a_n} \cdot t_n$, where $(t_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative reals, such that $t_n \xrightarrow{n \to \infty} t$ and in addition, $\sum_{n=1}^\infty |t_n - t| < \infty$.

Denoting by $\bar{\tau} := \sup\{|v| \mid v \in \tau\}$ the height of a tree τ, we have that $\bar{\tau} = \bar{h}(\tau) = \bar{c}(\tau)$. We therefore aim to apply (44) or (45) to sets of the form $A_n = \{\bar{h} > \frac{n}{a_n} t_n\}, A_{(n)} = \{\bar{h} > t_n\}$ and $A = \{\bar{h} > t\}$. In the previous Example 5.1, condition (A2) could have been verified as well with help of Lemma 5.3 in a similar way as we verify condition (A2) for this example in the following.

Again, we have $A_{(n),\varepsilon} = \{\bar{h} > t_n - \varepsilon\}$ and $A_\varepsilon = \{\bar{h} > t - \varepsilon\}$. This gives $A_{(n),\varepsilon} \cap A_\varepsilon = \{\bar{h} > (t_n \vee t) - \varepsilon\}$. Set $\lambda_n := \left(\frac{t-\varepsilon}{(t_n \vee t) - \varepsilon}\right)^{\frac{\alpha}{1-\alpha}}$, then we have

$$\widehat{\Phi}_{\lambda_n}(A_{n,\varepsilon} \cap A_\varepsilon) = \{\bar{h} > \lambda_n^{\frac{1}{\alpha}-1}((t_n \vee t) - \varepsilon)\} = \{\bar{h} > t - \varepsilon\} = A_\varepsilon.$$

With help of the scaling property in Lemma 5.3, one gets

$$\begin{aligned}\Delta(A_\varepsilon) &= \Delta(\widehat{\Phi}_{\lambda_n}\left(A_\varepsilon \cap A_{(n),\varepsilon}\right)) = \lambda_n^{\frac{1}{\alpha}} \Delta\left(A_\varepsilon \cap A_{(n),\varepsilon}\right) \\ &= \left(\frac{t-\varepsilon}{(t \vee t_n)-\varepsilon}\right)^{\frac{1}{1-\alpha}} \Delta(A_{(n),\varepsilon} \cap A_\varepsilon),\end{aligned}$$

and

$$g_n^\varepsilon = 1 - \left(\frac{t-\varepsilon}{(t_n \vee t)-\varepsilon}\right)^{\frac{1}{\alpha-1}} \leq \frac{1}{t-\varepsilon}(t_n - t)^+$$

as before. In the same way, one sees, that $h_n^\varepsilon \leq \frac{1}{t-\varepsilon}(t-t_n)^+$ and by the assumption $\sum_{n=1}^\infty |t_n - t| < \infty$, the sequences h_n^ε and g_n^ε are summable. In the same way as before, A is regular, measurable and open and condition (A1) holds thanks to $d_{\mathcal{H}}(A_{(n)}, A) = |t_n - t| \xrightarrow{n\to\infty} 0$. The application of Theorem 5.1 then yields Duquesne and Le Gall (2002, Prop. 2.5.2.).

Corollary 5.4 (Conditioning on the height)**.** *Conditioned on the height the scaled coding excursions $h^{(n)}$, $c^{(n)}$ converge to the conditional excursion measure Δ of the Lévy height process:*

$$\begin{aligned}\mathcal{L}\left(h^{(n)} \,\middle|\, \bar{\tau} > \frac{n}{a_n} t_n\right) &\xrightarrow{(d)} \Delta(\,\cdot \mid \bar{h} > t), \\ \mathcal{L}\left(c^{(n)} \,\middle|\, \bar{\tau} > \frac{n}{a_n} t_n\right) &\xrightarrow{(d)} \Delta(\,\cdot \mid \bar{h} > t).\end{aligned}$$

┘

Example 5.3 (Conditioning on the total path length). In the following new application of Theorem 5.1 we describe a general way how to use the scaling properties of Δ in the α-stable case in order to verify the Δ-regularity of some sets. We condition under the total path length L which is defined as the sum of all depths of vertices in a tree τ

$$L(\tau) = \sum_{v\in\tau} |v|.$$

The path length can also be expressed as the area under the height process coding the tree

$$L(\tau) = \sum_{k=0}^{\zeta(h)} h_k.$$

We choose $A_n = \{\int_0^{\zeta(h)} h_t\, dt > n^2/a_n\}$ respectively $A = A_{(n)} = \{\int_0^{\zeta(h)} h_t\, dt > 1\}$. Since the sequence of sets is constant and based on (48), we only need as in Corollary 4.2, to check the Δ-regularity of A in order to apply Theorem 5.1. For $h \in A_\varepsilon$ and $h' \in A$ with $d_{\mathcal{E}}(h, h') < \varepsilon$ we have that

$$\begin{aligned}1 &< \int_0^{\zeta(h')} h'_t\, dt \\ &\leq \int_0^{\zeta(h)} h_t + \varepsilon\, dt + \mathbb{1}_{\{\zeta(h)\leq\zeta(h')\}} \int_{\zeta(h)}^{\zeta(h')} \varepsilon\, dt \\ &= \int_0^{\zeta(h)} h_t\, dt + \varepsilon\zeta(h')\end{aligned}$$

Since $\zeta(h') \leq \zeta(h) + \varepsilon$, this implies that $\int_0^{\zeta(h)} h_t\, dt > 1 - \varepsilon(\varepsilon + \zeta(h))$ and therefore $A_\varepsilon \subseteq \{\int_0^{\zeta(h)} h_t\, dt > 1 - \varepsilon\zeta(h) - \varepsilon^2\}$. With the scaling operator Φ^*_λ as in Lemma 5.3 it then holds that

$$\int_0^{\zeta(\Phi^*_\lambda(h))} \Phi^*_\lambda(h)_t\, dt = \int_0^{\zeta(h)} \lambda^{1/\alpha-2} h_t\, dt > \lambda^{1/\alpha-2}(1 - \varepsilon\zeta(h) - \varepsilon^2)$$

for any $h \in A_\varepsilon$. This gives that

$$\Phi^*_\lambda(A_\varepsilon) \subseteq \{h \in \mathcal{E} \mid \int_0^{\zeta(h)} h_t\, dt > \lambda^{1/\alpha-2}(1-\varepsilon\lambda\zeta(h)-\varepsilon^2)\}.$$

For any $0 < K < \infty$ it then holds that

$$\begin{aligned}\Phi^*_\lambda(A_\varepsilon \cap \{\zeta \le K\}) &\subseteq \left\{\int_0^{\zeta(h)} h_t\, dt > (1-\varepsilon\lambda\zeta(h)-\varepsilon^2)\lambda^{1/\alpha-2},\ \zeta(h) \le \frac{1}{\lambda}K\right\}\\ &\subseteq \left\{\int_0^{\zeta(h)} h_t\, dt > \lambda^{1/\alpha-2}(1-\varepsilon K-\varepsilon^2)\right\}.\end{aligned}$$

Choosing $\lambda = \lambda^* := (1-\varepsilon K-\varepsilon^2)^{-\frac{\alpha}{1-2\alpha}}$ this gives

$$\Phi^*_{\lambda^*}(A_\varepsilon \cap \{\zeta \le K\}) \subseteq A.$$

Using the scaling property from Lemma 5.3 and (2) with $\rho = 1 - 1/\alpha$ we get

$$\begin{aligned}\lambda^{*\,1/\alpha}\Delta(A_\varepsilon) &= \Delta(\Phi^*_{\lambda^*}(A_\varepsilon))\\ &= \Delta(\Phi^*_{\lambda^*}(A_\varepsilon \cap \{\zeta > K\})) + \Delta(\Phi^*_{\lambda^*}(A_\varepsilon \cap \{\zeta \le K\}))\\ &\le \Delta(\Phi^*_{\lambda^*}(\{\zeta > K\})) + \Delta(\Phi^*_\lambda(A_\varepsilon \cap \{\zeta \le K\}))\\ &\le \Delta(\{\zeta > K/\lambda^*\}) + \Delta(A)\\ &= (K/\lambda^*)^{-1/\alpha}\Gamma(1-1/\alpha)^{-1} + \Delta(A).\end{aligned}$$

In total we have that for any $0 < K < \infty$ and $\varepsilon > 0$ it holds that

$$\Delta(A_\varepsilon) \le K^{-1/\alpha}\Gamma(1-1/\alpha)^{-1} + \lambda^{*-1/\alpha}\Delta(A).$$

Now choose $K = K(\varepsilon) \to \infty$ and $\varepsilon \searrow 0$ such that $K\varepsilon \searrow 0$. Then we have that $\lambda^* \to 1$. Since $\Delta(A_\varepsilon) \ge \Delta(A)$, this gives the regularity of the set A. The application of Theorem 5.1 yields:

Theorem 5.5 (Conditioning on the total path length). *Conditioned on the total path length the scaled height excursion $h^{(n)}$ converges to the conditional excursion measure Δ of the Lévy height process:*

$$\mathcal{L}\left(h^{(n)}\,\middle|\, L(\tau) > \frac{n^2}{a_n}\right) \xrightarrow{(d)} \Delta\left(\cdot\,\middle|\, \int_0^{\zeta(h)} h_t\, dt > 1\right).$$

Example 5.4 (Conditioning on the width). For a Galton–Watson tree τ the width of τ is defined as the maximum generation size

$$b(\tau) = \max_{k\ge 0} z_k. \tag{51}$$

$b(\tau)$ can be connected to the discrete height process since

$$z_k = \#\{j \ge 0 \mid h(\tau)_j = k\} \tag{52}$$

This connection suggests that the appropriate limit object related to the width of the tree could be the maximal Lévy height excursion local time $\sup_{a\ge 0}\Lambda^a_{\zeta(h)}(h)$ (for its definition see (Duquesne and Le Gall 2002, Cpt. 1.3.2)). A corresponding relation between the Brownian excursion local time and the profile process of a conditioned Galton–Watson tree had been established in Drmota and Gittenberger (1997). The following result confirms this conjecture.

We define the rescaled width $b^{(n)} = \sup_{a\ge 0} z^{(n)}_a$ and get the following

Theorem 5.6 (Conditioning on the width). *The rescaled height and contour processes converge conditioned under the width to the Lévy height excursion measure conditioned under the maximal local time, i.e.: For any $k > 0$ holds:*

$$\mathcal{L}(h^{(n)} \mid b(\tau) > a_n k) \xrightarrow{(d)} \Delta(\,\cdot \mid \sup_{a \geq 0} \Lambda^a_{\zeta(h)} > k) \tag{53}$$

and

$$\mathcal{L}(c^{(n)} \mid b(\tau) > a_n k) \xrightarrow{(d)} \Delta(\,\cdot \mid \sup_{a \geq 0} \Lambda^a_{\zeta(h)} > k) \tag{54}$$

Theorem 5.6 is obtained as a corollary of Theorem 5.1. The necessary regularity of the sets $\{\sup_{a\geq 0} \Lambda^a_{\zeta(h)} > k\}$ involving the maximal local time of the Lévy height excursion however needs a separate treatment and certain properties of these local times. In order to have this paper not too expanded we defer the detailed proof to a separate paper.

6. Proofs

Proof of Lemma 3.3. For $0 < t < K$ denote $g_t = \sup\{z \leq t \mid z \in \mathcal{Z}^K\}$ and $d_t = \inf\{z \geq t \mid z \in \mathcal{Z}^K\}$ and keep the same notation for g^n_t and d^n_t which are referring to $\mathcal{Z}^K_n$. Denote $h^t := (h_{g_t + t})_{0 \leq t \leq d_t - g_t}$ the excursion of h straddling t and keep the same notation for h^t_n which refer to the excursion of h_n straddling t.
Since $h_n \xrightarrow{n\to\infty} h$ in $\mathbb{D}$, there exists a sequence of time shifts $\lambda_n : \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing, satisfying (61) and such that for all $K \in \mathbb{N}$:

$$\sup_{t \leq K} |h_n(\lambda_n(t)) - h(t)| \to 0. \tag{55}$$

We aim to show that for each $0 < K < \infty$, it holds that

$$\sup_{t \leq K} |h^F_n(\lambda_n(t)) - h^F(t)| \xrightarrow{n\to\infty} 0. \tag{56}$$

We have

$$\begin{aligned} &\sup_{t \leq K} \left| h^F_n(\lambda_n(t)) - h^F(t) \right| \qquad (57)\\ &= \sup_{t \leq K} \left| F(h_n^{\lambda_n(t)})_{\lambda_n(t) - g^n_{\lambda_n(t)}} - F(h^t)_{t - g_t} \right| \mathbb{1}_{\{h_n(\lambda_n(t)) \neq 0,\ h(t) \neq 0\}} \\ &\quad + \sup_{t \leq K} \left| F(h_n^{\lambda_n(t)})_{\lambda_n(t) - g^n_{\lambda_n(t)}} \right| \mathbb{1}_{\{h_n(\lambda_n(t)) \neq 0,\ h(t) = 0\}} \\ &\quad + \sup_{t \leq K} \left| F(h^t)_{t - g_t} \right| \mathbb{1}_{\{h_n(\lambda_n(t)) = 0,\ h(t) \neq 0\}}, \end{aligned}$$

and we aim to show that the above terms tend to zero as n tends to infinity. For the first summand, because of the continuity of F, it remains to be shown that $h^{\lambda_n(t)} \to h^t$ in $\mathcal{E}$, uniformly in $t \in [0, K]$. By definition of these excursions, we need to show that there exists a sequence of time shifts $(\mu_n)_{n\in\mathbb{N}}$ satisfying (61) such that

$$\sup_{t \leq K} \left| h_n(\mu_n(g^n_{\lambda_n(t)} + t)) - h(g_t + t) \right| \xrightarrow{n\to\infty} 0 \tag{58}$$

and $\zeta(h_n^{\lambda_n(t)}) \to \zeta(h^t)$. The second statement is clear since

$$\sup_{t \leq K} |\zeta(h_n^{\lambda_n(t)}) - \zeta(h^t)| = \sup_{t \leq K} \left| d^n_{\lambda_n(t)} - g^n_{\lambda_n(t)} - d_t + g_t \right| \xrightarrow{n\to\infty} 0$$

thanks to (55) and $\mathcal{Z}_n^K \to \mathcal{Z}^K$. Since the convergence $\mathcal{Z}_n^K \xrightarrow{n\to\infty} Z^K$ with respect to $d_{\mathcal{H}}$ and the fact that $\sup_{t\geq 0}|\lambda_n(t) - t| \to 0$ gives $\sup_{t\leq K}|g^n_{\lambda_n(t)} - g_t| \xrightarrow{n\to\infty} 0$, the choice $\mu_n : g^n_{\lambda_n(t)} \mapsto \lambda_n(g_t + t)$ satisfies (61) and as a result, we have by using (55)

$$\sup_{t\leq K} \left| h_n(\mu_n(g^n_{\lambda_n(t)} + t)) - h(g_t + t) \right| = \sup_{t\leq K} |h_n(\lambda_n(g_t + t)) - h(g_t + t)| \xrightarrow{n\to\infty} 0. \tag{59}$$

Note that K was arbitrary in (55), such that we can enlarge it to $2K$ if needed, since then it holds that $g_t + t \leq 2K$ given $t \leq K$. So (58) holds and the first summand in (57) tends to zero. For the second summand in (57), the argument is similar.
We want to establish

$$\sup_{t\leq K} \left| F(h_n^{\lambda_n(t)})_{\lambda_n(t) - g^n_{\lambda_n(t)}} \right| \xrightarrow{n\to\infty} 0 \tag{60}$$

subject to $h(t) = 0$. Thanks to the continuity of F and by definition of $h^{\lambda_n(t)}$, it is enough to show $\sup_{t\leq K}|h_n(\lambda_n(t))| \to 0$. But this immediately follows with (55) and since $h(t) = 0$.

By the same argument, it follows that $\sup_{t\leq K}|F(h^t)_{t-g_t}| \to 0$ subject to $h_n(\lambda_n(t)) = 0$ and (56) follows. □

Proof of Lemma 4.2. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{D}$ be a sequence such that $f_n \xrightarrow{n\to\infty} f$ in J_1-topology. There are timeshifts $\lambda_n : \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing, satisfying

$$\lambda_n(0) = 0\,, \quad \lim_{t\to\infty} \lambda_n(t) = \infty, \quad \sup_{t\geq 0}|\lambda_n(t) - t| \xrightarrow{n\to\infty} 0, \tag{61}$$

such that for all $0 < K < \infty$: $\sup_{t\leq K}|f_n(\lambda_n(t)) - f(t)| \xrightarrow{n\to\infty} 0$. We may assume, that each λ_n is absolute continuous with respect to the Lebesgue measure and such that its differential $d\lambda_n$ satisfies $\sup_{t\leq K}|d\lambda_n(t) - 1| \xrightarrow{n\to\infty} 0$. The reason for this is that the topology generated by the Prokhorov distance d_0, which is defined via timeshifts of the latter kind, and the J_1-topology are the same, cf. Billingsley (1968, Thrm 14.1). Also note that as converging to a càdlàg function, the sequence $(f_n)_{n\in\mathbb{N}}$ necessarily has to satisfy $\sup_{t\leq K} |f_n(\lambda_n(t))| < C$ for some constant $C < \infty$ and n large enough. This gives

$$\begin{aligned}
&\sup_{t\leq K} \left| \int_0^t f(s)\, ds - \int_0^{\lambda_n(t)} f_n(s)\, ds \right| \\
&\quad = \sup_{t\leq K} \left| \int_0^t f(s)\, ds - \int_0^t f_n(\lambda_n(s))\, d\lambda_n(s)\, ds \right| \\
&\quad \leq \underbrace{\sup_{t\leq K} |d\lambda_n(t) - 1|}_{\to 0 \text{ for } n\to\infty} \cdot \underbrace{\int_0^t |f_n(\lambda_n(s))|\, ds}_{\leq C \text{ for } n \text{ large enough}} \\
&\qquad + \underbrace{\sup_{t\leq K} \int_0^t |f(s) - f_n(\lambda_n(s))|\, ds}_{\to 0 \text{ for } n\to\infty}
\end{aligned}$$

and as result, we have $I(f_n) \xrightarrow{n\to\infty} I(f)$ with respect to the J_1 topology which proves continuity of the map I.

□

Proof of Lemma 5.3. Using the equality $L(H) = L(Y)$, the first assertion follows in a straight way from (3). For the second assertion let $\tilde{Y}_t := \lambda^{-1/\alpha} Y_{\lambda t}$ and note that $\tilde{H} := \Phi^*{}_\lambda(H(Y))$ admits the same local time at zero as $H(\tilde{Y})$ because the linear scaling in time is the same for both. In conseqeunce

$$\Phi^*_\lambda(H(Y))_t = 0 \iff \lambda^{-1/\alpha} Y_{\lambda t} = 0 \iff H((\lambda^{-1/\alpha} Y_{\lambda t})_{t\geq 0}) = 0.$$

For $l > 0$, we have

$$
\begin{aligned}
N_l^H(A) &= \#\{t \in [0, l] \mid \exists k \geq 0 : \; h(H)_{L_k(Y)} \in A, \; L_k(Y) = t\} \\
&= \#\{t \in [0, l] \mid \exists k \geq 0 : \; h(\tilde{H})_{L_{k/\lambda}(\tilde{Y})} \in \Phi_\lambda^*(A), \; L_k(Y) = t\} \\
&= \#\{t \in [0, l] \mid \exists k \geq 0 : \; h(\tilde{H})_{\lambda^{-1/\alpha} L_k(Y)} \in \Phi_\lambda^*(A), \; L_k(Y) = t\} \\
&= \#\{t \in [0, \lambda^{-1/\alpha} \cdot l] \mid h(\tilde{H})_t \in \Phi_\lambda^*(A)\} \\
&\overset{(d)}{=} N^H_{\lambda^{-1/\alpha} l}(\Phi_\lambda^*(A)).
\end{aligned}
$$

Since for the distribution of the counting processes it holds that $N_l^H(A) \sim \mathrm{Pois}(l \cdot \Delta(A))$ and $N^H_{\lambda^{-1/\alpha} l}(\Phi_\lambda^*(A)) \sim \mathrm{Pois}(\lambda^{-1/\alpha} l \cdot \Delta(\Phi_\lambda^*(A)))$ the second assertion is proven. □

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