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Characterization of optimal risk allocations for convex risk functionals

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Abstract

In this paper we consider the problem of optimal risk allocation or risk exchange with respect to convex risk functionals, which not necessarily are monotone or cash invariant. General existence and characterization results are given for optimal risk allocations minimizing the total risk as well as for Pareto optimal allocations. We establish a general uniqueness result for optimal allocations. As particular consequence we obtain in case of cash invariant, strictly convex risk functionals the uniqueness of Pareto optimal allocations up to additive constants. In the final part some tools are developed useful for the verification of the basic intersection condition made in the theorems which are applied in several examples.

1 Introduction

In this paper we consider the optimal risk allocation resp. risk exchange problem defined as follows. Let$(\Omega, \mathfrak{A}, P)$ be a probability space and let $\varrho_i : L^\infty(P) \to (-\infty, \infty]$, $1 \leq i \leq n$, be convex, normed (i.e. $\varrho_i(0) = 0$), lower semicontinuous (lsc) risk functionals, describing the risk evaluation of n traders in the market. For $X \in L^\infty$ define

$$A(X) := \Big\{ (\xi_i)_{1 \leq i \leq n}; \xi_i \in L^\infty, \sum_{i=1}^n \xi_i = X \Big\} \tag{1.1}$$

to be the set of allocations of risk X to the n traders in the market endowed with risk measures ϱ_i. Let

$$\mathfrak{R} := \{(\varrho_i(X_i)); (X_i) \in A(X)\} = \mathfrak{R}(X) \tag{1.2}$$

denote the corresponding risk set. Our aim is to characterize Pareto-optimal (PO) allocations $(\xi_i) \in A(X)$ i.e. allocations such that the corresponding risk vectors are minimal elements of the risk set $\mathfrak{R}$ in the pointwise ordering. A related optimization problem is to characterize allocations (ξ_i) which minimize the total risk i.e.

$$\begin{aligned} \sum \varrho_i(\xi_i) &= \inf \Big\{ \sum \varrho_i(X_i); (X_i) \in A(X) \Big\} \\ &=: \wedge \varrho_i(X). \end{aligned} \tag{1.3}$$

In fact, in the special case of cash invariance of the risk functionals equivalence of the PO-property with total risk minimization in (1.3) holds true.

The optimal risk allocation is a classical problem in mathematical economics and insurance and is of considerable practical and theoretical interest. The early contributions to this problem go back to the treatment of risk sharing in insurance and reinsurance contracts (see the papers of Borch (1962), Gerber (1979), Bühlmann and Jewell (1979), Deprez and Gerber (1985), and Kaas et al. (2001). In more recent years this problem has been studied also in the context of financial risks, risk exchange, assignment of liabilities to daughter companies, individual hedging problems and many others (see the papers of Heath and Ku (2004), Barrieu and El Karoui (2005a,b), Burgert and Rüschendorf (2006, 2008), Jouini et al. (2005), Acciaio (2007), Filipović and Kupper (2007) and many references therein.

In comparison to previous work we characterize optimal allocations in the case of general convex lsc risk functionals ϱ_i, which are not necessarily monotone nor cash invariant. This generality allows to include into consideration some risk functionals of practical interest like mean variance or standard deviation or related one-sided risk functionals. We also investigate the question of uniqueness of solutions. For the case of cash invariant risk functionals ϱ_i f.e. it is obvious that for any allocation (ξ_i) of X with minimal total risk (or which is PO) also all decompositions $(\xi_i + c_i)$ with constants (c_i) such that $\sum_i c_i = 0$ minimize the total risk (or are PO). This nonuniqueness can be used to re-balance an optimal allocation in order to satisfy some further criteria like fairness, rationality, or some boundedness condition. Are there further optimal allocations? We derive a general uniqueness result which implies that in the case that the risk functionals are 'strictly convex' and cash invariant Pareto optimal risk allocations are unique up to constants.

The structure of the paper is the following. In Section 2 we introduce some of the basic notions on risk measures, convex duality and (weighted) infimal convolutions. We characterize in Section 3 optimal allocations, connect the PO property with the problem of minimizing total risk and establish an existence result. In Section 4 we prove a general uniqueness result for optimal risk allocations. Finally we discuss applications to some concrete risk allocation problems in Section 5 and establish some tools which are useful in the applications in order to verify the assumptions made in the characterization and existence theorems.

2 Infimal convolution and convex conjugates

Throughout this paper we consider convex, lower semicontinuous (lsc), proper risk functionals $\varrho : L^\infty(P) \to (-\infty, \infty]$, where proper means that $\operatorname{dom} \varrho = \{X \in L^\infty(P); \varrho(X) \in \mathbb{R}\} \neq \emptyset$ and that ϱ is nowhere $-\infty$. Lower semicontinuity is w.r.t. the norm topology or equivalently w.r.t. the weak topology on $L^\infty = L^\infty(P)$. Generally for a convex proper function on a locally convex space E we denote by

$$f^* : E^* \to \overline{\mathbb{R}}, f^*(x^*) = \sup_{x \in E}(\langle x^*, x\rangle - f(x)), \quad x^* \in E^* \tag{2.1}$$

the convex conjugate and by

$$f^{**}(x) = \sup_{x^* \in E^*} (\langle x, x^* \rangle - f^*(x^*)), \quad x \in E \tag{2.2}$$

the bi-conjugate of f. Let further

$$\partial f(x) = \{x^* \in E^*; f(y) - f(x) \geq \langle x^*, y - x \rangle, \ \forall y \in E\} \tag{2.3}$$

denote the set of subgradients of f in x.

Then

$$\partial f(x) = \{x^* \in E^*; \forall y \in E, \langle x^*, y \rangle \leq D(f, x)(y)\}, \tag{2.4}$$

where $D(f, x)(y)$ is the right directional derivative of f in x in direction y. This connection is useful in the applications in order to calculate the subgradient.

For proper convex functions and $\partial f(x) \neq \emptyset$ holds

$$\begin{aligned} x^* \in \partial f(x) &\Leftrightarrow \langle x^*, x \rangle = f(x) + f(x^*) \\ &\Leftrightarrow \langle x, x^* \rangle - f(x) \geq \sup_{y \in E} (\langle y, x^* \rangle - f(y)). \end{aligned} \tag{2.5}$$

By the Fenchel–Moreau theorem this is for lsc functions f further equivalent to

$$x \in \partial f^*(x^*). \tag{2.6}$$

For minimization problems of proper, convex, lsc functions f, as in our risk allocation problem (1.3), the following extension of Fermat's rule is of interest:

$$f(x) = \inf_{y \in E} f(y) \Leftrightarrow 0 \in \partial f(x). \tag{2.7}$$

For all results on convex duality we refer to Rockafellar (1974) and Barbu and Precupanu (1986).

The following proposition gives conditions which are needed to ensure properness of the infimal convolution $\wedge \varrho_i$.

Proposition 2.1 *For ϱ_i convex, lsc and proper holds*

$$\bigcap \operatorname{dom} \varrho_i^* \neq \emptyset \quad \Rightarrow \quad \operatorname{dom}(\wedge \varrho_i) \neq \emptyset. \tag{2.8}$$

Proof: For the proof we make use of the following results from convex analysis

1) As ϱ_i are proper, convex, and lsc it follows that ϱ_i^* are proper (see Barbu and Precupanu (1986, Cor. 1.4, Chapter 2)). For proper convex ϱ_i holds

$$(\wedge \varrho_i)^* = \sum \varrho_i^*. \tag{2.9}$$

This implies that $(\wedge \varrho_i)^*$ is nowhere $-\infty$.

2) Properness of f^* induces properness of f (see Barbu and Precupanu (1986, Cor. 1.4, Chapter 2)).

As consequence we obtain

$$\begin{aligned} \operatorname{dom}(\wedge\varrho_i)^* \neq \emptyset &\overset{1)}{\Longrightarrow} (\wedge\varrho_i)^* \text{ is proper} \\ &\overset{2)}{\Longrightarrow} (\wedge\varrho_i) \text{ is proper} \\ &\Longrightarrow \operatorname{dom}(\wedge\varrho_i) \neq \emptyset. \end{aligned}$$

□

Remark 2.2 a) For continuous risk functionals ϱ_i on L^∞ the infimal convolution $(\wedge\varrho_i)$ is continuous. Thus Barbu and Precupanu (1986, Cor. 1.4, Chapter 2) implies equivalence in (2.8). Thus the risk allocation problem defined as above does not make sense without the intersection condition (IS).

$$\text{(IS)} \qquad \bigcap_{i=1}^n \operatorname{dom} \varrho_i^* \neq \emptyset. \tag{2.10}$$

which is in the following generally assumed to hold true. A senseful modification of the risk allocation problem, in the continuous case, without assuming the intersection condition (IS) has been introduced and discussed in Burgert and Rüschendorf (2006, 2008).

b) For the case of coherent risk measures ϱ_i with representations $\varrho_i(X) = \sup_{\mu \in \mathcal{P}_i} E_\mu(-X)$ with sets of 'scenario' measures $\mathcal{P}_i \subset \operatorname{ba}(P)$ the conjugates are given by $\varrho_i^*(\mu) = 1_{\mathcal{P}_i}(\mu)$ and thus the intersection condition in (2.10) is equivalent to the condition

$$\bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset, \tag{2.11}$$

i.e. the traders have at least one scenario measure in common. In this case the result is due to Heath and Ku (2004) and Burgert and Rüschendorf (2008). It has been extended to convex risk measures in Jouini et al. (2005), Burgert and Rüschendorf (2006) and Acciaio (2007).

c) The results in this paper are given for risk measures on L^∞. Most of the results in this paper also can be given on L^p for $p \geq 1$. In this case the dual space is more pleasant and one obtains σ-additive measures (instead of finite additive measures) in the representation theorems (see Kaina and Rüschendorf (2008)). □

For the existence of optimal allocations $(\xi_i) \in A(X)$ minimizing the total risk, i.e.

$$\wedge\varrho_i(X) = \sum \varrho_i(\xi_i) \tag{2.12}$$

we need a strengthened version of the intersection condition IS.

$$\text{(SIS)} \qquad \operatorname{dom} \varrho_1^* \cap \bigcap_{i=2}^n \operatorname{int}(\operatorname{dom} \varrho_i^*) \neq \emptyset.$$

Further we assume that at least one risk functional ϱ_k is monotone (i.e. $X \leq Y$ implies $\varrho_k(X) \geq \varrho_k(Y)$). This assumption implies monotonicity of the infimal convolution $(\wedge \varrho_i)$ (Acciaio (2007, Lemma 3.3, i))).

Theorem 2.3 (Existence of optimal allocations) *Let ϱ_i be proper, convex, and lsc and assume that at least one of the ϱ_i is monotone and that the strong intersection property (SIS) holds. Then for all $X \in \operatorname{int}(\operatorname{dom} \wedge \varrho_i)$ there exists an allocation $(\xi_i) \in A(X)$ which minimizes the total risk i.e.*

$$\wedge \varrho_i(X) = \sum \varrho_i(\xi_i) \tag{2.13}$$

Proof: For proper, convex, and monotone functions f on a Banach lattice E holds $\partial f(x) \neq \emptyset$ for all $x \in \operatorname{int}(\operatorname{dom} f)$ (see Ruszczyński and Shapiro (2006, Corollary 3.1)). Thus we obtain for all $X \in \operatorname{int}(\operatorname{dom} \wedge \varrho_i)$ that $\partial(\wedge \varrho_i)(X) \neq \emptyset$. Let $\mu \in \partial(\wedge \varrho_i)(X)$ and thus

$$X \in \partial(\wedge \varrho_i)^*(\mu) = \partial\big(\sum \varrho_i^*\big)(\mu).$$

The strong intersection property (SIS) implies additivity of the subgradient mapping

$$\partial\big(\sum \varrho_i^*\big) = \sum \partial \varrho_i^* \tag{2.14}$$

(see Barbu and Precupanu (1986, Ch. 2, Cor. 2.5 and Rem. 2.8)). As consequence

$$X \in \sum \partial \varrho_i^*(\mu). \tag{2.15}$$

Thus there exists an allocation $(\xi_i) \in A(X)$ with $\xi_i \in \partial \varrho_i^*(\mu)$ which implies that (ξ_i) minimizes the total risk (see Theorem 3.1).

Remark 2.4 (Cash invariant risk measure) If the ϱ_i in Theorem 2.3 are additionally cash invariant, then under the intersection property (IS) $\operatorname{dom} \wedge \varrho_i = L^\infty$ and existence of optimal allocations (ζ_i) minimizing total risk does not need the stronger condition (SIS) (see Jouini et al. (2005), Acciaio (2007)). By the comonotone improvement result of Landsberger and Meijlison (1994) any allocation can be improved by a comonotone allocation, i.e. where $X_i = f_i(X)$ are monotone functions of X. This property combined with Dini's theorem allows to dismiss with the stronger intersection property (for details see Jouini et al. (2005)).

3 Characterization of optimal allocations

Risk allocations with minimal total risk are characterized by the following result which is an extension of the classical Borch (1962) theorem in insurance (see e.g. Gerber (1979)).

Theorem 3.1 (Characterization of minimal total risk allocations) *Let ϱ_i be proper convex, risk functionals. Then for $X \in L^\infty$ and $(\zeta_i) \in A(X)$ the following are equivalent.*

(i) (ζ_i) *has minimal total risk, i.e.* $\wedge \varrho_i(X) = \sum \varrho_i(\zeta_i)$

(ii) $\exists \mu \in \mathrm{ba}(P) : \zeta_i \in \partial \varrho_i^*(\mu), \quad 1 \le i \le n$ (3.1)

(iii) $\exists \mu \in \mathrm{ba}(P) : \mu \in \partial \varrho_i(\zeta_i), \forall i$ (3.2)

i.e. $\bigcap \partial \varrho_i(\zeta_i) \neq 0.$

Proof: In the case of convex risk measures Theorem 3.1 has been proved in Jouini et al. (2005) and extended to nonmonotone risk measures in Acciaio (2007). The proof in these papers however extends verbatim to the more general class of convex risk functionals considered here. □

(3.2) gives an important and very useful criterion for the calculation of optimal allocations. In the classical case of differentiable risk functionals this amounts to the condition that the Gateaux gradients $D\varrho_i(\zeta_i)$ are independent of i. The following proposition identifies this gradient with the subgradient of the infimal convolution.

Proposition 3.2 *Let $(\zeta_i) \in A(X)$ minimize the total risk, $\wedge \varrho_i(X) = \sum \varrho_i(\zeta_i)$.*

(a) $\forall (\eta_i) \in A(X)$ *holds* $\bigcap \partial \varrho_i(\eta_i) \subset \partial \wedge \varrho_i(X),$ (3.3)

(b) $\partial \wedge \varrho_i(X) = \bigcap \partial \varrho_i(\zeta_i).$ (3.4)

Proof:

a) For $\mu \in \bigcap \partial \varrho_i(\eta_i)$ we obtain by the Fenchel inequality using (2.9)

$$\wedge \varrho_i(X) \ge \langle \mu, X \rangle - \sum \varrho_i^*(\mu) = \sum (\langle \mu, \eta_i \rangle - \varrho_i^*(\mu)) = \sum \varrho_i(\eta_i).$$

By definition of the infimal convolution therefore equality holds and thus $\mu \in \partial \wedge \varrho_i(X)$.

b) If $\mu \in \partial \wedge \varrho_i(X)$, then

$$\begin{aligned} \sum (\langle \mu, \zeta_i \rangle - \varrho_i(\mu)) &= (\mu, X) - \sum \varrho_i^*(\mu) \\ &= \langle \mu, X \rangle - (\wedge \varrho_i)^*(\mu) = \wedge \varrho_i(X) = \sum \varrho_i(\zeta_i). \end{aligned}$$

Therefore, again from the Fenchel inequality we conclude that

$$\langle \mu, \zeta_i \rangle - \varrho_i(\mu) = \varrho_i(\zeta_i)$$

for all i and thus $\mu \in \bigcap \partial \varrho_i(\zeta_i)$. This implies that $\partial \wedge \varrho_i(X) \subset \bigcap \partial \varrho_i(\zeta_i)$. So (3.4) is a consequence of a).

□

To obtain a connection to Pareto optimality we need a property which we call non-saturation property (NS). We say that ϱ has the non-saturation property if

$$\text{(NS)} \quad \inf_{X \in L^\infty} \varrho(X) \text{ is not attained} \tag{3.5}$$

(NS) is a weak property on risk measures. It is implied in particular by the cash invariance property.

Under the non-saturation condition (NS) Pareto optimality is related to the problem of minimizing the total weighted risk. This is described by the *weighted minimal convolution* $(\wedge \varrho_i)_\gamma(X)$, defined for weight vectors $\gamma = (\gamma_i) \in \mathbb{R}^n$ by

$$(\wedge \varrho_i)_\gamma(X) := \inf \Big\{ \sum \gamma_i \varrho_i(X_i); (X_i) \in A(X) \Big\}. \tag{3.6}$$

This connection between Pareto optimality and minimizing total weighted risk goes back in more special situations to the early papers in insurance (see Gerber (1979)).

Theorem 3.3 (Characterization of Pareto optimal allocations) *Let ϱ_i be convex, lsc, proper risk functionals on L^∞ satisfying the non-saturation condition (NS). Then for $X \in L^\infty$ and $(\zeta_i) \in A(X)$ the following are equivalent:*

i) (ζ_i) *is PO*

ii) $\exists \gamma = (\gamma_i) \in \mathbb{R}^n_{>0}$ *such that* $\sum \gamma_i \varrho_i(\zeta_i) = (\wedge \varrho_i)_\gamma(X)$,

iii) $\exists \gamma \in \mathbb{R}^n_{>0}$ *and* $\exists \mu \in \mathrm{ba}(P)$ *such that* $\mu \in \gamma_i \partial \varrho_i(\zeta_i), \forall i$ (3.7)

or equivalently $\bigcap \gamma_i \partial \varrho_i(\zeta_i) \neq \emptyset$

iv) $\exists \mu \in \mathrm{ba}(P)$ *and* $\exists \gamma \in \mathbb{R}^n_{>0} : \zeta_i \in \partial(\gamma_i \varrho_i)^*(\mu), \quad \forall i.$ (3.8)

Proof: The equivalences of ii)–iv) follow from Theorem 3.1 applied to the convex risk functionals $\gamma_i \varrho_i$ using lsc and the property $\partial(\gamma_i \varrho_i) = \gamma_i \partial \varrho_i$.

i) ⇔ ii) For the proof of this equivalence we use the Hahn–Banach separation argument (cp. the proof of Theorem 3.1 in Jouini et al. (2005)) separating $B := \mathfrak{R} + \mathbb{R}^n_+$ from $C := (\varrho_i(\zeta_i)) - (\mathbb{R}^n_+ \setminus \{0\})$. By construction the vector $\lambda \in \mathbb{R}^n$ describing the separating hyperplane i.e. $\lambda^\top \cdot x \leq \lambda^\top \cdot y$ for all $x \in C$, $y \in B$ has non-negative coordinates, $\lambda_j \geq 0$. Now the (NS) condition is enough to imply that all components of λ are positive. For the proof assume that $\lambda_j = 0$ for some j. We assume w.l.g. $j = n$. Then (ζ_i) minimizes $\sum_{i=1}^{n-1} \lambda_i \varrho_i(\eta_i)$ over $(\eta_i) \in A(X)$. Thus since $\lambda_n = 0$ (ζ_i) minimizes $\sum_{i=1}^{n-1} \lambda_i \varrho_i(\eta_i)$ over all $\eta_i \in L^\infty$. By the (NS) condition this leads to a contradiction.

The direction ii) ⇒ i) is obvious. □

Remark 3.4 a) Using Proposition 3.2 Condition iii) in Theorem 3.3 can be reformulated as

iii') $\exists \gamma \in \mathbb{R}^n_{>0} : \partial(\wedge \varrho_i)_\gamma(X) \neq \emptyset$

b) The value of the optimal total weighted risk $(\wedge\varrho_i)_\gamma(X)$ is given by

$$(\wedge\varrho_i)_\gamma(X) = \langle \mu, X\rangle - \Sigma\gamma_i \varrho_i^*(\mu), \tag{3.9}$$

where μ is as in (3.7) or in (3.8) a solution of the dual problem.

For cash invariant convex risk functionals this characterization result implies by a rebalancing argument (see Jouini et al. (2005), Burgert and Rüschendorf (2006; 2008), Acciaio (2007)) that it is enough to consider the optimal risk allocation problem for the weight vector $\gamma = (1, \ldots, 1)$.

Corollary 3.5 *If ϱ_i are convex, lsc, proper cash invariant risk functionals on L^∞, then for $X \in L^\infty$, $(\zeta_i) \in A(X)$ the following are equivalent:*

i) (ζ_i) *is PO*

ii) (ζ_i) *minimizes the total risk, i.e.* $\sum \varrho_i(\zeta_i) = \wedge\varrho_i(X)$. (3.10)

iii) $\bigcap \partial\varrho_i(\zeta_i) \neq \emptyset$

iv) $\exists \mu \in \mathrm{ba}(P) : \zeta_i \in \partial\varrho_i^*(\mu), \quad \forall i$

v) $\exists \mu \in \mathrm{ba}(P) : \mu \in \partial\varrho_i(\zeta_i), \quad \forall i.$

Further for any μ as above holds

$$\wedge\varrho_i(X) = \langle \mu, X\rangle - \Sigma\varrho_i^*(\mu). \tag{3.11}$$

4 Uniqueness of optimal allocations

In this section we investigate the question under which conditions optimal allocations which minimize total risk are unique. This question leads naturally to the consideration of strictly convex risk measures. Then in the second part of this section we connect this to the characterization of PO-risk allocations and obtain as an interesting consequence, that PO-risk allocations in the case of *strictly convex* cash invariant risk measures are unique up to rebalancing. A function is called strictly convex if the inequality in the definition of convexity is strict for any $X, Y \in E$, $X \neq Y$, i.e. it holds $f(\alpha X + (1-\alpha)Y) < \alpha f(x) + (1-\alpha) f(Y)\ \forall \alpha \in (0,1)$. For the preparation of this uniqueness result we need some properties of subdifferentials.

Lemma 4.1 *Let ϱ be a finite strictly convex risk functional on L^∞.*

a) $\forall \mu \in \mathrm{ba}(P)$ *holds* $|\partial\varrho^*(\mu)| \leq 1$

b) For $X, Y \in L^\infty$, $X \neq Y$ holds $|\partial\varrho(X) \cap \partial\varrho(Y)| = 0$

Proof:

a) The conjugate of ϱ is defined by

$$\varrho^*(\mu) = \sup_{X \in L^\infty} \big(\langle \mu, X\rangle - \varrho(X)\big), \quad \mu \in \mathrm{ba}(P).$$

Strict convexity of ϱ implies that for any $\mu \in \mathrm{ba}(P)$ there is at most one maximizer X_μ of $\langle \mu, X\rangle - \varrho(X)$, i.e. $|\partial\varrho^*(\mu)| \leq 1$.

b) If $\mu \in \partial\varrho(X) \cap \partial\varrho(Y)$, then by (2.6) $X, Y \in \partial\varrho^*(\mu)$ in contradiction to a). □

The following property of subdifferentials is crucial for the uniqueness result.

Proposition 4.2 *If ϱ is convex, then for all $\mu \in \partial\varrho(X)$ and $\nu \in \partial\varrho(Y)$ by definition of the subgradient*

$$\langle \nu, X - Y\rangle \leq \varrho(X) - \varrho(Y) \leq \langle \mu, X - Y\rangle. \tag{4.1}$$

The inequalities in (4.1) *are strict if ϱ is strictly convex and $X - Y \neq const$.*

Proof: Based on the Fenchel inequality and (2.5) we have

$$\varrho(X) + \varrho^*(\mu) = \langle \mu, X\rangle \quad \text{and} \quad \varrho(Y) + \varrho^*(\mu) \geq \langle \mu, Y\rangle$$

as well as (4.2)

$$\varrho(Y) + \varrho^*(\nu) = \langle \nu, Y\rangle \quad \text{and} \quad \varrho(X) + \varrho^*(\nu) \geq \langle \nu, X\rangle.$$

As consequence we obtain

$$\varrho(X) - \varrho(Y) \leq \langle \mu, X - Y\rangle \quad \text{and} \quad \varrho(X) - \varrho(Y) \geq \langle \nu, X - Y\rangle. \tag{4.3}$$

In the case that $X - Y = c \neq 0$ the expressions in (4.3) depend on $\langle \mu, 1\rangle$ resp. $\langle \nu, 1\rangle$ which may be different. If $X - Y \neq c$ for all $c \in \mathbb{R}$ by the assumption of strict convexity the inequalities in (4.3) are strict. To show this assume that one of the inequalities in (4.2) would be an equality. Then this assumption would imply that $\partial\varrho(X) \cap \partial\varrho(Y) \neq \emptyset$ in contradiction to Lemma 4.1. □

As consequence we get for strictly convex risk functionals uniqueness of γ-infimal convolution allocations as defined in (3.6) up to rebalancing. We call (ζ_i) a *γ-optimal allocation* if it minimizes the weighted total risk $\sum \gamma_i \varrho_i(X_i)$ over $(X_i) \in A(X)$.

Theorem 4.3 *Let ϱ_n be strictly convex and for some $\gamma \in \mathbb{R}^n_{>0}$ let $(\zeta_i) \in A(X)$ be a γ-optimal allocation, i.e.*

$$(\wedge\varrho_i)_\gamma(X) = \sum_i \gamma_i \varrho_i(\zeta_i). \tag{4.4}$$

If (η_i) is a further γ-optimal allocation, then $\eta_n = \zeta_n + c_n$ for some constant c_n.

Proof: Assume that $\eta_n - \zeta_n \neq c$ for any $c \in \mathbb{R}$. By Theorem 3.1 applied to the risk functionals $\gamma_i \varrho_i$ there exists some $\mu \in \bigcap \partial(\gamma_i \varrho_i)(\zeta_i)$. As consequence of Proposition 4.2 we thus obtain

$$\begin{aligned} \lambda_n \varrho_n(\zeta_n) &< \langle \mu, \zeta_n - \eta_n \rangle + \lambda_n \varrho_n(\eta_n) \\ \lambda_i \varrho_i(\zeta_i) &\leq \langle \mu, \zeta_i - \eta_i \rangle + \lambda_i \varrho_i(\eta_i), \quad i \leq n-1. \end{aligned}$$

This implies

$$(\wedge \varrho_i)_\gamma(X) = \Sigma \gamma_i \varrho_i(\zeta_i) < \sum \lambda_i \varrho_i(\eta_i)$$

and thus (η_i) is not a γ-optimal allocation. □

Corollary 4.4 *If $\varrho_1, \dots, \varrho_n$ are strictly convex lsc risk functionals satisfying the NS condition as well as the strong intersection condition $\bigcap \operatorname{int}(\operatorname{dom}(\gamma_i \varrho_i^*)) \neq \emptyset$ for some $\gamma \in \mathbb{R}^n_{>0}$. Then there exists up to additive constants c_i a unique γ-optimal allocation rule.*

In fact by Theorem 4.3 it is enough to postulate strict convexity only for $\varrho_1, \dots, \varrho_{n-1}$. Theorem 4.3 does not imply a uniqueness result for cash invariant risk functionals ϱ_i since for any $X \in L^\infty(P)$, ϱ_i are not strictly convex on the affine supspace $X + \mathbb{R}$. To include this interesting case we define

$$L_0^\infty := \{X \in L^\infty : EX = 0\} \tag{4.5}$$

to be the class of equivalence classes of L^∞ modulo addition of constants. Cash invariance of ϱ implies for all $X \in L^\infty$, $c \in \mathbb{R}^1$

$$\partial \varrho(X) = \partial \varrho(X + c). \tag{4.6}$$

Therefore, we can extend Lemma 4.1 to the case where ϱ_i are cash invariant and strictly convex on L_0^∞. We obtain for cash invariant risk measures strictly convex on L_0^∞ the following strong uniqueness property. As noted before (see Remark 2.4) for cash invariant risk functionals the intersection property (IS) is necessary and sufficient for the existence of Pareto optimal allocations.

Corollary 4.5 (Uniqueness of PO-allocations) *Let $\varrho_1, \dots, \varrho_n$ be strictly convex on L_0^∞, cash invariant, lsc, finite risk functionals and assume the intersection condition $\bigcap \operatorname{dom} \varrho_i^* \neq \emptyset$. Then for any $X \in L^\infty$ there exists an up to additive constants unique Pareto optimal allocation (ζ_i) of X.*

Again it is enough that $\varrho_1, \dots, \varrho_{n-1}$ are strictly convex.

5 Applications and remarks

For the existence and characterization of Pareto optimal allocations the intersection property

$$\text{(IS)}_\gamma \quad \bigcap \operatorname{dom}(\gamma_i \varrho_i)^* \neq \emptyset$$

for some $\gamma \in \mathbb{R}^n_{>0}$ as well as the corresponding strong intersection property $(\text{SIS})_\gamma$ are important. To investigate this property in examples the following rules are useful.

Lemma 5.1 *Let ϱ be convex, lsc, then for $\alpha \in \mathbb{R}_{>0}$*

$$\begin{aligned} &\text{i)} && (\alpha\varrho)^*(\mu) &&= \alpha\varrho^*\big(\tfrac{\mu}{\alpha}\big)\\ &\text{ii)} && \operatorname{dom}(\alpha\varrho)^* &&= \alpha \operatorname{dom}\varrho^*\\ &\text{iii)} && \partial(\alpha\varrho)^*(\mu) &&= \partial\varrho^*\big(\tfrac{\mu}{\alpha}\big)\\ &\text{iv)} && \partial(\alpha\varrho)(X) &&= \alpha\partial\varrho(X). \end{aligned} \tag{5.1}$$

Proof:

i) $$\begin{aligned}(\alpha\varrho)^*(\mu) &= \sup_{X\in L^\infty}(\langle\mu, X\rangle - \alpha\varrho(X))\\ &= \alpha \sup_{X\in L^\infty}\big(\langle\tfrac{\mu}{\alpha}, X\rangle - \varrho(X)\big) = \alpha\varrho^*(\tfrac{\mu}{\alpha})\end{aligned}$$

ii) $$\begin{aligned}\operatorname{dom}(\alpha\varrho)^* &= \{\mu \in \text{ba}; (\alpha\varrho)^*(\mu) < \infty\} = \{\mu \in \text{ba}; \alpha\varrho^*(\tfrac{\mu}{\alpha}) < \infty\}\\ &= \{\alpha\mu \in \text{ba}; \alpha\varrho^*(\mu) < \infty\} = \alpha \operatorname{dom}\varrho^*\end{aligned}$$

iii) $$\begin{aligned}\partial(\alpha\varrho)^* &= \{X \in L^\infty; \alpha\varrho(X) \le \langle\mu, X\rangle - (\alpha\varrho)^*(\mu)\}\\ &= \{X \in L^\infty; \varrho(X) \le \langle\tfrac{\mu}{\alpha}, X\rangle - \varrho^*(\tfrac{\mu}{\alpha})\} = \partial\varrho^*(\tfrac{\mu}{\alpha})\end{aligned}$$

iv) is analogously to iii). □

The following proposition gives sufficient conditions to check the intersection property.

Proposition 5.2 *The intersection property (IS)$_\gamma$ is fulfilled under any of the following two conditions:*

a) $$\exists X \in L^\infty : \partial(\wedge\varrho_i)_\gamma(X) \neq \emptyset \tag{5.2}$$

b) $$\exists(\eta_i) \in A(X) \text{ such that } \bigcap \gamma_i\partial\varrho_i(\eta_i) \neq \emptyset. \tag{5.3}$$

Proof: By Proposition 3.2 and Lemma 5.1 holds for any $(\eta_i) \in A(X)$

$$\bigcap \gamma_i\partial\varrho_i(\eta_i) \subset \partial(\wedge\varrho_i)_\gamma(X).$$

Thus b) is a consequence of a). For proper convex functions f holds (see Barbu and Precupanu (1986, p. 101)):

If $X \in \operatorname{int}\operatorname{dom} f$, then $\partial f(X) \neq \emptyset$.
If $\partial f(X) \neq \emptyset$, then $X \in \operatorname{dom} f$.

This implies that

$$\operatorname{range}\partial f := \{X : \partial f(X) \neq \emptyset\} \subset \operatorname{dom} f^*. \tag{5.4}$$

As consequence we obtain for any $(\eta_i) \in A(X)$ the relation

$$\operatorname{range}(\partial(\wedge\varrho_i)_\gamma) \subset \operatorname{dom}(\wedge\varrho_i)^*_\gamma = \bigcap \gamma_i \operatorname{dom}\varrho_i^*. \tag{5.5}$$

Thus a) follows. □

5.1 Expected risk

Let ϱ be an expected risk functional

$$\varrho(X) = Er(X), \tag{5.6}$$

where r is a convex, strictly decreasing, differentiable function $r : \mathbb{R} \to \mathbb{R}$. This is the typical case considered in the calculation of convex principles in premium calculation (see Deprez and Gerber (1985)). Then ϱ is Gateaux differentiable and

$$\partial\varrho(X) = r' \circ X \tag{5.7}$$

is given by the Gateaux-gradient of ϱ. As consequence we obtain from Theorem 3.3 a classical result of Borch (1962), see also Deprez and Gerber (1985) and Barrieu and Scandolo (2007):

$$(\xi_i) \text{ is a PO-allocation } \Leftrightarrow \exists\gamma \in \mathbb{R}^n_> : \gamma_i r'_i \circ \xi_i = \gamma_j r'_j \circ \xi_j, \quad \forall i, j. \tag{5.8}$$

5.2 Dilated risk functionals

Let ϱ be a convex risk functional and define the class of dilated risk functionals ϱ_λ for $\lambda > 0$ by

$$\varrho_\lambda(X) = \lambda\varrho\Big(\frac{X}{\lambda}\Big). \tag{5.9}$$

Then as in Lemma 5.1 one obtains

$$\begin{array}{ll} \text{i)} & \operatorname{dom}\varrho_\lambda = \lambda \operatorname{dom}\varrho \\ \text{ii)} & \varrho_\lambda^* = \lambda\varrho^* \\ \text{iii)} & \partial\varrho_\lambda(X) = \partial\varrho\Big(\frac{X}{\lambda}\Big) \\ \text{iv)} & \partial\varrho_\lambda^*(\mu) = \lambda\partial\varrho^*(\mu). \end{array} \tag{5.10}$$

As consequence we obtain from the characterization Theorem 3.3 (see also Proposition 3.5 of Barrieu and El Karoui (2005a) for the case of convex risk measures).

Theorem 5.3 (Dilated risk functionals) *Let ϱ be a convex, lsc, proper risk functional on L^∞ satisfying condition NS and let $\varrho_i = \varrho_{\lambda_i}$, $1 \leq i \leq n$, corresponding dilated risk functionals, $\lambda_i > 0$. Let $\wedge := \sum \lambda_i$ and let $X \in L^\infty$ with $\frac{X}{\wedge} \in \operatorname{int}(\operatorname{dom} \varrho)$, then $\xi_i := \frac{\lambda_i}{\wedge} X$ defines a PO-allocation of X.*

Proof: For the proof it is enough to check condition iii) of Theorem 3.1. Using (5.10) and (5.3) we obtain

$$\bigcap \partial \varrho_i(\xi_i) = \bigcap \partial \varrho\Big(\frac{\xi_i}{\lambda_i}\Big) = \bigcap \partial \varrho\Big(\frac{X}{\wedge}\Big) = \partial \varrho\Big(\frac{X}{\wedge}\Big) \neq \emptyset.$$

Thus by Theorem 3.1 and Remark 5.5 (ξ_i) is a PO allocation of X. □

5.3 Mean variance vs. standard-deviation

The *mean variance principle* ϱ_δ^{mv} is defined for $\delta > 0$ by

$$\varrho_\delta^{mv}(X) := E(-X) + \delta \operatorname{Var}(X). \tag{5.11}$$

The parameter δ reflects the degree of risk aversion. ϱ_δ^{mv} is a convex, lsc, cash invariant risk measure but ϱ_δ^{mv} is not monotone. The following lemma describes the conjugate $(\varrho_\delta^{mv})^* =: v_\delta^{mv}$ and the subgradient. We identify P continuous measures with their densities. The subgradients were already given in Acciaio (2007) for the essentially equivalent utility formulation. Some related calculations of subdifferentials are also given in Ruszczyński and Shapiro (2006). We include a sketch of the proof since it shows some typical arguments for the calculation of subgradients useful also for related applications.

Lemma 5.4 i) $v_\delta^{mv}(\mu) = \frac{\operatorname{Var}\mu}{4\delta}$ *for all* $\mu \in \operatorname{dom}(v_\delta^{mv}) = \{\mu \in L^1; E(-\mu) = 1\} \cap L^2$

ii) $\partial \varrho_\delta^{mv}(X) = \{2\delta(X - EX) - 1\}, \quad \forall X \in L^\infty$

iii) $\partial v_\delta^{mv}(\mu) = \{\frac{\mu}{2\delta} + c; c \in \mathbb{R}\}, \quad \forall \mu \in \operatorname{dom}(v_\delta^{mv})$

Proof:

ii) ϱ_δ^{mv} is easily seen to be Gateaux differentiable with Gateaux gradient $\nabla \varrho_\delta^{mv}(X) = 2\delta(X - EX) - 1$. This implies ii).

iii) By lsc of ϱ_δ^{mv} we have $\mu \in \partial \varrho_\delta^{mv}(X)$ is equivalent to $X \in \partial v_\delta^{mv}(\mu)$. Since, for $X \in L^\infty$ with $EX = 0$, $\partial \varrho_\delta^{mv}(X + c) = \partial \varrho_\delta^{mv}(X) = \{2\delta X - 1\}$ by Corollary 4.1 in Aubin (1993) we obtain $\partial v_\delta^{mv}(\mu) = (\partial \varrho_\delta^{mv})^{-1}(\mu) = \{\frac{\mu}{2\delta} + c; c \in \mathbb{R}\}$ for all $\mu \in \operatorname{dom}(v_\delta^{mv})$.

i) Using that $\partial v_\delta^{mv}(\mu) = \arg\max(\langle \mu, X\rangle - \varrho_\delta^{mv}(X), X \in L^\infty)$ we obtain for $X_0 \in \partial v_\delta^{mv}(\mu)$ using iii)

$$\begin{aligned} v_\delta^{mv}(\mu) &= \langle \mu, X_0\rangle - \varrho_\delta^{mv}(X_0) \\ &= E\mu X_0 - E(-X_0) - \delta \operatorname{Var}(X_0) \\ &= EX_0(1+\mu) - \delta \operatorname{Var}(X_0) \\ &= E\frac{\mu}{2\delta}(1+\mu) + cE(1+\mu) - \delta \operatorname{Var}\left(\frac{\mu}{2\delta}\right) \\ &= -\frac{1}{2\delta} + \frac{1}{2\delta}E\mu^2 - \frac{1}{4\delta}E\mu^2 + \frac{1}{4\delta} \\ &= \frac{\operatorname{Var}\mu}{4\delta}. \end{aligned}$$

Since ϱ_δ^{mv} is law invariant and thus Fatou continuous we can restrict above to P-continuous measures and any $\mu \in \operatorname{dom} v_\delta^{mv}$ must satisfy $\operatorname{Var}\mu < \infty$ and thus $\mu \in L^2$. □

The *standard deviation principle* ϱ_δ^{sd} is defined for $\delta > 0$ by

$$\varrho_\delta^{sd}(X) = E(-X) + \delta\sqrt{\operatorname{Var} X}. \tag{5.12}$$

Compared to the mean variance principle it is less sensitive concerning the variance. Similarly to Lemma 5.4 one obtains the subgradients (see also Acciaio (2007)).

Lemma 5.5 *For the standard deviation principle ϱ_δ^{sd} and the corresponding conjugate v_δ^{sd} holds (again we identify $\mu \in M(P)$ with its density)*

1) $$\partial\varrho_\delta^{sd}(X) = \begin{cases} \operatorname{dom} v_\delta^{sd} & \text{if } X = const. \\ \delta\frac{X-EX}{\|X-EX\|_2} - 1 & \text{else} \end{cases} \tag{5.13}$$

2) $$\partial v_\delta^{sd}(\mu) = \mathbb{R} \cup \left\{X \in L^\infty : \mu = \delta\frac{X-EX}{\|X-EX\|_2} - 1\right\}. \tag{5.14}$$

In the following we determine all Pareto optimal allocations between an SDP ϱ_1 (with $\delta_1 > 0$) and an MVP ϱ_2 (with $\delta_2 > 0$) and conjugates v_1, v_2.

Proposition 5.6 (SDP vs. MVP) *Let ϱ_1 be an SDP and ϱ_2 an MVP with $\delta_i > 0$. Then for any $X \in \operatorname{dom}(\varrho_1 \wedge \varrho_2)$, $X \not\equiv const.$ there exists a up to constants unique Pareto optimal allocation (ξ_1, ξ_2) which is given by:*

1) If $0 < \frac{\delta_1}{2\delta_2\sigma(X)} < 1$, *then*

$$\xi_1 = \Big(1 - \frac{\delta_1}{2\delta_2\sigma(X)}\Big)X + c, \quad \xi_2 = \frac{\delta_1}{2\delta_2\sigma(X)}X - c \tag{5.15}$$

and the optimal dual measure μ is given by $\mu = \delta_1\frac{X-EX}{\sigma(X)} - 1$.

2) If $\frac{\delta_1}{2\delta_2\sigma(X)} \geq 1$, then the allocation $(0, X)$ is up to constants the unique PO-allocation.

Proof: Both risk functionals ϱ_1, ϱ_2 are convex, lsc, cash invariant and law invariant. Thus the intersection property is fulfilled and for all $X \in \operatorname{dom}(\varrho_1 \wedge \varrho_2)$ and $\delta_1, \delta_2 > 0$ there exists a Pareto optimal allocation.

To construct a PO-allocation, we use the characterization result finding $(\xi_1, \xi_2) \in A(X)$ and a measure μ (identified with its density) such that $\mu \in \partial\varrho_1(\xi_1) \cap \partial\varrho_2(\xi_2)$, i.e. by Lemmas 5.4 and 5.5

$$\left\{\delta_1 \frac{\xi_1 - E\xi_1}{\|\xi_1 - E\xi_1\|_2} - 1\right\} = \partial\varrho_1(\xi_1) = \partial\varrho_2(\xi_2) = \{2\delta_2(\xi_2 - E\xi_2) - 1\}. \quad (5.16)$$

In case 1) one finds a nontrivial solution as given in (5.15). In case 2) the following argument leads to the conclusion. Let $\eta_1, \eta_2 \in A(X)$ be an allocation of X, then

$$\begin{aligned}
\varrho_1(\eta_1) + \varrho_2(\eta_2) &= E(-X) + \delta_1\sigma(\eta_1) + \delta_2 \operatorname{Var}(\eta_2) \\
&\geq E(-X) + \delta_2 \operatorname{Var}(\eta_2) + 2\delta_2\sigma(X)\sigma(\eta_1) \quad (\text{as } \delta_1 \geq 2\delta_2\sigma(X)) \\
&\geq E(-X) + \delta_2 \operatorname{Var}(\eta_2) + 2\delta_2 \operatorname{Cov}(X, \eta_1) \\
&= E(-X) + \delta_2 \operatorname{Var}(X - \eta_1) + 2\delta_2 \operatorname{Cov}(X, \eta_1) \\
&= E(-X) + \delta_2(\operatorname{Var}(X) + \operatorname{Var}(\eta_1)) \\
&\geq E(-X) + \delta_2 \operatorname{Var}(X) = \varrho_2(X).
\end{aligned}$$

Thus $(0, X)$ is an optimal allocation. Uniqueness follows from our uniqueness result in Corollary 4.5. □

The characterization and uniqueness result allows to deal in a similar way with further convex risk functionals without assuming the monotonicity or the cash invariance property. One can e.g. consider risk functionals of the form

$$\varrho_1(X) = E(-X) + \delta E((X - q_\alpha(X))_-^p, \quad p \geq 1,$$

where $q_\alpha(X)$ is the α-quantile of X. ϱ is convex and monotone but not cash invariant. Similarly,

$$\varrho_2(X) = E(-X) + \delta E|X - q_\alpha(X)|^p, \quad p \geq 1,$$

which is neither monotone nor cash invariant. For some explicit further examples in the case of nonmonotone risk measures we refer to Acciaio (2007). There some more cases are considered which have as solutions the typical insurance contracts (stop-loss contracts). Many examples of this type are to be found in the relevant insurance literature. We refer to the classical literature Gerber (1979), Deprez and Gerber (1985), Kaas et al. (2001), and the references therein.

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