

Value-at-Risk bounds with variance constraints

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October 18, 2013

Abstract

Recent literature deals with bounds on the Value-at-Risk (VaR) of risky portfolios when only the marginal distributions of the components are known. In this paper we study Value-at-Risk bounds when the variance of the portfolio sum is also known, a situation that is of considerable interest in risk management.

We provide easy to calculate Value-at-Risk bounds with and without variance constraint and show that the improvement due to the variance constraint can be quite substantial. We discuss when the bounds are sharp (attainable) and point out the close connections between the study of VaR bounds and convex ordering of aggregate risk. This connection leads to the construction of a new practical algorithm, called Extended Rearrangement Algorithm (ERA), that allows to approximate sharp VaR bounds. We test the stability and the quality of the algorithm in several numerical examples.

We apply the results to the case of credit risk portfolio models and verify that adding the variance constraint gives rise to significantly tighter bounds in all situations of interest. However, model risk remains a concern and we criticize regulatory frameworks that allow financial institutions to use internal models for computing the portfolio VaR at high confidence levels (e.g., 99.5%) as the basis for setting capital requirements.

Key-words: Value-at-Risk, Convex order, Comonotonicity, Model risk, Rearrangement algorithm.

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1 Introduction

In this paper, we study bounds for the Value-at-Risk (VaR) of sums of risks with known marginal distributions (describing the stand-alone risks) under the additional constraint that the variance of the sum is known (which partially describes the dependence between risks). This setting is of significant interest as in many practical situations it corresponds closely to the maximum information at hand when assessing the VaR of a portfolio. For example, in the context of credit risk portfolio models one usually has knowledge about the marginal risks (through so-called PD, EAD and LGD models) and the variance of the aggregate risk (sum of the individual losses) is also often available (as obtained from default correlation models or through statistical analysis of observed credit losses). The same setting also appears in the context of risk aggregation and solvency calculations (Basel III and Solvency II). Banks and insurers usually have models to estimate risk distributions and VaR per risk type (credit risk, market risk, operational risk,...) and per business line, and next they rely on a correlation matrix to obtain the VaR of the aggregated portfolio. Taking into account this information we assess how much model risk is left in the computation of VaR.

Bounds on Value-at-Risk in the unconstrained case (only the marginal distributions are fixed) have been studied by Rüschen-dorf (1982) (in the case of two risks) and more recently by Denuit et al. (1999), Embrechts and Puccetti (2006a), Puccetti and Rüschen-dorf (2012a,b), and Embrechts et al. (2013). In particular, there exist several explicit results on VaR bounds when the marginals are identically distributed. In the inhomogeneous case, however, the analysis becomes rapidly more involved and explicit results are scarce. Puccetti and Rüschen-dorf (2012a) propose the rearrangement algorithm (RA) as a practical way to approximate bounds on the Value-at-Risk of a sum of individual risks when the marginal distributions are known; see also Embrechts et al. (2013). So far, numerical experiments have shown that the RA presents very good accuracy.

There are already results regarding VaR bounds in the presence of partial information on the dependence of the underlying risks in the sum; see e.g. Rüschen-dorf (1991), Embrechts and Puccetti (2009), Embrechts et al. (2013) for results when some of the bivariate distributions are known. However, the bounds that are proposed in these papers are often hard to compute numerically, especially for higher dimensions and in the inhomogeneous case. Moreover, in practice, bivariate distributions are usually not known. In contrast, the variance of the entire portfolio sum can often be statistically estimated with sufficient degree of accuracy or its value can be implied by the availability of the correlations between risks. Intuitively, as the variance measures the average spread of the aggregate portfolio loss around the mean, one could expect that its knowledge has a significant impact on the maximum possible VaR. Hence, in this paper we study bounds on the Value-at-Risk of a sum with a known maximum variance and with fixed marginals.

In Section 2 we give simple upper and lower VaR bounds in terms of the tail Value-at-Risk in the unconstrained case when only marginal information is available. These unconstrained bounds are also valid for the case of heterogeneous portfolios and several examples illustrate that they turn out to be reasonably sharp. Some asymptotic sharpness results on these bounds are given in Puccetti and Rüschen-dorf (2012d) in the homogeneous case and extended in Puccetti et al. (2013) to some inhomogeneous cases. We find that the upper and lower bounds are sharp if it is possible to construct random variables, which are mixing on the upper part resp. on the lower part of the distribution, i.e. when their sum is constant. We state a general version of an asymptotic sharpness result in this paper indicating that approximate sharpness of the unconstrained bounds can be expected for large portfolios. In Section 2 we also obtain a connection between the problem of obtaining good VaR bounds and results on convex ordering. This connection leads to further improved bounds in certain cases.

In Section 3, we consider an additional constraint on the variance of the joint portfolio and give an analytical bound in this case. We show that these (constrained) VaR bounds can be significantly tighter than the bounds in the unconstrained case. We describe conditions to ensure sharpness of these bounds. To obtain sharp VaR bounds for the constrained case, we show that it is necessary to make the distribution of the aggregate risk as flat as possible in the upper part as well as in the lower part while considering at the same time the variance constraint.

This insight gives the intuition to develop in Section 4 a new algorithm to determine approximate sharp bounds in the constrained case. This algorithm extends the rearrangement algorithm, which was proposed to approximate sharp VaR bounds in the unconstrained case by Embrechts et al. (2013). We simultaneously rearrange the upper and the lower part of the distribution of the sum and move in some systematic way through the domain of the random sum in order to fulfill the variance constraint. A series of examples show that the extended rearrangement algorithm (ERA) works well. In particular, we see under which conditions the additional variance constraint leads to essentially improved bounds. The development of this ERA seems to be an essential tool for applications.

Finally, in Section 5 we apply the results to the case of a credit risk portfolio. In this context, the variance constraint significantly improves the unconstrained bounds. We also show that the models that are used in the industry and regulatory frameworks underestimate risk. We criticize the way internal models in regulatory frameworks are used for establishing capital requirements. In particular, we do not recommend using them for computing portfolio VaR at high confidence levels (e.g., 99.5%) as the basis for setting capital requirements. It seems more effective to impose additional restrictions on the internal models used for setting the capital requirements, or even to enforce the use of a single model to this purpose. By doing so, the capital requirements become better comparable across different institutions, which also enhances fair competition. Final remarks are presented in Section 6. Some proofs are given in the appendix.

1.1 Problem description

Consider a portfolio with n individual risks X_j with finite mean and variance. Assume that the marginal distributions F_j of X_j for $j = 1, \dots, n$ are also given: we write $X_j \sim F_j$. Since the marginal distributions are fixed, the mean of the aggregate portfolio loss, $S_n = X_1 + X_2 + \dots + X_n$, is known and equal to

$$\mu := E(X_1 + X_2 + \dots + X_n). \quad (1.1)$$

In the main part of this paper we derive bounds on the Value-at-Risk of the sum S_n when its variance stays below some level s^2 . The dependence between the different X_i that attain the VaR bounds in the absence of a variance constraint may indeed give rise to a too high variance of the sum that is compared to the observed variance of a portfolio. This feature will be confirmed later with several examples.

Let us denote the Value-at-Risk of the portfolio sum S_n at q -confidence level ($0 < q < 1$) by $\text{VaR}_q[S_n]$.

$$\text{VaR}_q[S_n] = \inf \{x \in \mathbb{R} \mid F_{S_n}(x) \geq q\}, \quad q \in (0, 1),$$

where $F_{S_n}(x)$ is the distribution function of S_n . The VaR is thus defined as the left inverse of the distribution function and we may also write that $\text{VaR}_q[S_n] = F_{S_n}^{-1}(q)$. Similarly, we define the upper Value-at-Risk as an upper q -quantile, i.e.

$$\text{VaR}_q^+[S_n] = \sup \{x \in \mathbb{R} \mid F_{S_n}(x) \leq q\}.$$

In this paper, we are interested in finding the minimum possible VaR of the sum

$$\begin{aligned} m(s^2) &= \inf \text{VaR}_q[S_n] \\ &\text{subject to } X_j \sim F_j, \text{var}(S_n) \leq s^2. \end{aligned} \tag{1.2}$$

and the maximum possible VaR (in terms of the upper quantile)

$$\begin{aligned} M(s^2) &= \sup \text{VaR}_q^+[S_n] \\ &\text{subject to } X_j \sim F_j, \text{var}(S_n) \leq s^2. \end{aligned} \tag{1.3}$$

Note that we allow for $s^2 = \infty$ to include the absence of variance constraint. For ease of exposition we sometimes write m and M instead of $m(s^2)$ and $M(s^2)$. In Section 2 we consider the unconstrained case (thus without the constraint on the variance) and derive an upper bound for $M(\infty)$ and a lower bound for $m(\infty)$. We also discuss conditions for sharpness of these bounds. In Section 3 we derive bounds in the constrained case. We describe conditions that ensure sharpness of the bounds. All these bounds are usually not sharp. In Section 4 we introduce an algorithm called ERA, which allows to obtain approximations to sharp bounds in all circumstances.

In what follows we shall make extended use of the convex ordering, \leq_{cx} between random variables X, Y defined by

$$X \leq_{\text{cx}} Y \quad \text{if } E(f(X)) \leq E(f(Y))$$

for all convex functions $f(\cdot)$ such that the expectation exists. Note that $E(X) \leq_{\text{cx}} X$ and $X \leq_{\text{cx}} Y$ implies in particular that Y has the same mean as X and has larger variance than X . The convex order is thus a device that allows comparing the variability of random variables. For details of this ordering see Müller and Stoyan (2002) or Denuit et al. (2005).

2 VaR bounds of a portfolio with fixed marginal distributions

In this section we derive easy to calculate analytical VaR bounds for the portfolio sum, discuss their sharpness and provide a connection with convex ordering.

2.1 Unconstrained bounds

The first theorem provides bounds on the Value-at-Risk of a sum of risks with given marginals without considering a variance constraint on the portfolio sum (formally, solving Problems (1.2) and (1.3) with $s^2 = \infty$). The bounds in this section need finite first moments of the risks X_i but not necessarily finite second moments.

For random variables $X_i \sim F_i$, $1 \leq i \leq n$ and a random variable $U \sim U(0, 1)$, define $X_i^c = F_i^{-1}(U)$, $1 \leq i \leq n$. Then $X_i^c \sim F_i$ and (X_1^c, \dots, X_n^c) is a comonotonic vector associated with F_1, \dots, F_n . Let $S_n^c = \sum_{i=1}^n X_i^c$ denote the comonotonic sum. For $q \in (0, 1)$ we denote by $\text{TVaR}_q(X_i)$ the Tail Value-at-Risk at level q ,

$$\text{TVaR}_q(X_i) = \frac{1}{1-q} \int_q^1 \text{VaR}_u[X_i] du,$$

and by $\text{LTVaR}_q(X_i)$ the left Tail Value-at-Risk

$$\text{LTVaR}_q(X_i) = \frac{1}{q} \int_0^q \text{VaR}_u[X_i] du.$$

In the following theorem we state simple upper and lower bound for VaR in terms of TVaR and LTVaR. The upper bound can be found in the literature (see e.g. Puccetti and Rüschendorf (2012d, Introduction to Section 2)) but has not been stated there in explicit form.

Theorem 2.1 (unconstrained bounds). *Let $q \in (0, 1)$, $X_i \sim F_i$ ($i = 1, 2, \dots, n$), $S_n = \sum_{i=1}^n X_i$ and $S_n^c = \sum_{i=1}^n X_i^c$. Then,*

$$\begin{aligned} A := \sum_{i=1}^n \text{LTVaR}_q(X_i) = \text{LTVaR}_q(S_n^c) &\leq \text{VaR}_q[S_n] \\ &\leq \text{VaR}_q^+[S_n] \leq B := \sum_{i=1}^n \text{TVaR}_q(X_i) = \text{TVaR}_q(S_n^c). \end{aligned}$$

Proof. By definition of TVaR_q we have the well-known inequality

$$\text{VaR}_q^+[S_n] \leq \text{TVaR}_q(S_n). \quad (2.1)$$

Furthermore, by the classical result of Meilijson and Nadas (1979), $S_n \leq_{\text{cx}} S_n^c$, i.e. S_n is smaller than the comonotonic sum S_n^c in convex order. Since TVaR_q is a convex risk measure, this implies

$$\text{TVaR}_q(S_n) \leq \text{TVaR}_q(S_n^c) = \sum_{i=1}^n \text{TVaR}_q(X_i) = B. \quad (2.2)$$

This last conclusion is also a direct consequence of the sub-additivity of TVaR_q .

For the lower bound we make use of the fact that for any random variable X

$$q \text{LTVaR}_q(X) + (1 - q) \text{TVaR}_q(X) = E(X).$$

This implies as in the first part that $\text{VaR}_q[S_n] \geq \text{LTVaR}_q(S_n) \geq \sum_{i=1}^n \text{LTVaR}_q(X_i) = A$. \square

The bounds A and B are explicit and can be computed from the marginal distributions directly. Note that they are also valid (and as easy to compute) for heterogeneous portfolios (i.e. when individual risks do not have the same cdf). This feature contrasts with earlier results in the literature in which the bounds are usually harder to compute and are not always available when portfolios exhibit heterogeneity. The bounds A and B have been derived without considering a constraint on the variance of the portfolio and are thus also bounds in the presence of such constraint. Hence, from Theorem 2.1 it follows immediately that $A \leq m \leq M \leq B$.

The proof of Theorem 2.1 also allows to conclude under which conditions the bounds A and B are sharp (attained). To this purpose it is useful to represent (without loss of generality) the risks X_i as $X_i = f_i(U)$ ($i = 1, 2, \dots, n$) for some $U \sim U(0, 1)$ such that $\{S_n \geq \text{VaR}_q[S_n]\} = \{U \in [q, 1]\}$, i.e. the upper q -part of the distribution of S_n is given by the upper q -part $\{U \geq q\}$ of the random variable U . As $f_i(U)$ and $F_i^{-1}(U)$ have the same distribution we say that f_i is a ‘rearrangement’ of F_i^{-1} on $[0, 1]$ and write $f_i \sim_r F_i^{-1}$. Furthermore, if $f_i(V)$ and $F_i^{-1}(V)$ have the same distribution for some random variable V that is uniformly distributed on a subset T of $[0, 1]$, then we say that f_i is a rearrangement of F_i^{-1} on T and we write $f_i|_T \sim_r F_i^{-1}|_T$.

Theorem 2.2 (Sharpness of the unconstrained bounds). *Let $X_i = f_i(U) \sim F_i$, $1 \leq i \leq n$ and let $S_n = \sum_{i=1}^n X_i$ be the portfolio sum. Then:*

- a) *The upper bound B in Theorem 2.1 is attained by S_n if and only if the two following conditions are satisfied:*

- 1) $f_i|_{[q,1]} \sim_r F_i^{-1}|_{[q,1]}$, $1 \leq i \leq n$, i.e. the f_i are rearrangements of F_i^{-1} on $[q, 1]$
- 2) X_1, X_2, \dots, X_n are mixing on $\{U \geq q\}$, i.e. $\sum_{i=1}^n f_i(u) = c$, $\forall u \in [q, 1]$ for some $c \in \mathbb{R}$.

b) The lower bound A in Theorem 2.1 is attained by S_n if and only if the two following conditions are satisfied:

- 1) $f_i|_{(0,q)} \sim_r F_i^{-1}|_{(0,q)}$, $1 \leq i \leq n$
- 2) X_1, X_2, \dots, X_n are mixing on $\{U < q\}$, i.e. $\sum_{i=1}^n f_i(u) = c$, $\forall u \in [q, 1]$ for some $c \in \mathbb{R}$.

Proof. The upper bound B is sharp if and only if the inequalities in (2.1) and (2.2) are in fact equalities. Equality in (2.2) is by definition of TVaR_q equivalent to the fact that the f_i are rearrangements of F_i^{-1} on $[q, 1]$ (see for example Puccetti and Rüschendorf (2012b)). Furthermore, equality holds in (2.1) if and only if $S_n = \sum_{i=1}^n f_i(u) = c$ on $[q, 1]$, i.e. if and only if the random variables $X_i = f_i(U)$ are mixing on $[q, 1]$ (see also Figure 2.1). The argument for the sharpness in b) is similar. \square

Note that the constant c is equal to B resp. A in statement a) resp. b) of the theorem. A geometric interpretation of the bounds is given in Figure 2.1. The bound B – as an average of the upper VaRs

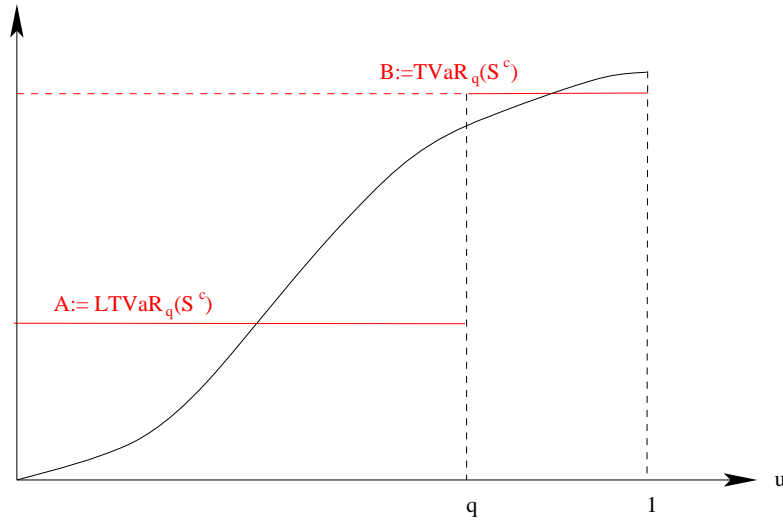


Figure 2.1 Illustration of Theorem 2.1. Representation of Value-at-Risk VaR_q as a function of the level $q \in (0, 1)$. Note that the VaR bounds at the level q corresponds to the TVaR and LTVaR of a comonotonic sum: $S^c := X_1^c + X_2^c + \dots + X_n^c$ where $(X_1^c, X_2^c, \dots, X_n^c) =_d (F_1^{-1}(U), F_2^{-1}(U), \dots, F_n^{-1}(U))$.

(from level q onwards) in the comonotonic case – clearly dominates the VaR_q^+ of the comonotonic sum. It is not possible that S_n has an (upper) quantile function that takes a value in q that is strictly larger than B as this would imply that the corresponding $\text{TVaR}_q(S_n)$ would be larger than B , which is not possible. Hence, B is a sharp bound if and only if the quantile function of S_n takes the value B from q onwards.

The mixing property on $[q, 1]$ may be too strong to achieve but a weakened form of asymptotic mixability for large sample sizes (formally, for $n \rightarrow \infty$) may hold true. This condition then implies that our upper bound is asymptotically sharp. The idea of this result can be found in Theorems 2.3 and 2.5 of Puccetti and Rüschendorf (2012c). In the homogeneous case an asymptotic sharpness result has been given in Puccetti and Rüschendorf (2012d); for some extensions see Puccetti et al. (2012). Precisely, the following result says that in great generality one can expect approximative sharpness of the unconstrained bounds A (for m) and B (for M) in the case of large portfolios.

Theorem 2.3 (Asymptotic sharpness of unconstrained bound). *Let X_i be integrable random variables, $X_i \sim F_i$. Assume that there exists $X_i^n = f_i^n(U) \sim F_i$, $1 \leq i \leq n$ such that for some sequences $\alpha_n \downarrow q$, $\beta_n \uparrow 1$ it holds*

$$(AM) \quad \sum_{i=1}^n X_i^n = c_n + R_n \quad \text{on } [\alpha_n, \beta_n],$$

where c_n are constants and $R_n = r_n(U)$ with $Er_n(U_{[q,1]}) \xrightarrow{n \rightarrow \infty} 0$, i.e. $S_n^a := \sum_{i=1}^n X_i^n$ is ‘asymptotically mixing’ on $[q, 1]$.

Then

$$\frac{\text{VaR}_q^+[S_n^a]}{\text{TVaR}_q(S_n^c)} \rightarrow 1,$$

i.e. the upper tail Value-at-Risk bound is asymptotically sharp.

The proof of this result is similar as the proof of the sharpness under the mixing condition in Theorem 2.2 and therefore omitted.

The asymptotic mixability condition (AM) can be verified in particular in the homogeneous case with decreasing densities (see Wang and Wang (2011)) but also in cases where F_i can take only finitely many distribution functions of this type say G_1, \dots, G_m (see also Puccetti et al. (2012, 2013)).

2.2 VaR bounds and convex order

Sharpness of the unconstrained bounds requires that the portfolio sum is constant on the upper part resp. on the lower part of the distribution. This feature suggests that there is a connection between studying “the variability” of sums and finding VaR bounds. Specifically, in this subsection we describe that the problem of obtaining good VaR bounds is closely related to the problem of determining lower minimal elements for the portfolio sum on the upper respectively on the lower part with respect to convex order. This connection is useful as it leads to improved bounds in certain cases.

In what follows we denote by F_i^q the distribution of F_i when restricted to the upper q -part of F_i , i.e. formally F_i^q is the distribution of $F_i^{-1}(U)$, where U is uniformly distributed on $[q, 1]$. The VaR upper bound problem can indeed be equivalently described by restricting to the upper q -part of the distributions (see Puccetti and Rüschendorf (2012b)). The following theorem states that the problem of obtaining a sharp upper bound for the Value-at-Risk of a sum is closely related to determine minimal sums in convex order (for given marginal distributions). More precisely, an improvement of a sum with respect to convex order leads to an improvement of the Value-at-Risk. Thus sharp upper bounds for the Value-at-Risk of a sum are given by a sum minimal in convex order.

Theorem 2.4 (VaR bounds and convex order). *Let $X_i, Y_i \sim F_i$ and let $X_i^q, Y_i^q \sim F_i^q$ be the restrictions to the upper q -parts. If*

$$S^q = \sum_{i=1}^n Y_i^q \leq_{\text{cx}} \sum_{i=1}^n X_i^q, \tag{2.3}$$

then

$$\text{VaR}_q^+ \left[\sum_{i=1}^n X_i \right] \leq \text{VaR}_0^+ [S^q] \leq B.$$

For the proof see the Appendix.

As a consequence we get again sharpness in the case that the distributions are mixing on the upper part.

Corrolary 2.5. Assume there exist $Y_i^q \sim F_i^q$ with $S^q = \sum_{i=1}^n Y_i^q = c$. Then for all $X_i \sim F_i$ it holds

$$\text{VaR}_q^+ \left[\sum_{i=1}^n X_i \right] \leq \text{VaR}_0^+[S^q] = B.$$

Remark 2.6.

- (i) In general, as stated in (2.3) the worst value for $\text{VaR}_q[S]$ is attained for some minimal element in the class \mathcal{F}^q of sums of the components of random vectors (Y_1, \dots, Y_n) such that $Y_i \sim F_i^q$ in convex order. In dimension $n = 2$ a smallest element in convex order on the upper part exists and is known to be the countermonotonic distribution, i.e.

$$X_1^{cm} = F_1^{-1}(U), \quad X_2^{cm} = F_2^{-1}(1 - U), \quad (2.4)$$

where $U = U_{[q,1]}$ is uniformly distributed on $[q, 1]$ (see Rüschemdorf (1983)). The resulting VaR bound is by Theorem 2.4 a sharp upper bound and is identical to the solution of this case in Rüschemdorf (1982).

- (ii) It is well-known that the comonotonic sum is the largest possible element in \mathcal{F}^q in convex order (Meilijson and Nadas (1979)),

$$S \leq_{\text{cx}} S^c \quad (2.5)$$

for $S = \sum_{i=1}^n Y_i$, $S^c = \sum_{i=1}^n Y_i^c$, $Y_i \sim F_i^q$. Hence, the comonotonic dependence does not yield a valid solution of our VaR upper bound¹.

- (iii) Similarly to Theorem 2.4 we get an improved lower bound for $\text{VaR}_q[\sum_{i=1}^n X_i]$. Let $F_{i,q}$ be the distribution of $F_i^{-1}(U)$ for $U \sim U(0, q)$ and let for $X_i, Y_i \sim F_i$, $X_{i,q}, Y_{i,q} \sim F_{i,q}$ be the restrictions to the lower q -part. Then we get:

If $S_q = \sum_{i=1}^n Y_{i,q} \leq_{\text{cx}} \sum_{i=1}^n X_{i,q}$, then

$$A \leq \text{VaR}_1 S_q \leq \text{VaR}_q \left[\sum_{i=1}^n X_i \right]. \quad (2.6)$$

Thus from convex ordering we get an improved lower bound for the Value-at-Risk given by the upper bound of S_q . In particular, sharp lower bounds for the Value-at-Risk of a sum are given by a sum minimal in convex order. If the $Y_{i,q}$ ($i = 1, 2, \dots, n$) can be chosen mixing on $[0, q]$, then obviously $S_q = \sum_{i=1}^n Y_{i,q} \leq_{\text{cx}} S = \sum X_{i,q}$ and we obtain as a consequence sharp lower bounds (see Theorem 2.2).

3 VaR bounds with a variance constraint on the portfolio sum

3.1 Variance-constrained bounds

We now propose to improve the bounds by constraining the variance of the sum $X_1 + X_2 + \dots + X_n$ to be below a maximum level s^2 . From the introduction we recall that this setting is highly relevant as in many practical situations the use of historical data on observed portfolio losses allows to estimate (a bound on) the variance of the portfolio sum. We first provide an example that gives some intuition as to how we can deal with the constrained problem.

¹An exception occurs when all risks are right bounded and the probability level q used for assessing the VaR is sufficiently large.

Example 3.1. Let X_1 and X_2 be both uniformly distributed on the unit interval and assume that $s^2 \geq \frac{3}{16}$ (e.g., $s^2 = \infty$). We are interested in finding VaR bounds for $X_1 + X_2$ with $q = 0.75$. From Theorem 2.1, $A = \frac{3}{4}, B = \frac{7}{4}$. Next, consider Y_1 and Y_2 , uniformly distributed random variables over $(0, 1)$ such that $Y_2 = \frac{3}{4} - Y_1$ if $Y_1 < \frac{3}{4}$ and $Y_2 = \frac{7}{4} - Y_1$ if $Y_1 \geq \frac{3}{4}$. Note that $Y_1 + Y_2 = \frac{3}{4} = A$ for $Y_1 < \frac{3}{4}$ and $Y_1 + Y_2 = \frac{7}{4} = B$ for $Y_1 \geq \frac{3}{4}$. Thus, $\text{VaR}_q[Y_1 + Y_2] = A$ and $\text{VaR}_q^+[Y_1 + Y_2] = B$. Furthermore, $\text{var}(Y_1 + Y_2) = q(A - 1)^2 + (1 - q)(B - 1)^2 = \frac{3}{16} \leq s^2$, thus the variance constraint is satisfied. Hence, for $s^2 \geq \frac{3}{16}$ we have that $m = A$ and $M = B$ and we have also shown that both m and M are attained by (Y_1, Y_2) .

The example shows that the unconstrained bounds A and B can also be sharp for the constrained problems (1.2) and (1.3) respectively. More surprisingly, the dependence structures to attain the upper resp. the lower bound are identical. When $s^2 < \frac{3}{16}$, the bounds A and B will not be sharp for the respective problems (1.2) and (1.3).

Inspired by Example 3.1 and Figure 2.1, we define a random variable X^* that takes two possible values corresponding to the bounds A and B that we derived in Theorem 2.1:

$$X^* = \begin{cases} A & \text{with probability } q, \\ B & \text{with probability } 1 - q. \end{cases} \quad (3.1)$$

Then, the cdf F of X^* verifies $F(x) = 0$ if $x < A$, $F(x) = q$ if $A \leq x < B$ and $F(x) = 1$ if $x \geq B$. Note that $E(X^*) = \mu$ whereas $\text{var}(X^*)$ is given as $\text{var}(X^*) = q(A - \mu)^2 + (1 - q)(B - \mu)^2$. This distribution is going to play a key role in solving the constrained problems (1.2) and (1.3).

In the presence of an additional variance constraint on the portfolio sum, A and B (as in Theorem 2.1) are still bounds for $\text{VaR}_q[X_1 + X_2 + \dots + X_n]$ and they may still be attained in which case they are best possible. For example, assume that the lowest value that $X_1 + X_2 + \dots + X_n$ takes is A with probability q . In this case, $X_1 + X_2 + \dots + X_n$ has minimum variance² if $X_1 + X_2 + \dots + X_n =_d X^*$ and thus A may be attained depending on the value s^2 . Similarly, the upper bound for the Value-at-Risk (with confidence q) is reached when B is the largest value that $X_1 + X_2 + \dots + X_n$ can take (with probability $1 - q$). Thus, when the variable X^* satisfies the variance constraint, i.e., $\text{var}(X^*) \leq s^2$, then the bounds A and B cannot be readily improved. However, if $\text{var}(X^*) > s^2$, then A and B are generally too wide and better bounds are to be constructed. It is then intuitive that better bounds can be found by constructing a variable Y taking two values a (larger than A) and b (smaller than B) with respective probabilities q and $1 - q$ in such a way that the variance constraint of the portfolio sum is satisfied.

Pursuing this idea further, let us define the function

$$B(\alpha) := \frac{1}{1 - q} \int_{q-\alpha}^{1-\alpha} \text{VaR}_u[S^c] \, du \quad (3.2)$$

on the interval $[0, q]$ and note that $B(0) = B$. Let us also define the variable X_α^* taking the values $A(\alpha)$ and $B(\alpha)$,

$$X_\alpha^* = \begin{cases} A(\alpha) & \text{with probability } q, \\ B(\alpha) & \text{with probability } 1 - q, \end{cases} \quad (3.3)$$

in which $A(\alpha) := \frac{\mu - B(\alpha)(1 - q)}{q}$. Note that $A(0) = A$ and that $X_0^* =_d X^*$. We have the following property.

²It is easy to prove that X^* is of minimum variance among all the random variables Z that take the value A with probability q and that satisfy $E(Z) = \mu$.

Proposition 3.2. *There exists $0 < \beta < q$ such that $\alpha \rightarrow \text{var}(X_\alpha^*) := q(A(\alpha) - \mu)^2 + (1 - q)(B(\alpha) - \mu)^2$ is continuously decreasing on $[0, \beta]$ with minimum value given by $\text{var}(X_\beta^*) = 0$ and with maximum value given by $\text{var}(X_0^*) = q(A - \mu)^2 + (1 - q)(B - \mu)^2$.*

The following theorem shows that the construction as outlined above gives bounds on Value-at-Risk in the presence of a variance constraint.

Theorem 3.3. *Let $q \in (0, 1)$, $X_i \sim F_i$ ($i = 1, 2, \dots, n$), $S_n = \sum_{i=1}^n X_i$ satisfying $\text{var}(S_n) \leq s^2$. Then we have*

$$a) \quad a := A(\alpha^*) \leq m \leq \text{VaR}_q[S_n] \leq \text{VaR}_q^+[S_n] \leq M \leq b := B(\alpha^*).$$

where α^* is defined as

$$\alpha^* := \min \{ \alpha \mid 0 \leq \alpha \leq q, \text{var}(X_\alpha^*) \leq s^2 \}, \quad (3.4)$$

b) *If $s^2 \geq q(A - \mu)^2 + (1 - q)(B - \mu)^2$ then $a = A$ and $b = B$.*

$$\text{Otherwise,} \quad a = \mu - s \sqrt{\frac{1 - q}{q}} \text{ and } b = \mu + s \sqrt{\frac{q}{1 - q}}. \quad (3.5)$$

The proof of Theorem 3.3 is given in the Appendix.

The presence of the variance constraint does not always strengthen the bounds A and B . Indeed, the variable X^* ($= X_0^*$), taking two values A and B may also satisfy the variance constraint, i.e. $\text{var}(X^*) \leq s^2$ and in this case $a = A$ and $b = B$. Hence, we conclude that if s^2 is not too large (i.e. when $s^2 \leq q(A - \mu)^2 + (1 - q)(B - \mu)^2$) then the bounds a and b that are obtained for the constrained case strictly outperform the bounds in the unconstrained case.

The question thus rises what is meant with “too large”. This aspect pertains to the characteristics of the problem and the data at hand. However, a few observations are of interest. When all risks are identically distributed then the bounds A and B grow linearly with the size of the portfolio but on the other hand, as the standard deviation of a portfolio is sub-additive, the condition $s^2 \geq q(A - \mu)^2 + (1 - q)(B - \mu)^2$ becomes harder to satisfy, meaning that it becomes more likely that the bounds a and b are better than A and B . For example, when the risks (e.g. in a life insurance context) are approximately independent, then the bounds a and b will strictly improve upon A and B for moderate portfolio sizes. On the other hand, when all risks are positively and equally correlated, then when n gets large a new risk X_{n+1} , that is added to an existing portfolio loss S_n , becomes perfectly correlated with this portfolio, thus resulting in standard deviations that add up meaning that a and b may be identical to A and B again.

Remark 3.4. (i) When there is no information on the dependence available we are still able to conclude

$$\mu - \sum_{i=1}^n \sigma_i \sqrt{\frac{q}{1 - q}} \leq \text{VaR}_q \left[\sum_{i=1}^n X_i \right] \leq \text{VaR}_q^+ \left[\sum_{i=1}^n X_i \right] \leq \mu + \sum_{i=1}^n \sigma_i \sqrt{\frac{q}{1 - q}}. \quad (3.6)$$

where $\sigma_i^2 = \text{var}(X_i)$, $1 \leq i \leq n$. Indeed, as the standard deviation is sub-additive it is clear that $s \leq \sum_{i=1}^n \sigma_i$. Thus (3.6) is a consequence of Theorem 3.3.

- (ii) Sometimes one knows some of the correlations between the risks X_i . This partial information can then be used to give an upper bound on the variance of the portfolio sum, which may or may not sharpen the unconstrained Value-at-Risk bounds that we derived in Theorem 3.3. For example, assume that $\text{var}(X_1 + X_2 + \cdots + X_i) \leq s_1^2$ and $\text{var}(X_{i+1} + X_2 + \cdots + X_n) \leq s_2^2$. Then,

$$\mu - (s_1 + s_2) \sqrt{\frac{q}{1-q}} \leq \text{VaR}_q \left[\sum_{i=1}^n X_i \right] \leq \text{VaR}_q^+ \left[\sum_{i=1}^n X_i \right] \leq \mu + (s_1 + s_2) \sqrt{\frac{q}{1-q}}. \quad (3.7)$$

Note however that $\sqrt{\text{var}(X_1 + X_2 + \cdots + X_n)} = s \leq s_1 + s_2$ implying that the knowledge of some correlations does not readily allow to improve upon the bounds in Theorem 3.3.

3.2 Sharpness and convex order

By Theorem 3.3 b), the bounds in the variance-constrained case are given by a two-point distribution in a, b for small variance respectively A, B for large variance. Similarly as in Theorem 2.2 for the unconstrained case we show that the constrained bounds are sharp if and only if risks X_1, X_2, \dots, X_n have a mixing property.

Theorem 3.5 (Sharpness of variance-constrained bounds). *Let $X_i = f_i(U) \sim F_i$, $1 \leq i \leq n$ where U is uniformly distributed on $(0, 1)$, and $S_n = \sum_{i=1}^n X_i$ satisfies the variance constraint, i.e. $\text{var}(S_n) \leq s^2$. Without loss of generality let $\{S_n \geq \text{VaR}_q[S_n]\} = \{U \geq q\}$. Then, the upper bound b in Theorem 3.3 b) is attained if and only if the lower bound a is attained and equivalently, if*

$$S_n = b \text{ on } \{U \geq q\} \text{ and } S_n = a \text{ on } \{U < q\}, \quad (3.8)$$

i.e. S_n is mixing on the upper q -part $\{U \geq q\}$ and on the lower q -part $\{U < q\}$ with mixing constants b, a .

In general, the bounds proposed in Theorem 3.3 are not sharp. Theorem 3.5 however suggests how to obtain sharp VaR bounds when there is a constraint on the variance of the sum. Loosely speaking, the outcomes of the variables should be rearranged to produce a dependence between the risks such that the outcomes for the sum are as concentrated as possible around the two values a and b that occur with respective probabilities $1 - q$ and q . This idea is concordant with the aim of finding convex order bounds. Indeed, the improvement result in Theorem 2.4 based on convex order that was valid for the unconstrained case extends in a similar way also to the variance-constrained case. Let $Y_i = f_i(U) \sim F_i$, $i = 1, 2, \dots, n$, and let $S_n = \sum_{i=1}^n Y_i$ with $\text{var}(S_n) \leq s^2$, be an admissible solution for the constrained VaR $_q$ upper bound problem. Let the upper q -part of the distribution of S_n , $\{S_n \geq \text{VaR}_q[S_n]\}$, be without loss of generality identical to $\{U \geq q\}$.

Then we get the following generalization of Theorem 3.5 saying that optimal solutions of the constrained problem should be as flat as possible on their upper and on their lower q -part. In what follows, for given random variables X, Y and subset T of Ω , $X|_T \leq_{\text{cx}} Y|_T$ denotes that the conditional distribution of the restriction $X|_T$ is smaller in convex order than $Y|_T$.

Theorem 3.6 (Variance-constrained bounds and convex order). *If $X_i \sim F_i$ and $\bar{S}_n = \sum_{i=1}^n X_i$ satisfy*

$$\bar{S}_n|_{U \geq q} \leq_{\text{cx}} S_n|_{U \geq q} \quad \text{and} \quad \bar{S}_n|_{U < q} \leq_{\text{cx}} S_n|_{U < q}, \quad (3.9)$$

then \bar{S}_n is admissible, $\text{var}(\bar{S}_n) \leq s^2$, and \bar{S}_n is an improvement of S_n in the sense that

$$\text{VaR}_q^+[\bar{S}_n] \geq \text{VaR}_q^+[S_n] \quad \text{and} \quad \text{VaR}_q[\bar{S}_n] \leq \text{VaR}_q[S_n].$$

The proof of Theorem 3.6 is given in the Appendix. Theorem 3.6 says that in order to get sharp upper bounds of $\text{VaR}_q^+[S_n]$ and sharp lower bounds for $\text{VaR}_q[S_n]$ one should try to rearrange the random variables Y_i on the upper q -part $T = \{U \geq q\}$ of S_n and also on their lower q -part $T^c = \{U < q\}$ such that the distribution is as flat as possible (in convex order) on T and on T^c . In particular this holds true for an optimal solution of the variance-constrained problem. Both flattenings can be done separately or simultaneously.

This basic idea is developed in the next section into an algorithm that allows to approximate sharp VaR_q bounds in the variance-constrained case.

4 The extended rearrangement algorithm (ERA)

This section extends the rearrangement algorithm (RA) of Puccetti and Rüschendorf (2012a) and Embrechts, Puccetti and Rüschendorf (2013) for finding approximate sharp bounds on the distribution of a function of n dependent random variables having fixed marginal distributions. The basic idea of the ERA algorithm is based on Section 3.2 and aims at making the distribution of S_n as flat as possible on the upper and lower part by applying the RA algorithm on both parts and by moving in a systematic way through the domains in order to satisfy the variance constraint.

The algorithm requires random variables that are discretely distributed. Let d be the number of points used to discretize the random variables and assume that each risk X_j ($j = 1, 2, \dots, n$) is sampled into d equiprobable values.

To avoid confusion, when we compute the bounds A, B resp. a, b for the discretized (sampled) risks, then we use the notations A_d, B_d resp. a_d, b_d .

$$B_d = \sum_{j=1}^n \frac{1}{d-k} \sum_{i=k+1}^d x_{ij}, \quad A_d = \sum_{j=1}^n \sum_{i=1}^d x_{ij} - \frac{d-k}{k} B_d. \quad (4.1)$$

Denote by μ_d the mean of the sum of the discretized risks. We first describe briefly the RA and next describe in detail the extended rearrangement algorithm that we propose as a suitable way to compute numerical VaR bounds of portfolios in the presence of a variance constraint. We point out that unlike the theoretical VaR bounds the algorithm does not require that the risks have finite mean (unconstrained bounds) or variance (constrained bounds).

4.1 The Rearrangement Algorithm (RA)

The rearrangement algorithm (RA) can be seen as a method to construct dependence between variables X_j ($j = 1, 2, \dots, n$) such that the distribution of S_n becomes as small as possible in convex order. For each j , define $x_{ij} := F_j^{-1}(\frac{i}{d+1})$ ($i = 1, \dots, d; j = 1, 2, \dots, n$) to obtain a $d \times n$ matrix $\mathbf{X} = (x_{ij})$, corresponding to a multivariate vector (X_1, X_2, \dots, X_n) that is comonotonic. Let us denote the matrix after rearrangement by $\mathbf{X}^* = (x_{ij}^*)$. In order to make the distribution of $S_n^* = X_1^* + X_2^* + \dots + X_n^*$ as small as possible in convex order, one needs to rearrange the elements within each column j ($j = 1, 2, \dots, n$) the d values x_{ij} ($i = 1, \dots, d$) such that the function $i \rightarrow \sum_{j=1}^n x_{ij}^*$ is “as flat as possible” corresponding to the objective of making the distribution of $S_n^* = X_1^* + X_2^* + \dots + X_n^*$ as small as possible in convex order. Note that as the rearrangements are only done within columns, X_j and X_j^* will have the same distribution. Of course, when the distribution is S_n is the smallest possible in the sense of convex order it must also have minimum variance. Hence, it must hold that for all $\ell = 1, 2, \dots, n$, X_ℓ is anti-monotonic with $\sum_{k=1, k \neq \ell}^n X_k^*$ (Puccetti and Rüschendorf (2012a), Theorem

2.1)). This observation is at the core of their rearrangement method: The subsequent columns of the matrix are rearranged such that they become anti-monotonic with the sum of all other columns until convergence is reached.

This algorithm was successfully used to compute (approximate) VaR bounds on the sum of n dependent risks with given marginal distributions. Indeed, to maximize the Value-at-Risk of a sum of dependent risks, Embrechts et al. (2013) applies the RA on the last rows of the matrix (corresponding to the highest values for each risk), accounting for a probability $1 - q$. To fix the ideas let us assume that there exists $k \in \{0, 1, \dots, d\}$, such that

$$q = \frac{k}{d}.$$

Embrechts, Puccetti and Rüschendorf (2013) then apply the RA on a $(d - k) \times n$ sub-matrix consisting of the rows $k + 1, \dots, d$ of the original matrix \mathbf{X} and accounting for a probability $1 - q$. By doing so the quantile function of S_n^* becomes “as flat as possible” on $[q, 1]$ while preserving the distributions of X_j . Then, in the ideal situation where the quantile function is constant on $[q, 1]$, one obtains a sum S_n^* with a VaR (at confidence level q) that is equal to the TVaR of the comonotonic sum $X_1 + X_2 + \dots + X_n$ and thus it is the maximum possible. It is clear that the minimum VaR can be obtained approximately by applying the RA on a $k \times n$ sub-matrix consisting of the rows $1, \dots, k$ of the original matrix \mathbf{X} .

As suggested by Theorem 3.6, for the solution of the constrained VaR bound problem it is necessary to minimize the distributions of S_n in convex order on the upper and on the lower q -part while satisfying the variance constraint. The following extension of the RA algorithm is consistent with this idea by choosing in a suitable way rearrangements that are admissible for the variance constraint and applying then the RA algorithm to the corresponding upper and lower q -parts separately.

4.2 Extended Rearrangement Algorithm

Based on Theorems 3.5 and 3.6, it is natural to modify the rearrangement algorithm used by Embrechts et al. (2013) and Puccetti and Rüschendorf (2012a) to construct numerically the minimum and maximum bounds.

- Step 1: For each $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, d\}$, define $x_{ij} := F_j^{-1}(\frac{i}{d+1})$. Let $\mu_d = \frac{1}{d} \sum_{i=1}^d \sum_{j=1}^n x_{ij}$
- Step 2: Calculate A_d and B_d as in (4.1) (Theorem 2.1) and b_d (Theorem 3.3) for the sampled variables.
- Step 3: If $q(A_d - \mu_d)^2 + (1 - q)(B_d - \mu_d)^2 \leq s^2$ then go directly to step 4 and the mixing constants³ are A_d and B_d . Otherwise,

- Step 3a: for all $m = 1, 2, \dots, k$ compute $b_d(m) := \frac{\sum_{i=k+1-m}^{d-m} \sum_{j=1}^n x_{ij}}{d-k}$.
- Step 3b: $m^* = \min\{m | b_d(m) \leq b_d\}$.

³We know that A_d and B_d are also going to be the constrained bounds in this case and that the variance constraint is satisfied automatically (see the second part of Theorem 3.3).

– Step 3c: replace M by a new matrix

$$M := (m_{ij})_{ij} \leftarrow \begin{bmatrix} x_{(d-m^*+2)1} & x_{(d-m^*+2)2} & \cdots & x_{(d-m^*+2)n} \\ x_{(d-m^*+3)1} & x_{(d-m^*+3)2} & \cdots & x_{(d-m^*+3)n} \\ \vdots & \vdots & & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dn} \\ x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{(d-m^*+1)1} & x_{(d-m^*+1)2} & \cdots & x_{(d-m^*+1)n} \end{bmatrix}$$

Note that when $m^* = 1$ then $m_{ij} = x_{ij}$, for all $i = 1, \dots, d$ and $j = 1, \dots, n$.

- Step 4: Initiate $r \leftarrow 0$ and $v_{-1} \leftarrow +\infty$.
- Step 5: Apply the RA on $(I) := \{(m_{ij})_{i=1, \dots, k, j=1, \dots, n}\}$, and obtain $(x_{ij}^*)_{i=1..k, j=1..n}$.
- Step 6: Apply the RA on $(II) := \{(m_{ij})_{i=(k+1), \dots, d, j=1, \dots, n}\}$, and obtain $(x_{ij}^*)_{i=(k+1)..d, j=1..n}$.
- Step 7: Compute $S_i^* = \sum_j x_{ij}^*$ for each $i = 1, \dots, d$.
- Step 8: Compute $v_r := \frac{1}{d} \sum_i (S_i^*)^2 - (\sum_i \frac{S_i^*}{d})^2$.
 - If $v_r < s^2$ then the approximate solutions to Problems (1.2) and (1.3) have been found; the lower bound is given by $\max_{i \leq k} (x_{i1}^* + \dots + x_{in}^*)$ and the upper bound by $\min_{i \geq k+1} (x_{i1}^* + \dots + x_{in}^*)$. The algorithm must stop here.
 - If $v_r > v_{r-1}$ the variance starts to increase and the algorithm will not converge (as the variance bound can not be satisfied). The algorithm must stop here.
 - Otherwise, $r \leftarrow r + 1$ and go back to step 5 with the following new matrix M

$$M := (m_{ij})_{ij} \leftarrow \begin{bmatrix} m_{d1} & m_{d2} & \cdots & m_{dn} \\ m_{11} & m_{12} & \cdots & m_{1n} \\ \vdots & \vdots & & \vdots \\ m_{(d-1)1} & m_{(d-1)2} & \cdots & m_{(d-1)n} \end{bmatrix}.$$

At each step 5 and 6, the algorithm makes the quantile function of the sum as flat as possible to the left (respectively to the right) of the cutoff point corresponding to desired probability level q , used to assess the VaR. By construction, one expects the first value obtained for v to be slightly larger than s^2 . As r increases, the average value on the left increases but the average value on the right decreases. Hence the new value v will decrease and may become smaller than s^2 . If s^2 is a “feasible” variance constraint, then the algorithm will usually stop. However, due to the discretization and the fact that there is no formal guarantee that the algorithm provides the true solution, it may happen that v never becomes smaller than s^2 or that the values for v start increasing again. The algorithm stops as soon as one of these two events happens.

To further improve the quality of the algorithm, it appears very useful (especially when risks are heavily tailed) to compute an upper bound (as an approximation for b_d) and a lower bound (as an

approximation for a_d) separately. To this end, one can use the property that for any random variable X it holds that

$$-\text{VaR}_{1-q}^+[-X] = \text{VaR}_q[X].$$

Hence, one runs the extended RA twice: one time as it is described above and one time with X replaced by $-X$ and q replaced by $1 - q$. Next one takes the best upper and lower bounds among the two algorithms. Typically the algorithm as described above works better for obtaining an upper bound whereas replacing the role of X by $-X$ and q by $1 - q$ allows to get a better approximation for the lower bound. Finally, note that by setting the constraint $s > \sum_{i=1}^n \sigma_i$ the ERA behaves as the RA and thus allows to determine numerical unconstrained bounds.

The bounds that are obtained by running the algorithm are within the interval (a_d, b_d) . If the obtained values are close to the boundary values a_d and b_d then this means that the algorithm is indeed able to identify a dependence that gives rise to (almost) sharp VaR bounds for the portfolio sum. In the next section we assess the performance of the ERA using some examples. These examples suggest that the algorithm that we propose to compute approximate VaR bounds is performing very well.

4.3 Example with normally distributed risks

Assume that (X_1, X_2, \dots, X_n) is a vector of dependent standard normally distributed random variables with a correlation matrix (ρ_{ij}) such that $\rho_{ii} = 1$ for $i = 1, \dots, n$ and $\rho_{ij} = \rho$ for all $i \neq j$. Note that the dependence structure of (X_1, X_2, \dots, X_n) is only partially specified (through the knowledge of the pairwise correlations) so that we cannot compute the VaR of the portfolio sum $S_n = X_1 + X_2 + \dots + X_n$ precisely. Hence, we apply the ERA to compute numerical upper and lower bounds and we compare them with the theoretical bounds of Theorem 2.1 and Theorem 3.3.

In Table 4.1, we assess the VaRs at 95%, 99%, 99,5% for three levels of correlation $\rho = 0$, $\rho = 0.15$ and $\rho = 0.3$ and the portfolio sizes $n = 10$ and $n = 100$. Panel A gives the constrained lower and upper VaR bounds using the ERA for different discretization levels, namely $d = 1,000$, $d = 10,000$ and $d = 100,000$. We denote these constrained lower and upper bound by m_d and M_d , respectively. Panel B gives the corresponding constrained bounds a_d and b_d as an application of Theorem 3.3, whereas Panel C gives the unconstrained bounds A_d and B_d using Theorem 2.1. The last line in Panel B and C gives the values for $a_\infty := a$ and $b_\infty := b$, respectively $A_\infty := A$ and $B_\infty := B$. These bounds are based on the original (non discretized) distributions and are explicitly given as

$$A = -n \frac{\phi(\Phi^{-1}(q))}{q}, \quad B = n \frac{\phi(\Phi^{-1}(q))}{1 - q}$$

and

$$a = \max \left(-s \sqrt{\frac{1 - q}{q}}, A \right), \quad b = \min \left(s \sqrt{\frac{1 - q}{q}}, B \right),$$

where ϕ and Φ denote the standard normal density and distribution function and where $s^2 = n + n(n - 1)\rho$ is the variance of S_n . Finally, Panel D assumes that (X_1, X_2, \dots, X_n) is multivariate normally distributed (the dependence is thus assumed to be Gaussian) in which case the VaR numbers of the portfolio sum can be computed exactly.

There are several interesting observations. *First*, when comparing the results of Panel A and Panel B, we observe that the ERA is performing remarkably well. The obtained numerical bounds m_d and M_d are very close to their theoretical counterparts a_d and b_d , showing that the ERA is able to construct the dependence between the risks such that the sum is (almost exactly) concentrated on two values a_d and b_d .

Panel A: Approximate sharp bounds obtained by the ERA as presented in Section 4.2

(m_d, M_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(-0.709; 13.69)	(-1.073; 20.43)	(-1.073; 20.43)	(-2.284; 43.15)	(-9.131; 172.3)	(-10.75; 204.4)
VaR _{99%} , $d = 1,000$	(-0.260; 26.00)	(-0.260; 25.99)	(-0.260; 26.00)	(-0.993; 98.49)	(-2.627; 260.2)	(-2.627; 260.2)
VaR _{99.5%} , $d = 1,000$	(-0.136; 27.86)	(-0.136; 27.85)	(-0.136; 27.85)	(-0.695; 139.9)	(-1.400; 278.9)	(-1.400; 278.9)
VaR _{95%} , $d = 10,000$	(-0.721; 13.77)	(-1.084; 20.60)	(-1.084; 20.60)	(-2.293; 43.58)	(-9.133; 173.3)	(-10.84; 206.0)
VaR _{99%} , $d = 10,000$	(-0.268; 26.56)	(-0.268; 26.56)	(-0.268; 26.56)	(-1.003; 99.40)	(-2.683; 265.6)	(-2.683; 265.6)
VaR _{99.5%} , $d = 10,000$	(-0.144; 28.76)	(-0.144; 28.76)	(-0.144; 28.76)	(-0.706; 141.0)	(-1.445; 287.6)	(-1.445; 287.6)
VaR _{95%} , $d = 100,000$	(-0.725; 13.78)	(-1.085; 20.62)	(-1.085; 20.62)	(-2.294; 43.58)	(-9.133; 173.5)	(-10.85; 206.2)
VaR _{99%} , $d = 100,000$	(-0.269; 26.64)	(-0.269; 26.64)	(-0.269; 26.64)	(-1.005; 99.50)	(-2.691; 266.4)	(-2.691; 266.4)
VaR _{99.5%} , $d = 100,000$	(-0.145; 28.90)	(-0.145; 28.90)	(-0.145; 28.90)	(-0.709; 141.1)	(-1.452; 289.0)	(-1.452; 289.0)

Panel B: Constrained bounds as obtained in Theorem 3.3

(a_d, b_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(-0.721; 13.70)	(-1.076; 20.44)	(-1.076; 20.44)	(-2.280; 43.33)	(-9.079; 172.5)	(-10.76; 204.4)
VaR _{99%} , $d = 1,000$	(-0.263; 26.02)	(-0.263; 26.02)	(-0.263; 26.02)	(-0.999; 98.90)	(-2.628; 260.2)	(-2.628; 260.2)
VaR _{99.5%} , $d = 1,000$	(-0.140; 27.89)	(-0.140; 27.89)	(-0.140; 27.89)	(-0.705; 140.2)	(-1.402; 278.9)	(-1.402; 278.9)
VaR _{95%} , $d = 10,000$	(-0.725; 13.77)	(-1.084; 20.60)	(-1.084; 20.60)	(-2.292; 43.55)	(-9.126; 173.4)	(-10.84; 206.0)
VaR _{99%} , $d = 10,000$	(-0.268; 26.56)	(-0.268; 26.56)	(-0.268; 26.56)	(-1.004; 99.42)	(-2.683; 265.6)	(-2.683; 265.6)
VaR _{99.5%} , $d = 10,000$	(-0.144; 28.76)	(-0.144; 28.76)	(-0.144; 28.76)	(-0.708; 141.0)	(-1.445; 287.6)	(-1.445; 287.6)
VaR _{95%} , $d = 100,000$	(-0.725; 13.78)	(-1.085; 20.62)	(-1.085; 20.62)	(-2.294; 43.58)	(-9.133; 173.5)	(-10.85; 206.2)
VaR _{99%} , $d = 100,000$	(-0.269; 26.64)	(-0.269; 26.64)	(-0.269; 26.64)	(-1.005; 99.49)	(-2.691; 266.4)	(-2.691; 266.4)
VaR _{99.5%} , $d = 100,000$	(-0.145; 28.9)	(-0.145; 28.90)	(-0.145; 28.90)	(-0.709; 141.1)	(-1.452; 289.0)	(-1.452; 289.0)
VaR _{95%} , $d = +\infty$	(-0.725; 13.78)	(-1.086; 20.63)	(-1.086; 20.63)	(-2.294; 43.59)	(-9.134; 173.5)	(-10.86; 206.3)
VaR _{99%} , $d = +\infty$	(-0.269; 26.65)	(-0.269; 26.65)	(-0.269; 26.65)	(-1.005; 99.50)	(-2.692; 266.5)	(-2.692; 266.5)
VaR _{99.5%} , $d = +\infty$	(-0.145; 28.92)	(-0.145; 28.92)	(-0.145; 28.92)	(-0.709; 141.1)	(-1.453; 289.2)	(-1.453; 289.2)

Panel C: Unconstrained bounds as obtained in Theorem 2.1

(A_d, B_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(-1.076; 20.44)	(-1.076; 20.44)	(-1.076; 20.44)	(-10.76; 204.4)	(-10.76; 204.4)	(-10.76; 204.4)
VaR _{99%} , $d = 1,000$	(-0.262; 26.02)	(-0.263; 26.02)	(-0.263; 26.02)	(-2.628; 260.2)	(-2.628; 260.2)	(-2.628; 260.2)
VaR _{99.5%} , $d = 1,000$	(-0.140; 27.89)	(-0.140; 27.89)	(-0.140; 27.89)	(-1.402; 278.9)	(-1.402; 278.9)	(-1.402; 278.9)
VaR _{95%} , $d = 10,000$	(-1.084; 20.60)	(-1.084; 20.60)	(-1.084; 20.60)	(-10.84; 206.0)	(-10.84; 206.0)	(-10.84; 206.0)
VaR _{99%} , $d = 10,000$	(-0.268; 26.56)	(-0.268; 26.56)	(-0.268; 26.56)	(-2.683; 265.6)	(-2.683; 265.6)	(-2.683; 265.6)
VaR _{99.5%} , $d = 10,000$	(-0.145; 28.76)	(-0.144; 28.76)	(-0.144; 28.76)	(-1.445; 287.6)	(-1.445; 287.6)	(-1.445; 287.6)
VaR _{95%} , $d = 100,000$	(-1.085; 20.62)	(-1.085; 20.62)	(-1.085; 20.62)	(-10.85; 206.2)	(-10.85; 206.2)	(-10.85; 206.2)
VaR _{99%} , $d = 100,000$	(-0.269; 26.64)	(-0.269; 26.64)	(-0.269; 26.64)	(-2.691; 266.4)	(-2.691; 266.4)	(-2.691; 266.4)
VaR _{99.5%} , $d = 100,000$	(-0.145; 28.90)	(-0.145; 28.90)	(-0.145; 28.90)	(-1.452; 289.0)	(-1.452; 289.0)	(-1.452; 289.0)
VaR _{95%} , $d = +\infty$	(-1.086; 20.63)	(-1.086; 20.63)	(-1.086; 20.63)	(-10.86; 206.3)	(-10.86; 206.3)	(-10.86; 206.3)
VaR _{99%} , $d = +\infty$	(-0.269; 26.65)	(-0.269; 26.65)	(-0.269; 26.65)	(-2.692; 266.5)	(-2.692; 266.5)	(-2.692; 266.5)
VaR _{99.5%} , $d = +\infty$	(-0.145; 28.92)	(-0.145; 28.92)	(-0.145; 28.92)	(-1.453; 289.2)	(-1.453; 289.2)	(-1.453; 289.2)

Panel D: Exact VaR numbers when the risks are multivariate normally distributed

	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.30$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.30$
VaR _{95%}	5.201	7.974	10.005	16.449	65.485	91.137
VaR _{99%}	7.357	11.277	14.151	23.263	92.617	128.897
VaR _{99.5%}	8.145	12.487	15.668	25.758	102.549	142.720

Table 4.1 Bounds on Value-at-Risk of sums of normally distributed risks

Second, the distance between the upper and lower bounds as reported in the different Panels A, B and C is usually significant. For example, Panel B shows that the true 95%-VaR of a portfolio of 100 uncorrelated (but not independent) normally distributed risks is in the interval $(-2.294; 43.59)$. Considering that the given portfolio has zero mean and a standard deviation of 10, this interval

appears to be rather wide. Note indeed that when the risks are independent then the true 95%-VaR can be computed and is given by 16.449. In other words, when the risks are known to be independent the 95%-VaR is approximately 3 times smaller than the reported upper bound (i.e. 43.59) that is valid when we only know that the correlations are equal to zero. When we ignore the variance constraint, then the upper bound is as high as 206.3. We also observe that the distance between the bounds becomes wider when increasing the level of the probability q used to assess VaR. These observations already suggest that misspecification of models is a significant concern, especially when the VaRs are assessed at high probability levels (which is the case in solvency frameworks such as Solvency II and Basel III where $q = 99.5\%$).

Third, when comparing the results of Panel B and Panel C, we observe that adding a variance constraint may have a significant impact on the unconstrained bounds. When the portfolio exhibits low to moderate correlation, then the constraint bounds a and b that we propose improve upon the unconstrained ones, A and B . It is straightforward to show that

$$a > A, \quad b < B \quad \Leftrightarrow \quad \rho \leq \hat{\rho}(n, q) := \frac{n(\phi(\Phi^{-1}(q)))^2}{q(1-q)(n-1)} - \frac{1}{n-1} \quad (4.2)$$

The relation (4.2) allows us to derive the critical correlation values $\hat{\rho}(n, q)$ as a function of portfolio size and probability level q . If the correlation ρ of the portfolio is lower than the critical value then the constrained bounds improve upon the unconstrained ones. We report some values here: $\hat{\rho}(10, 0.95) = 0.138$, $\hat{\rho}(10, 0.995) = -0.0644$, $\hat{\rho}(100, 0.95) = 0.216$, and $\hat{\rho}(100, 0.995) = 0.0324$.

4.4 Examples with Pareto distributed risks

We assume that (X_1, X_2, \dots, X_n) is an homogeneous portfolio of dependent Pareto distributed random variables (of type II). Hence, $F_i(x) = 1 - (1+x)^{-\theta}$ ($i = 1, 2, \dots, n$) with $x > 0$ and with a tail parameter $\theta > 0$. The correlation matrix (ρ_{ij}) is such that $\rho_{ii} = 1$ for $i = 1, \dots, n$ and $\rho_{ij} = \rho$ for all $i \neq j$.

We first consider the case $\theta = 3$ so that the first two moments exist, which allows us to compute the different VaR bounds that we discussed in the previous sections. We first calculate,

$$\begin{aligned} E(X_i) &= \frac{1}{\theta - 1}, \quad \text{var}(X_i) = \frac{2}{(\theta - 1)(\theta - 2)} - \frac{1}{(\theta - 1)^2}, \quad F_{X_i}^{-1}(p) = (1 - p)^{-1/\theta} - 1, \\ \text{TVaR}_q(X_i) &= \frac{(1 - q)^{-1/\theta}}{(1 - \frac{1}{\theta})} - 1, \quad \text{LTVaR}_q(X_i) = \frac{1}{q}(E(X_i) - (1 - q)\text{TVaR}_q(X_i)). \end{aligned}$$

When applying Theorem 2.1, we find that the absolute unconstrained bounds are

$$B = n \frac{(1 - q)^{-1/\theta}}{(1 - \frac{1}{\theta})} - n, \quad A = \frac{n - 1}{q\theta - 1} - B \frac{(1 - q)}{q}.$$

and from Theorem 3.3,

$$a = \max\left(-s\sqrt{\frac{1 - q}{q}}, A\right), \quad b = \min\left(s\sqrt{\frac{1 - q}{q}}, B\right),$$

where $s^2 = (n + n(n - 1)\rho) \left(\frac{2}{(\theta - 1)(\theta - 2)} - \frac{1}{(\theta - 1)^2}\right)$.

We present the results of our calculations in Tables 4.2 in a similar way as in the previous example. Panel A shows the numerical sharp lower and upper bounds obtained by using the ERA (note that the discretization involves the computation of $x_{ij} = (1 - i/(d + 1))^{-1/\theta} - 1$ for $i = 1, \dots, d$; $j = 1, 2, \dots, n$).

Panel A: Approximate sharp bounds obtained by the ERA as presented in Section 4.2

(m_d, M_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(4.387; 14.57)	(4.118; 19.93)	(3.894; 24.24)	(47.47; 80.69)	(42.55; 174.0)	(39.99; 222.7)
VaR _{99%} , $d = 1,000$	(4.883; 26.69)	(4.594; 38.25)	(4.472; 48.40)	(48.40; 121.2)	(46.22; 336.4)	(45.10; 447.5)
VaR _{99.5%} , $d = 1,000$	(5.960; 36.56)	(4.868; 53.99)	(4.616; 63.88)	(48.61; 151.4)	(47.07; 457.6)	(46.61; 550.4)
VaR _{95%} , $d = 10,000$	(4.401; 15.72)	(4.091; 21.85)	(3.863; 26.19)	(47.96; 84.72)	(42.48; 188.9)	(39.61; 243.3)
VaR _{99%} , $d = 10,000$	(5.486; 28.69)	(4.591; 43.45)	(4.492; 53.22)	(48.99; 129.5)	(46.61; 366.0)	(45.36; 489.5)
VaR _{99.5%} , $d = 10,000$	(6.820; 39.48)	(5.471; 59.60)	(4.850; 73.11)	(49.23; 162.8)	(47.54; 499.1)	(46.68; 671.5)

Panel B: Variance-constrained bounds as obtained in Theorem 3.3

(a_d, b_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(4.383; 14.98)	(4.100; 20.35)	(3.893; 24.28)	(47.45; 80.97)	(42.45; 175.9)	(39.84; 225.5)
VaR _{99%} , $d = 1,000$	(4.680; 27.90)	(4.557; 40.15)	(4.466; 49.12)	(48.39; 121.8)	(46.20; 338.5)	(45.06; 451.9)
VaR _{99.5%} , $d = 1,000$	(4.749; 37.50)	(4.662; 54.87)	(4.615; 64.06)	(48.61; 152.2)	(47.06; 459.4)	(46.26; 620.1)
VaR _{95%} , $d = 10,000$	(4.398; 16.03)	(4.089; 21.92)	(3.861; 26.23)	(47.96; 84.74)	(42.48; 188.9)	(39.61; 243.4)
VaR _{99%} , $d = 10,000$	(4.725; 30.20)	(4.589; 43.64)	(4.490; 53.50)	(48.99; 129.6)	(46.59; 367.3)	(45.33; 491.7)
VaR _{99.5%} , $d = 10,000$	(4.800; 40.74)	(4.705; 59.80)	(4.634; 73.77)	(49.23; 162.9)	(47.54; 500.0)	(46.65; 676.3)
VaR _{95%} , $d = +\infty$	(4.372; 16.94)	(4.037; 23.30)	(3.791; 27.96)	(48.01; 87.75)	(42.09; 200.3)	(38.99; 259.2)
VaR _{99%} , $d = +\infty$	(4.725; 32.25)	(4.578; 46.77)	(4.470; 57.41)	(49.13; 136.2)	(46.53; 393.1)	(45.18; 527.4)
VaR _{99.5%} , $d = +\infty$	(4.806; 43.63)	(4.702; 64.22)	(4.634; 77.72)	(49.39; 172.2)	(47.56; 536.4)	(46.60; 726.9)

Panel C: Unconstrained bounds as obtained in Theorem 2.1

(A_d, B_d)	$n = 10$			$n = 100$		
	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$	$\rho = 0$	$\rho = 0.15$	$\rho = 0.3$
VaR _{95%} , $d = 1,000$	(3.642; 29.05)	(3.642; 29.05)	(3.642; 29.05)	(36.42; 290.5)	(36.42; 290.5)	(36.42; 290.5)
VaR _{99%} , $d = 1,000$	(4.435; 52.22)	(4.435; 52.22)	(4.435; 52.22)	(44.35; 522.2)	(44.35; 522.2)	(44.35; 522.2)
VaR _{99.5%} , $d = 1,000$	(4.615; 64.06)	(4.615; 64.06)	(4.615; 64.06)	(46.15; 640.6)	(46.15; 640.6)	(46.15; 640.6)
VaR _{95%} , $d = 10,000$	(3.646; 30.33)	(3.646; 30.33)	(3.646; 30.33)	(36.46; 303.3)	(36.46; 303.3)	(36.46; 303.3)
VaR _{99%} , $d = 10,000$	(4.447; 57.76)	(4.447; 57.76)	(4.447; 57.76)	(44.47; 577.6)	(44.47; 577.6)	(44.47; 577.6)
VaR _{99.5%} , $d = 10,000$	(4.633; 74.11)	(4.633; 74.11)	(4.633; 74.11)	(46.33; 741.1)	(46.33; 741.1)	(46.33; 741.1)
VaR _{95%} , $d = +\infty$	(3.647; 30.72)	(3.647; 30.72)	(3.647; 30.72)	(36.47; 307.2)	(36.47; 307.2)	(36.47; 307.2)
VaR _{99%} , $d = +\infty$	(4.448; 59.62)	(4.448; 59.62)	(4.448; 59.62)	(44.48; 596.2)	(44.48; 596.2)	(44.48; 596.2)
VaR _{99.5%} , $d = +\infty$	(4.635; 77.72)	(4.635; 77.72)	(4.635; 77.72)	(46.35; 777.2)	(46.35; 777.2)	(46.35; 777.2)

Table 4.2 Bounds on Value-at-Risk of sums of Pareto distributed risks ($\theta = 3$)

Panel B gives the corresponding absolute constrained bounds and Panel C gives the absolute bounds in the unconstrained case. The results are in line with the results that we obtain in the case of normally distributed risks. Also in this case the ERA gives rise to numerical bounds that are close to ones that were obtained theoretically. In other words, the absolute bounds are “nearly sharp” in this case. The Pareto distribution has heavy tails and hence one observes a significant difference between A_d (given by (4.1)) and A (respectively B_d and B) for small values of d as it appears in Table 4.2. Note also that the difference between upper and lower bounds is again significant, confirming that in the case when there is no or limited information on the dependence, the model risk that goes along with a particular model is an issue. Note also that the impact of the variance constraint is more significant than in the normal case. For example, when $q = 99.5\%$, $n = 100$ and $\rho = 0.15$ we find a numerical sharp bound $m_{10,000} = 499.1$, (close to the absolute bound $b_{10,000} = 500.01$) whereas the unconstrained bound amounts to $B_{10,000} = 741.1$.

5 VaR bounds of credit risk portfolios

The financial crisis has increased the pressure on financial institutions and regulators to continuously assess the adequacy of risk management models. As credit risk is a significant concern for the stability

of the financial system, it is clear that credit risk portfolio models are in the scope of such critical review. In this section we apply the theory on VaR bounds to the case of credit risk portfolios and we discuss the results in the context of model risk assessment. Here, we merely define model risk as the risk that the computed portfolio-VaR is incorrect as a result of using a misspecified model. Note that a related issue concerns the risk of using model parameters that are estimated, a concern that arises because of the statistical uncertainty on these estimations.

First, we show that the theoretical set-up of the paper is of particular relevance to the case of credit risk. Next, we discuss credit portfolio models that are used in practice and we compare their VaR numbers with the unconstrained and constrained bounds. These results show that adding the variance constraint significantly improves the unconstrained bounds. Nevertheless, model risk remains an issue that is hard to avoid. We provide some guidelines that can be useful for improving the regulation regarding the way capital requirements are to be established by financial institutions.

5.1 Credit Risk Portfolio models

5.1.1 Description

Let us consider a portfolio (X_1, X_2, \dots, X_n) containing non-negative risks $X_i = v_i \mathcal{B}(p_i) \sim F_i$, ($i = 1, 2, \dots, n$). Here, each X_i can be seen as a representation of the risk the bank runs when providing a loan to company “ i ”. Specifically, p_i is the probability that the i -th company defaults and in the case of a default the loss incurred is equal to v_i . We denote by p_{ij} the pairwise default probability that both company i and company j default. The pairwise default correlation ρ_{ij}^D ($i, j = 1, 2, \dots, n$) is then given as

$$\rho_{ij}^D = \frac{p_{ij} - p_i p_j}{\sqrt{p_i(1-p_i)}\sqrt{p_j(1-p_j)}}. \quad (5.1)$$

The variance of the portfolio sum $S = X_1 + X_2 + \dots + X_n$ thus depends on the exposures v_i (net of recoveries), the single default probabilities p_i and the pairwise default probabilities p_{ij} ($i, j = 1, 2, \dots, n$). As there is an intrinsic lack of sufficient default statistics (joint defaults are inherently very rare events), it becomes clear that in practice the knowledge of the above mentioned parameters is the maximum⁴ amount of information available when building models. In other words, all models that compute risk measures for credit risk portfolios require some further ad-hoc assumptions for describing the full dependence (e.g., the specification of the probabilities that 3 or more loans default together). This line of reasoning shows that the problem setting that we discussed in this paper is particularly relevant for credit risk. To compare our VaR bounds with the VaRs calculated from various standard models we consider a homogeneous portfolio of credit risks with net exposures $v_i = 1$ ($i = 1, 2, \dots, n$). Let p and ρ^D denote the default probability and pairwise default correlation (between two risks X_i and X_j , $i \neq j$).

Many industry credit risk portfolio models rely on the “Merton’s model of the firm” when computing the VaR of a portfolio (see also the survey of McKinsey ((2009))). Also Basel III and Solvency II regulatory frameworks rely on the same model when setting their VaR-based capital requirements. The very basic idea of the Merton approach is to model a default as the event when *the asset value drops below a threshold value*. Formally, after normalization, a default for the i -th risk occurs for the event $\{N_i < c\}$ where N_i is the normalized asset return and c is the constant threshold value so that $p = p_i = P(N_i < c)$. Merton’s model further assumes that the joint asset (log-)returns are multivariate normally distributed. Hence, for an homogeneous portfolio one can conveniently express

⁴In fact, it is already ambitious to have all pairwise correlations available. It is more realistic to assume knowledge of the portfolio variance (for example, based on an analysis of the aggregate default statistics).

the standardized asset return for the i -th obligor N_i as

$$N_i = \sqrt{\rho^A}M + \sqrt{1 - \rho^A}\varepsilon_i, \quad (5.2)$$

Here, M and ε_i ($i = 1, 2, \dots, n$) are independent standard normally distributed random variables and $\rho^A \geq 0$ is the (asset) correlation coefficient. The variable M corresponds to a systemic factor (describing the global economy) and the ε_i reflect the idiosyncratic (individual) risks. Furthermore, for a given realization for $M = m$ the portfolio sum has a binomial distribution with parameters n and $p(m)$, where $p(m)$ is the conditional default probability given as

$$p(m) = P(N_i < c | M = m) = P\left(\varepsilon_i < \frac{c - \sqrt{\rho^A}m}{\sqrt{1 - \rho^A}}\right) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho^A}m}{\sqrt{1 - \rho^A}}\right), \quad (5.3)$$

where Φ is the distribution of the standard normal random variable. The VaRs can now be easily estimated using e.g. Monte-Carlo simulation. As conditionally on the state of the economy M , the default events are independent it follows that in the limit ($n \rightarrow \infty$) the portfolio loss $\frac{S}{n}$ has the same distribution as $p(M)$. We thus obtain that

$$\lim_{n \rightarrow \infty} \text{VaR}_q \left[\frac{S}{n} \right] = \Phi\left(\frac{\Phi^{-1}(p) + \sqrt{\rho^A} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho^A}}\right), \quad (5.4)$$

see also Vasicek (2001). In fact, the model as described above is an example of a one-factor mixture model in which the default event of the obligor is assumed to be driven by a common economic factor M . It can also be seen as the one-factor version of the KMV model that is highly used in the industry and also appears in regulatory frameworks. For example, the Basel III standard framework relies on formula (5.4) to determine the required capital that banks need to hold for their credit portfolios; see the Basel Committee on Banking Supervision (2006), (2010). Also the Solvency II framework uses this formula to decide on the amount of capital that insurers need to hold as a buffer against the adverse consequences if one or more of their reinsurance or derivative counterparts fail. We refer to Committee of Insurance and Occupational Pension Supervisors (2008) for an overview of the technical Solvency II guidelines as well as to Doff (2008) for a detailed critical analysis of the whole Solvency II framework.

It is clear that other distributions for $p(M)$ can also be used and other choices that have been made in the literature include a logit-normal mixing distribution (one obtains the one-factor version of the CreditMetrics approach) and a Beta distribution. Note that this last model is intimately related to the actuarial approach for credit risk portfolio modelling, which is also known as the one-factor CreditRisk⁺ model; see Vandendorpe et al. (2008) for a study. We recall that in all these cases, conditionally on $M = m$ the number of defaults is the sum of n independent Bernoulli variables with parameter $p(m)$ and hence has a binomial distribution with parameters n and $p(m)$, which implies that upon the specification of the mixing distribution used the VaRs can be readily obtained.

5.1.2 Parameterization

The natural parameterization of Merton's model consists in the knowledge of the default probability and the asset correlation. The latter parameter can be estimated using asset value data, but unfortunately these values are not readily observable. One way to deal with this issue is to generate pseudo asset values that are based on equity value data or other data series that can be used (after suitable transformation) as a proxy for asset values; see e.g. Duellmann et al. (2008), Pitts(2004) and Chernih et al. (2010). Another approach consists in estimating the default correlation ρ^D using

default statistics and next inferring from ρ^D and p the implied asset correlation ρ^A (using the KMV model⁵). Papers in the latter category include Gordy (2000), Frey and McNeil (2003) and Dietsch and Petey (2004).

We remark that it is common to use the asset correlations (and not the default correlations) as the basis for discussing and comparing the different findings. The different studies mentioned report different values for the asset correlations, depending on loan quality (level of default probability), the sector of activity of the obligor and so on. As a conclusion, one can state that the reported values range between 1% and 30% with an average of about 8–10%. This range for asset correlations is also consistent with the values that are used in regulatory frameworks (for example, Basel III uses asset correlations up to 30%). In Table 5.3, we report the default correlation as a function of the default probability and the asset correlation.

	$p = 0.05\%$	$p = 0.50\%$	$p = 1\%$	$p = 5\%$
$\rho^A = 0\%$	0.00%	0.00%	0.00%	0.00%
$\rho^A = 4\%$	0.03%	0.19%	0.32%	0.94%
$\rho^A = 8\%$	0.08%	0.44%	0.71%	1.99%
$\rho^A = 12\%$	0.15%	0.75%	1.18%	3.14%
$\rho^A = 16\%$	0.24%	1.13%	1.75%	4.40%
$\rho^A = 20\%$	0.38%	1.60%	2.41%	5.78%
$\rho^A = 24\%$	0.56%	2.17%	3.19%	7.28%

Table 5.3 The table provides default correlations ρ^D for given levels of the default probability p and the asset correlation ρ^A .

Whereas in the case of the KMV model the natural parameterization consists in a direct specification of the default probability p and the asset correlation ρ^A , in the other models the input parameters here are the default probability p and the default correlation ρ^D (obtained though an historical data analysis or inferred from the asset correlation ρ^A). Hence the mean and the variance of $p(M)$ are specified since $E(p(M)) = p$ and $\text{var}(p(M)) = \rho^D p(1 - p)$. Next, one infers the two parameters of the mixing distribution at hand. In the case of a Beta distribution these parameters can be obtained analytically and for the CreditMetrics approach one can use numerical techniques. We provide some more details in the appendix.

5.1.3 Comparing the VaRs and assessing the impact of the variance constraint

We first consider a portfolio of 10,000 loans. Using Table 8.6 on page 365 from McNeil et al. (2005), we fix the default probability $p = 0.049$ and $\rho^D = 0.0157$. The variance s^2 of the portfolio sum of n correlated Bernoulli risks is thus equal to

$$s^2 = np(1 - p) + n(n - 1)p(1 - p)\rho^D.$$

Consider for instance $d = 1,000$. Each X_j ($j = 1, 2, \dots, n; n = 10,000$) takes the value 0 in 951 states and 1 in 49 states, so that effectively each X_j has a Bernoulli distribution with parameter 0.049. We consider thus a matrix $(x_{ij})_{i=1, \dots, d, j=1, \dots, n}$ of $d \times n$ entries. Let q the probability level that is used to compute the Value-at-Risk. We then apply Theorem 2.1 to find A_d and B_d ,

$$A_d = n \frac{(p - \min(p, (1 - q)))}{q}, \quad B_d = n \min(p/(1 - q), 1).$$

⁵Indeed, note that $\rho^D = \frac{p_{\text{pair}} - p^2}{p(1 - p)}$ where the pairwise default probability of two different obligors write as $p_{\text{pair}} = \Phi_{\rho^A}(\Phi^{-1}(p), \Phi^{-1}(p))$ in which $\Phi_{\rho^A}(\cdot, \cdot)$ denotes the distribution function of a bivariate standard normal random couple with correlation coefficient ρ^A .

Furthermore, using Theorem 3.3 we derive a_d and b_d as

$$a_d = \max \left(np - s \sqrt{\frac{1-q}{q}}, A_d \right), \quad b_d = \min \left(np + s \sqrt{\frac{q}{1-q}}, B_d \right).$$

Similarly as in the case of a portfolio of normally and Pareto distributed risks, we also apply the extended RA to find numerical VaR bounds. Finally, we also report the results that are obtained using different mixture models that we apply asymptotically. The results are reported in Table 5.4. All numbers are normalized as percentage of the maximum possible total loss, i.e. n . In other words, in Table 5.4 the outputs A_d , B_d , a_d , b_d and the approximate sharp bounds that are obtained by applying the extended RA are divided by n and multiplied by 100. The example shows that adding a variance constraint has a significant impact on the level of the VaR bounds. For example, the unconstrained upper bound of the 95%-VaR is 98%, but the constrained one is only 16.73%. As expected the difference between the upper and lower bounds is increasing significantly when increasing the probability level used for VaR assessments. Clearly, when using $q = 99.5\%$ as the basis for calculating VaR and capital requirements, then the results of the models are typically within the big range of possible values of VaR but as they only use information on the default probability p and the default correlation ρ^D they seem difficult to justify. The model risk appears more limited when using lower probability levels for assessing the VaR. For example, when using the 90%-VaR we find that the distance between the upper bound b_d and the lower bound a_d becomes more limited and the different industry models provide outcomes that are nearly in the middle of the interval (a_d, b_d) .

	(A_d, B_d)	(a_d, b_d)	(m_d, M_d)	KMV	Beta	CreditMetrics
VaR _{0.8}	(0%; 24.50%)	(3.54%; 10.33%)	(3.63%; 10%)	6.84%	6.95%	6.71%
VaR _{0.9}	(0%; 49.00%)	(4.00%; 13.04%)	(4.00%; 13%)	8.51%	8.54%	8.41%
VaR _{0.95}	(0%; 98.00%)	(4.28%; 16.73%)	(4.32%; 16%)	10.10%	10.01%	10.11%
VaR _{0.995}	(4.42%; 100.00%)	(4.71%; 43.18%)	(4.73%; 40%)	15.15%	14.34%	15.87%

Table 5.4 The table provides VaR bounds and VaR computed in different models (KMV, Beta, CreditMetrics).

Finally, we compute the bounds A , a , b and B as well as the VaRs in a KMV framework for an infinitely big portfolio assuming a relevant range of default probabilities and asset correlations. The results are reported in Table 5.5 and confirm the findings of the previous example.

	$p = 0.25\%$			$p = 1\%$		
	(A, B)	(a, b)	KMV	(A, B)	(a, b)	KMV
$\rho^A = 0\%$	(0%; 50%)	(0.25%; 0.25%)	0.25%	(0.50%; 100%)	(1.00%; 1.00%)	1.0%
$\rho^A = 6\%$	(0%; 50%)	(0.23%; 3.27%)	1.2%	(0.50%; 100%)	(0.95%; 10.98%)	4.0%
$\rho^A = 12\%$	(0%; 50%)	(0.23%; 5.05%)	2.1%	(0.50%; 100%)	(0.92%; 16.27%)	6.3%
$\rho^A = 18\%$	(0%; 50%)	(0.23%; 6.84%)	2.9%	(0.50%; 100%)	(0.90%; 21.18%)	8.7%
$\rho^A = 24\%$	(0%; 50%)	(0.21%; 8.76%)	3.8%	(0.50%; 100%)	(0.87%; 26.09%)	11.1%
$\rho^A = 30\%$	(0%; 50%)	(0.20%; 10.85%)	4.8%	(0.50%; 100%)	(0.85%; 31.13%)	13.7%

Table 5.5 Unconstrained and constrained upper and lower 0.995-VaR bounds for several combinations of default probability and correlation and the VaR in the (one-factor) KMV model.

In particular, they show the significant impact of the variance constraint on the VaR bounds. For example, when the asset correlation $\rho^A = 6\%$ and $p = 1\%$, one has that the unconstrained upper bound for the 99.5%-VaR is 100% whereas the constrained bound is only 11.1%. These findings also confirm that computing capital requirements based on the 99.5% VaRs is prone to significant model

error as the distance between the upper and lower VaR bounds is big. For example, for an asset correlation $\rho^A = 6\%$ and a default probability $p = 0.25\%$, the most optimistic model may report a 99.5%-VaR that is equal to 0.2% whereas the most pessimistic one provides in this instance a 99.5%-VaR that is equal to 3.3%. These results thus show that knowledge of the marginal distributions and the correlations is not enough to estimate true portfolio VaRs with confidence.

6 Conclusions

Recent literature has been dealing with the problem of finding sharp bounds on the Value-at-Risk of risky portfolios when the distributions of the risky components are known. This problem is challenging and there are few theoretical results available that allow to deal with situations in which also some information on the dependence is available.

In this paper we consider a variance constraints for the portfolio sum as source of information on the dependence and we propose some simple bounds that are easy to compute. These bounds are usually not sharp, but their construction as well as some theoretical results on convex ordering and mixability provide us with the intuition for proposing a new algorithm that allows us to approximate the sharp bounds. Several numerical examples show that the algorithm performs well and confirm that a variance constraint can significantly improve the unconstrained bounds of Embrechts et al. (2013). This algorithm can thus be seen as a practical way to deal with a problem that is otherwise hard to solve theoretically.

We believe that our results are useful for studying model risk. In the paper we touch on this issue by discussing the adequacy of credit risk portfolio models. We show that the VAR computed in typical credit models that the financial institutions report do not necessarily reflect the true risk and are hard to compare. In this respect we note that under the internal model approach of Basel III and Solvency II, the financial institutions are allowed to use their own model for setting their capital requirements. However, it is hard, if not impossible, to show which model is better than the others, as they might be all consistent with the available amount of information (namely, default probabilities and default correlations in the credit risk context). When applied to similar portfolios, the models may thus give rise to significantly different VaRs. Therefore, in the context of setting capital requirements, it might be useful to impose some additional constraints on the internal models used for this purpose. For example, one may use the computed VaR bounds (or a weighted average of them) as a yardstick to set a minimum value on the VaR that is obtained by the internal model, as for example $a_d + 0.7(b_d - a_d)$ resp. $m_d + 0.7(M_d - a_d)$. Another possibility consists in imposing a particular model when setting capital requirements, so that at least capital levels can be readily compared across institutions, and fair competition will be enhanced.

A Appendix

Proof of Theorem 2.4, page 7.

Let us first remark that

$$M = \sup_{X_i \sim F_i} \text{VaR}_q^+ \left[\sum_{i=1}^n X_i \right] = \sup_{Y_i \sim F_i^q} \text{VaR}_0^+ \left[\sum_{i=1}^n Y_i \right]. \quad (\text{A.1})$$

Note also that

$$\text{VaR}_0^+ \left[\sum_{i=1}^n Y_i \right] = \text{ess inf} \left(\sum_{i=1}^n Y_i \right) \quad (\text{A.2})$$

is the minimal support of the distribution of $\sum_{i=1}^n Y_i$.

So the problem of maximizing the VaR is equivalent to maximizing the minimal support of $\sum_{i=1}^n Y_i$ over all possible $Y_i \sim F_i^q$. This problem in turn is closely related to convex ordering: Let $Y = \sum_{i=1}^n Y_i$, $Z = \sum_{i=1}^n Z_i$ with $Y_i, Z_i \sim F_i^q$ have dfs F, G , then it holds:

$$\begin{aligned} Y \leq_{\text{cx}} Z &\Leftrightarrow \int_0^q F^{-1}(p) dp \geq \int_0^q G^{-1}(p) dp \quad \text{for all } 0 < q < 1 \\ &\Leftrightarrow F^{-1} \prec_s G^{-1} \end{aligned} \quad (\text{A.3})$$

where \prec_s is the Schur order (see Rüschendorf (1983b; 2013, Corollary 3.26) and Dhaene et al. (2006)). In particular (A.3) implies that

$$F^{-1}(0) = \text{ess inf} \left(\sum_{i=1}^n Y_i \right) \geq G^{-1}(0) = \text{ess inf} \left(\sum_{i=1}^n Z_i \right) \quad (\text{A.4})$$

and $F^{-1}(1) \leq G^{-1}(1)$. The minimal support of Y is larger than the minimal support of Z and the maximal support of Y is smaller than the maximal support of Z . As a consequence, the convex ordering in (A.3) implies ordering of the VaR.

Proof of Theorem 3.3, page 10. Variance-constrained bounds

- a) If $\text{var}(X_{\alpha^*}^*) < s^2$, this means that $\alpha^* = 0$. Hence, $A(\alpha^*)$ and $B(\alpha^*)$ correspond to the absolute bounds and there is nothing to prove. We further assume that $\text{var}(X_{\alpha^*}^*) = s^2$ and denote by G the distribution of $X_{\alpha^*}^*$. We first prove that b is an absolute upper bound for feasible solutions of (1.3). Hence, assume there exist (X_1, X_2, \dots, X_n) such that $\text{VaR}_q^+ [X_1 + X_2 + \dots + X_n] > b$. One has that, $\forall a \leq x < b$, $F_{X_1+X_2+\dots+X_n}(x) \leq G(x) = q$. When $b \leq x$, $F_{X_1+X_2+\dots+X_n}(x) \leq G(x) = 1$. Since $G(x) = 0$ for $x < a$ this implies that,

$$\begin{cases} \forall x < a, & F_{X_1+X_2+\dots+X_n}(x) \geq G(x), \\ \forall x \geq a, & F_{X_1+X_2+\dots+X_n}(x) \leq G(x). \end{cases} \quad (\text{A.5})$$

In other words, the distribution function $F_{X_1+X_2+\dots+X_n}$ crosses G once from above. Since $E(X_1 + X_2 + \dots + X_n) = \mu$ this implies that $X_{\alpha^*}^* \leq_{\text{cx}} X_1 + X_2 + \dots + X_n$ (see Karlin and Novikoff (1963), Müller and Stoyan (2002)). Since $\text{var}(X_{\alpha^*}^*) = s^2$, the feasibility of (X_1, X_2, \dots, X_n) requires that $\text{var}(X_1 + X_2 + \dots + X_n) = \text{var}(X_{\alpha^*}^*)$. In view of the convex ordering between $X_1 + X_2 + \dots + X_n$ and $X_{\alpha^*}^*$, this is only possible when $X_1 + X_2 + \dots + X_n \stackrel{d}{=} X_{\alpha^*}^*$ (here, $\stackrel{d}{=}$ means that there is equality in distribution), which is a contradiction.

The proof that a is an absolute lower bound can be given in a similar way. Let now (X_1, X_2, \dots, X_n) be such that $\text{VaR}_q [X_1 + X_2 + \dots + X_n] < a$. One has that, $\forall x \leq a$, $F_{X_1+X_2+\dots+X_n}(x) \geq G(x) = 0$. When $a \leq x < b$, $F_{X_1+X_2+\dots+X_n}(x) \geq G(x) = q$. Since $G(x) = 1$ for $x \geq b$ this implies that,

$$\begin{cases} \forall x < b, & F_{X_1+X_2+\dots+X_n}(x) \geq G(x), \\ \forall x \geq b, & F_{X_1+X_2+\dots+X_n}(x) \leq G(x). \end{cases} \quad (\text{A.6})$$

In other words, the distribution function $F_{X_1+X_2+\dots+X_n}$ crosses G once from above. By symmetry of the argument the result follows from the first part of the proof.

- b) If $s^2 \geq q(A - \mu)^2 + (1 - q)(B - \mu)^2$ then the result is obvious from Theorem 2.1 and Proposition 3.2. In the other case, the proposition implies that there exists α^* such that $\text{var}(X_{\alpha^*}^*) = s^2$. Hence, a and b can be seen as the mass points from a 2-point distribution satisfying the mean constraint μ and the variance constraint s^2 . This yields the desired expressions for a and b immediately. \square

Proof of Theorem 3.6, page 11. Sharpness of variance-constrained bounds

We note that by the convex ordering assumption in (3.9) we get for the upper q -part $T = \{U \geq q\}$ of S_n .

$$\begin{aligned} E(\overline{S}_n)^2 &= E((\overline{S}_n)^2 | T)P(T) + E((\overline{S}_n)^2 | T^c)P(T^c) \\ &\leq E(S_n^2 | T)P(T) + E(S_n^2 | T^c)P(T^c) \\ &= ES_n^2 \end{aligned}$$

and thus $\text{var}(\overline{S}_n) \leq \text{var}(S_n) \leq s^2$. The argument for the increase of $\text{VaR}_q^+[\overline{S}_n]$ (resp. decrease of $\text{VaR}_q[\overline{S}_n]$) compared to $\text{VaR}_q^+[S_n]$ (resp. $\text{VaR}_q[S_n]$) is similar as in Theorem 2.4. \square

Mixing Distributions, page 20.

Note that all mixture models that we discuss require 2 parameters only and are thus completely specified once the default probability and the default correlation (or the asset correlation) are known.

Beta distribution

We say that the mixing variable $p(M)$ has a Beta distribution with parameters $a, b > 0$ if it has a density given as

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \text{ for } 0 < x < 1, \quad (\text{A.7})$$

where $\Gamma(x)$ is the Gamma function. It readily shows that

$$p = \frac{a}{a+b}, \quad \rho^D = \frac{1}{a+b+1},$$

which allows us to get a and b as a function of ρ^D and p in explicit form:

$$a = p \left(\frac{1}{\rho} - 1 \right), \quad b = (1-p) \left(\frac{1}{\rho} - 1 \right).$$

Probit-Norm distribution (KMV)

The mixing variable $p(M)$ is said to have a Probit-Norm distribution if it writes as $p(M) = \Phi(\mu + \sigma M)$ where Φ is the distribution function of a standard normal random variable. In this case one has that

$$\mu = \frac{\Phi^{-1}(p)}{\sqrt{1 - \rho^A}}, \quad \sigma^2 = \frac{\rho^A}{1 - \rho^A}.$$

Note that if the default correlation is provided and not the asset correlation then one first needs to back out the asset correlation.

Logit-Norm distribution (KMV)

Finally, in the case of the Logit-Normal mixing distribution one has that $p(M) = F(\mu + \sigma M)$ with $F = \frac{1}{1 + \exp(-x)}$. As the moments of $p(M)$ are not known analytically, parameters μ and σ can only be obtained numerically.

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