Mathematics of Deep Learning, Summer Term 2020

Week 6

Signal Analysis

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Overview of Week 6

Acknowledgement of Sources

Sources for this lecture:

- Christensen (2016): An introduction to frames and Riesz bases
- Dahlke, De Mari, Grohs, Labatte (2015): Harmonic and Applied Analysis
- Feichtinger Gröchenig (1988): A unified approach to atomic decompositions
- Folland (2016): A course in abstract harmonic analysis

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 1

Coorbit Theory, Signal Analysis, and Deep Learning

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Harmonic Analysis

- Setting: $\pi\colon G \to U(H)$ is a strongly continuous irreducible unitary representation of a locally compact group G on a Hilbert space H such that $\int |\langle \pi_g f, f\rangle_H|^2 dg < \infty$ for some $\psi \in H$.

- Voice transform: For any $\psi \in H$, the voice transform is the linear map

$$V_\psi\colon H \to C(G), \qquad V_\psi f(g) = \langle f, \pi_g \psi\rangle_H.$$

- Admissibility: the voice transform V_ψ is isometric for all $\psi \in D(A)$ with $\|A\psi\|_H = 1$, where A is the Duflo–Moore operator. These ψ are called admissible.

- Reproducing kernel spaces: for any admissible ψ, the voice transform is an isometric isomorphism onto the space

$$\{F \in L^2(G) : F * V_\psi \psi = F\}$$

with reproducing kernel $V_\psi \psi$.

Coorbit Theory

- Weighted spaces: for exponents $p \in [1, \infty]$ and w-moderate weight functions $m\colon G \to \mathbb{R}_+$, one defines weighted spaces $L^p_w(G)$ and $L^p_m(G)$, respectively.
- Analyzing vectors are defined as admissible ψ with $V_\psi \psi \in L^1_w(G)$.
- Coorbit spaces $H_{p,m}$ are constructed by requiring the voice transform to be an isomorphism for some (equivalently, all) analyzing vectors ψ:

$$V_\psi \colon H_{p,m} \xrightarrow{\cong} \{F \in L^p_m(G) : F * V_\psi \psi = F\}.$$

- Banach frames: for suitable analyzing vectors $\psi \in D(A)$ and group elements $(g_k)_{k\in\mathbb{N}}$, one obtains a Banach frame $(\pi_{g_k}\psi)_{k\in\mathbb{N}}$ for the coorbit space $H_{p,m}$ with respect to a weighted sequence space ℓ^p_m.
- Proof by correspondence principle: $(L_{g_k} V_\psi \psi)_{k\in\mathbb{N}}$ is a Banach frame for $\{F \in L^p_m(G) : F * V_\psi \psi = F\}$ with respect to ℓ^p_m.

Abelian Groups are not Interesting for Coorbit Theory

Theorem

Abelian groups have only one-dimensional irreducible representations.

Lemma (Schur)

$\pi : G \to U(H)$ *is irreducible if and only if its centralizer is trivial, i.e.,*

$$\{T \in L(H) : \pi_g T = T \pi_g \text{ for all } g \in G\} = \mathrm{span}\{\mathrm{Id}_H\}.$$

Proof of the Theorem:

- The centralizer of π is trivial because π is irreducible.
- The operators π_g belong to the centralizer because G is Abelian.
- Thus, the operators π_g are multiples of the identity.
- Thus, all one-dimensional subspaces are invariant. □

Signal analysis and Deep Learning

Signal Analysis:

- There are many different group representations with associated voice transforms.
- These have a variety of applications in signal analysis such as time-frequency analysis, multi-resolution analysis, and edge detection.
- The interpretation varies strongly from case to case.

Deep learning inherits many of the strengths of signal analysis:

- Many voice transforms are implementable via shallow nets with activation function equal to the analyzing function.
- Alternatively, via dictionary learning, they are implementable via deep nets with other activation functions.
- In this case, deep learning can adaptively select (i.e., learn) a suitable analyzing function.

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Refresh your memory of the voice transform and the construction of coorbit spaces.
- Check: As the translation group is Abelian, its representation on $L^2(\mathbb{R}^d)$ must be reducible—can you find a subrepresentation?
- Check: Same question for the modulation group. Hint: apply the Fourier transform.
- Check: How can dictionary learning be applied to implement signal transforms via deep networks?
- Background: Look up the proof of Schur's lemma. For instance, in [Christensen], [Dahlke e.a.], or [Folland].

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 2

Heisenberg Group

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Structure

Definition

The Heisenberg group is the set $G := \mathbb{R}^d \times \mathbb{R}^d \times S^1$ equipped with the product topology and the composition

$$(a_1, b_1, t_1) \cdot (a_2, b_2, t_2) := (a_1 + a_2, b_1 + b_2, t_1 t_2 e^{2\pi i b_1 a_2})\,.$$

Properties:

- The Heisenberg group is not Abelian.
- The Haar measure is the product measure of the three involved Lebesgue measures.
- The Heisenberg group is unimodular.

Representation

Definition

The Schrödinger representation $\pi\colon G \to U(L^2(\mathbb{R}^d))$ is defined as

$$\pi(a,b,t)f(x) := te^{2\pi i b(x-a)} f(x-a)\,,$$

where $f \in L^2(\mathbb{R}^d)$, $(a,b,t) \in G$, and $x \in \mathbb{R}^d$.

Remark:

- π can be expressed in terms of translation and modulation as

  $$\pi(a,b,t)f = te^{-2\pi i ab} E_b T_a f\,.$$

  Translations are time shifts, and modulations are frequency shifts.
- π is irreducible and integrable.
- All unit vectors in $L^2(\mathbb{R}^d)$ are admissible because G is unimodular.

Gabor Transform

Remark:

- The Gabor transform or short-time Fourier transform is the voice transform of the Schrödinger representation.
- The torus component $t \in S^1$ can (and will) be ignored for all practical purposes.

Definition

For any admissible $\psi \in L^2(\mathbb{R}^d)$, the Gabor transform $V_\psi \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is given by

$$V_\psi f(a,b) := \int_{\mathbb{R}^d} f(x)\psi(x-a)e^{-2\pi i x b}dx = \langle f, E_b T_a \psi \rangle_{L^2(\mathbb{R}^d)},$$

where $f \in L^2(\mathbb{R}^d)$ and $a, b \in \mathbb{R}^d$.

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe the Schrödinger representation of the Heisenberg group. Think about a way of memorizing the group structure.
- Check: Why can the torus component be ignored for the purpose of signal analysis?

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 3

Modulation Spaces

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Analyzing Functions

Setting: We consider the Schrödinger representation π of the Heisenberg group G on $L^2(\mathbb{R}^d)$.

Lemma

Let w be a weight function on G. A function $\psi \in L^2(\mathbb{R}^d)$ is an analyzing vector for w if and only if $\|\psi\| = 1$ and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle \psi, E_b T_a \psi \rangle| \, w(a,b) \, da \, db < \infty \, .$$

Remark:

- The Feichtinger algebra $\mathcal{S}_0$ is defined as the subspace of $L^2(\mathbb{R}^d)$ described by the above integrability condition with $w \equiv 1$.
- The Gauss function is analyzing[1] for all polynomial weight functions $w(a,b) := (1 + \|b\|)^{|s|}$, $s \in \mathbb{R}$.

[1] See [Feichtinger Gröchenig 1988, Section 7.1].

Gabor coorbit spaces

Remark: Gabor coorbit spaces are called modulation spaces:

Definition

Let $d \in \mathbb{N}$, let m be a w-moderate weight, and let ψ be an analyzing vector for w. For any $1 \leq p, q \leq \infty$, the modulation space $M_m^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'$ such that

$$\int \left(\int |\langle f, E_b T_a \psi \rangle|^p \, m(a,b)^p da \right)^{q/p} db < \infty \,,$$

with the usual modifications for $p, q \in \{\infty\}$.

Remark:

- This definition is independent of the choice of w and ψ.
- For $p = q$, we write $M_m^p := M_m^{p,p}$.

Properties and Examples

The Feichtinger algebra provides a rich repertoire of analyzing vectors because it

- Contains all $f \in C_c(\mathbb{R}^d)$ with $\mathcal{F}f \in L^1(\mathbb{R}^d)$.
- Contains the Schwartz space of rapidly decreasing functions.
- Is invariant under the Heisenberg group and the Fourier transform.

Modulation spaces with constant weights $m \equiv 1$:

- M^1_m is the Feichtinger algebra $\mathcal{S}_0$.
- M^2_m is the space $L^2(\mathbb{R}^d)$.

Modulation spaces with polynomial weights $m(a,b) \coloneqq (1 + \|b\|)^s$:

- M^2_m is the Sobolev (aka. Bessel potential) space $H^s(\mathbb{R}^d)$, for any $s \in \mathbb{R}$. This follows from the respective characterization via frames.

Gabor Frames

Theorem

Let $p \in [1, \infty)$*, let* $s \in \mathbb{R}$*, let* $w(a,b) := (1 + \|b\|)^{|s|}$*, and let* $m(a,b) := (1 + \|b\|)^s$*. For any* $\psi \in M^{1,1}_w \setminus \{0\}$ *and sufficiently small* $\alpha, \beta > 0$*, the vectors* $(E_{\beta b} T_{\alpha a} \psi)_{a,b \in \mathbb{Z}^d}$ *form a Banach frame for* M^p_m *with respect to the sequence space*

$$\ell^p_m := \left\{ (\lambda_{a,b})_{a,b \in \mathbb{Z}^d} : \|\lambda\|^p_{\ell^p_m} := \sum_{a,b \in \mathbb{Z}^d} |\lambda_{a,b}|^p (1 + \|b\|)^{sp} < \infty \right\}.$$

Proof: For this choice of weight function, no further conditions[2] on the analyzing vector ψ are needed. □

Remark: The result is independent of the enumeration of $a, b \in \mathbb{Z}^d$ because the sum in the ℓ^p_m norm converges unconditionally.

[2]See Theorem 3.19 in Dahlke, De Mari, Grohs, Labatte (2015).

Gabor Frames for Time-Frequency Analysis

Remark: Gabor frames (equivalently, the short-time Fourier transform) define a uniform tiling of the time-frequency domain:

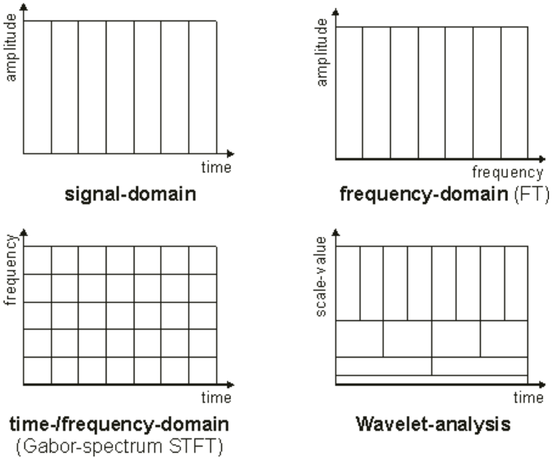

Figure: [www.ndt.net/article/v07n09/08]

Gabor Frames for Time-Frequency Analysis

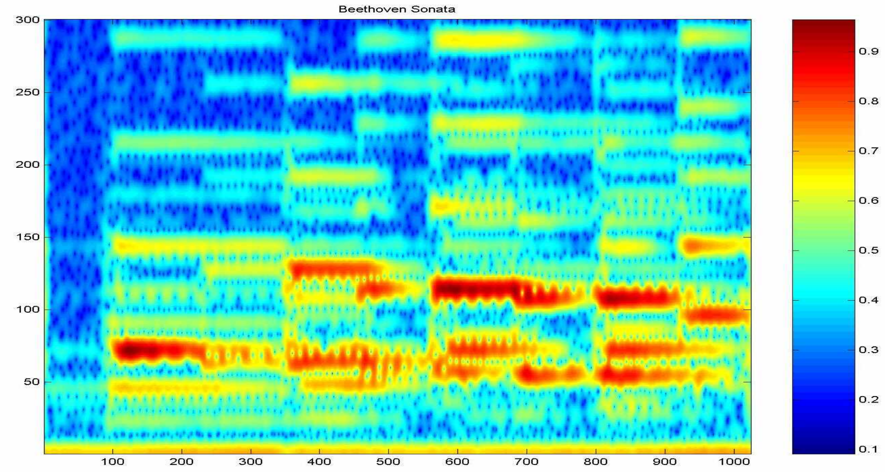

Figure: Intensity (color-coded) of an audio signal, plotted over time (horizontal) and frequency (vertical). [Feichtinger (2015): Wiener Amalgams and Gabor Analysis]

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe the Gabor transform, modulation spaces, and their role in signal analysis.
- Check: Compute the analyzing condition more explicitly. Hint: express the integral da by a convolution and apply the Fourier transform; see [Feichtinger Gröchenig (1988), Section 7.1].
- Background: Read up on the Gabor transform and short-time Fourier transform.

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Week 6, Video 4

Affine Group

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Structure

Definition

The affine group is the set $G := (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ equipped with the product topology and the composition

$$(a', b') \cdot (a, b) := (a'a, a'b + b') \,.$$

Properties:

- This corresponds to the composition of affine maps.
- The affine group is not Abelian.
- The left Haar measure is $\frac{1}{|a|^2} da\, db$, and the right Haar measure is $\frac{1}{|a|} da\, db$, where $da\, db$ denotes the Lebesgue measure on $\mathbb{R}^2$.
- In particular, the group is not unimodular.

Representation

Definition

The affine representation $\pi \colon G \to U(L^2(\mathbb{R}))$ is defined as

$$\pi(b,a)f(y) \coloneqq \frac{1}{\sqrt{|a|}} f\Big(\frac{y-b}{a}\Big), \quad f \in L^2(\mathbb{R}), \quad (b,a) \in G, \quad y \in \mathbb{R}\,.$$

Remark:

- π can be expressed in terms of translation and dilation as

$$\pi(a,b)f = T_b D_a f\,.$$

- The representation π is irreducible and integrable.[1]

[1] Irreducibility fails for the connected subgroup $\mathbb{R}_{>0} \times \mathbb{R}$.

Admissibility

Lemma

The Duflo–Moore operator associated to π is given by

$$Af(\xi) := \frac{\mathcal{F}f(\xi)}{\sqrt{|\xi|}}, \qquad \xi \in \mathbb{R},$$

and is defined for all f in

$$D(A) := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \frac{|\mathcal{F}f(\xi)|^2}{|\xi|}\, d\xi < \infty \right\}.$$

Remark: Thus, a function $\psi \in L^2(\mathbb{R})$ is admissible if and only if it satisfies the Calderón equation[2]

$$\int_{\mathbb{R}} \frac{|\mathcal{F}\psi(\xi)|^2}{|\xi|} d\xi = 1.$$

[2] See [Dahlke e.a., Example 2.48.]

Wavelet Transform

Remark:

- Admissible vectors are called wavelets.
- The wavelet transform is the voice transform of the affine representation.

Definition

For any admissible $\psi \in L^2(\mathbb{R})$, the wavelet transform $V_\psi \colon L^2(\mathbb{R}) \to L^2(G)$ is given by

$$V_\psi f(a,b) := \frac{1}{\sqrt{|a|}} \int_{\mathbb{R}} f(x) \overline{\psi\Big(\frac{x-b}{a}\Big)}\, dx \,.$$

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe the representation of the affine group.
- Background: Read the computation of the Duflo–Moore operator. See [Dahlke e.a. (2015), Example 2.48].
- Check: What goes wrong when the affine group is replaced by the connected subgroup $\mathbb{R}_{>0} \times \mathbb{R}$? Hint: see the computation of the Duflo–Moore operator.
- Check: What goes wrong for affine groups in higher dimension. Hint: see the computation of the Duflo–Moore operator.
- Discussion: Can you think of a sub-group of the affine group which has an integrable representation in higher dimension? Hint: restrict to scalar multiples of orthogonal matrices.

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Week 6, Video 5

Wavelet Spaces

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Analyzing functions

Setting: We consider the representation π of affine group G on $L^2(\mathbb{R})$.

Lemma

Let w be a weight function on G. A function $\psi \in L^2(\mathbb{R})$ is an analyzing vector for w if and only if $\|A\psi\| = 1$ and

$$\int_G |\langle \psi, T_b D_a \psi \rangle| \, w(a,b) \, \frac{da\, db}{|a|^2} < \infty \, .$$

Examples:[1]

- Schwartz functions whose Fourier transform is compactly supported in $\mathbb{R} \setminus \{0\}$ are analyzing for any weight function.
- Compactly supported functions with sufficient smoothness and sufficiently many vanishing moments are analyzing for weight functions of the form $w(a,b) := |a|^s + |a|^{-s}$.

[1] See [Dahlke e.a., Theorems 3.24 and 3.35].

Wavelet Coorbit Spaces

Definition

Let m be a w-moderate weight, and let ψ be an analyzing vector for w. For any $p \in [1, \infty]$, the wavelet coorbit space $H_{p,m}$ consists of all tempered distributions $f \in \mathcal{S}'$ such that

$$\int_G |\langle f, T_b D_a \psi \rangle|^p \, m(a,b)^p \frac{da\, db}{|a|^2} < \infty \,,$$

with the usual modification for $p = \infty$.

Remark:

- This definition is independent of the choice of w and ψ.
- The main example is $m(a,b) = |a|^{-s}$ with $s \in \mathbb{R}$, and in this case $H_{p,m}$ coincides[2] with the homogeneous Besov space $\dot{B}^{s-1/2-1/p}_{p,p}$.

[2]See [Feichtinger Gröchenig 1998] or [Dahlke e.a. 2015]

Wavelet Frames

Theorem

Let $p \in [1, \infty)$, $s \in \mathbb{R}$, $w(a,b) := |a|^s + |a|^{-s}$, and $m(a,b) := |a|^{-s}$. For any w-admissible symmetric ψ subject to some further conditions[3] and sufficiently small $\alpha > 1$ and $\beta > 0$, the vectors $(T_{\alpha^a \beta b} D_{\alpha^a} \psi)_{a,b \in \mathbb{Z}}$ form a Banach frame for $H_{p,m}$ with respect to the sequence space

$$\ell_m^p := \left\{ (\lambda_{a,b})_{a,b\in\mathbb{Z}} : \|\lambda\|_{\ell_m^p}^p := \sum_{a,b\in\mathbb{Z}} |\lambda_{a,b}|^p \alpha^{-asp} < \infty \right\}.$$

Proof: For any given U and sufficiently small $\alpha > 1$ and $\beta > 0$, the sequence $(\epsilon\alpha^a, \epsilon\alpha^a\beta b)_{\epsilon\in\{-1,1\},a\in\mathbb{Z},b\in\mathbb{Z}}$ is U-dense and relatively separated. □

[3] See Theorem 3.19 in Dahlke, De Mari, Grohs, Labatte (2015).

Wavelet Frames for Multi-Resolution Analysis

Remark: Wavelet frames define a non-uniform tiling of the time-frequency domain, which corresponds to fast sampling of high frequencies and slow sampling of low frequencies.

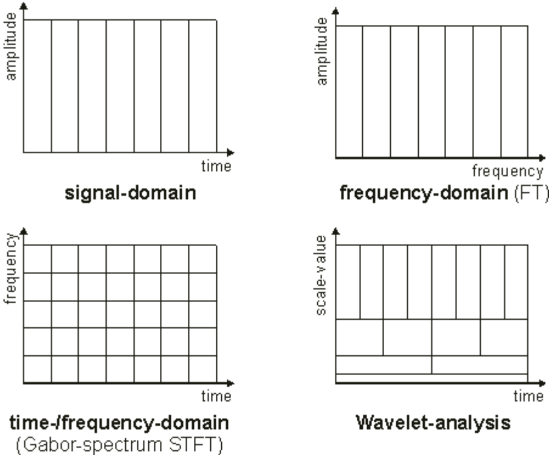

Figure: [www.ndt.net/article/v07n09/08]

Wavelet Frames for Multi-Resolution Analysis

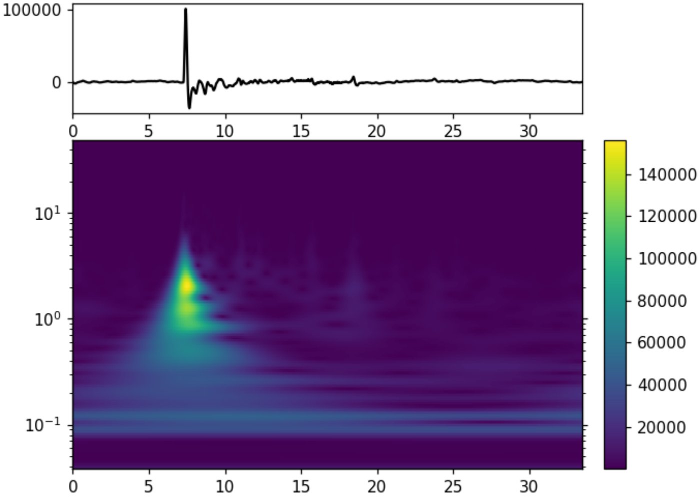

Figure: Top: A seismic signal. Bottom: The signal intensity (color-coded) plotted over time (horizontal) and scale (vertical). From `obspy.org`

Application to Image Analysis

Remark: The JPEG2000 standard uses lossy compression based on Cohen–Daubechies–Feauveau (CDF) wavelets.

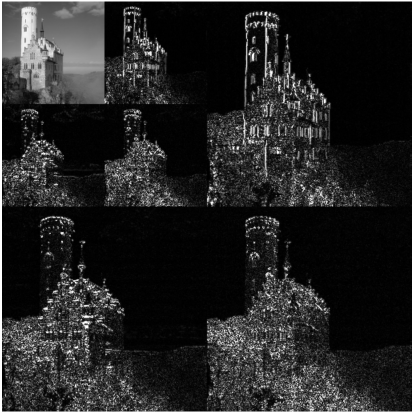

Figure: Wavelet coefficients at scale $a = 1$ (top left), differences to scale $a = 1/2$ (neighboring squares), and differences to scale $a = 1/4$ (neighboring squares).
From en.wikipedia.org/wiki/JPEG_2000

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe wavelet spaces and the wavelet transform.
- Check: Draw the locations of the group elements in the definition of wavelet frames.
- Check: These group elements accumulate near $a = 0$; why are they still relatively separated?
- Check: Verify that $m(a,b) := |a|^s$ is moderate for $w(a,b) := |a|^s + |a|^{-s}$.
- Background: Read up on wavelets and multi-resolution analysis.

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 6

Shearlet Group

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Structure

Notation: For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, let

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \operatorname{sign}(a)\sqrt{|a|} \end{pmatrix} \quad \text{and} \quad S_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

denote the parabolic scaling matrix and the shear matrix, respectively.

Definition

The full shear group is the set $G := \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ equipped with the product topology and the composition

$$(a_1, b_1, t_1) \cdot (a_2, b_2, t_2) := (a_1 a_2, b_1 + b_2\sqrt{|a_1|}, t_1 + S_{b_1} A_{a_1} t_2) \, .$$

Properties:

- The full shearlet group is not Abelian.
- The left Haar measure is given by $|a|^{-3} \, da \, db \, dt$.

Representation

Definition

The shearlet representation $\pi\colon G \to U(L^2(\mathbb{R}^2))$ is defined as

$$\pi(a,b,t)f(x) := |a|^{-\frac{3}{4}} f(A_a^{-1}S_b^{-1}(x-t)),$$

where $f \in L^2(\mathbb{R}^2)$, $(a,b,t) \in G$, and $x \in \mathbb{R}^2$.

Remark:

- It can be written in terms of translations and the left-regular representation of parabolic scaling and shear matrices:

$$\pi(a,b,t)f(y) = T_t L_{S_b A_a} f\,.$$

- The representation π is irreducible and square-integrable.
- However, as an aside, the representation of the reduced shear group $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2$ is reducible.

Admissibility

Lemma

The Duflo–Moore operator associated to π is given by

$$Af(\xi,\eta) := \frac{\mathcal{F}f(\xi,\eta)}{|\xi|}, \qquad (\xi,\eta) \in \mathbb{R}^2,$$

and is defined for all f in

$$D(A) := \left\{ f \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} \frac{|\mathcal{F}f(\xi,\eta)|^2}{|\xi|^2} < \infty \right\}.$$

Remark: Thus, a function $\psi \in L^2(\mathbb{R}^2)$ is admissible if and only if

$$\int_{\mathbb{R}^2} \frac{|\mathcal{F}\psi(\xi,\eta)|^2}{|\xi|^2} d\xi \, d\eta = 1.$$

Shearlet Transform

Remark:

- Admissible vectors are called shearlets.
- The shearlet transform is the voice transform of the shearlet representation.

Definition

For any admissible $\psi \in L^2(\mathbb{R}^2)$, the shearlet transform $V_\psi \colon L^2(\mathbb{R}^2) \to L^2(G)$ is given by

$$V_\psi f(g) = \langle f, \pi_g f \rangle \,.$$

Remark: Generalizations to higher dimensions are possible.

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe the shearlet group and its representation.
- Check: Draw the action of a shear matrix on a rectangle.
- Background: Skim through the computation of the Haar measure and the admissibility condition. Hint: this can be found in [Dahlke e.a. (2015), Lemma 3.27 and Proposition 3.30].

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 7

Shearlet Coorbit Spaces

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Analyzing Functions

Setting: We consider the representation of the shearlet group G on $L^2(\mathbb{R}^2)$.

Examples of analyzing functions:[1]

- Schwartz functions whose Fourier transform is compactly supported in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$ are analyzing for every locally integrable weight function $w(a, b, t) = w(a, b)$.
- Compactly supported functions with sufficient smoothness and sufficiently many vanishing moments are analyzing for weight functions $w(a, b, t) = w(a) = |a|^r + |a|^{-r}$ with $r \in \mathbb{R}$.

[1] See [Dahlke e.a., Theorems 3.33 and 3.35]

Shearlet Coorbit Spaces

Definition

Let m be a w-moderate weight, and let ψ be an analyzing vector for w. For any $p \in [1, \infty]$, the shearlet coorbit space $H_{p,m}$ consists of all tempered distributions $f \in \mathcal{S}'$ such that

$$\int_G |\langle f, \pi_g \psi \rangle|^p \, m(g)^p dg < \infty \,,$$

with the usual modification for $p = \infty$.

Remark:

- This definition is independent of the choice of w and ψ.
- In the most important case $m(a, b, t) = |a|^{-s}$ with $s \in \mathbb{R}$, there are comparison results to Besov spaces.

Shearlet Frames

Theorem

Let $p \in [1,\infty)$, $s \in \mathbb{R}$, $w(a,b,t) = |a|^s + |a|^{-s}$, *and* $m(a,b,t) = |a|^{-s}$. *For suitable*[2] ψ *and sufficiently small* $\alpha > 1$, $\beta > 0$, *and* $\tau > 0$, *the vectors*

$$\Big(\pi_g \psi : g = (\alpha^a, \alpha^{a/2}\beta b, S_{\alpha^{a/2}\beta b} A_{\alpha^a} \tau t)\Big)_{a\in\mathbb{Z}, b\in\mathbb{Z}, t\in\mathbb{Z}}$$

form a Banach frame for $H_{p,m}$ *with respect to the sequence space*

$$\ell^p_m := \Big\{(\lambda_{a,b,t})_{a,b,t\in\mathbb{Z}} : \|\lambda\|^p_{\ell^p_m} := \sum_{a,b,t\in\mathbb{Z}} |\lambda_{a,b,t}|^p \alpha^{-asp} < \infty\Big\}.$$

Proof:[2] For any given U and sufficiently small $\alpha > 1$, $\beta > 0$, and $\tau > 0$, the following group elements are U-dense and relatively separated:

$$(\epsilon\alpha^a, \alpha^{a/2}\beta b, S_{\alpha^{a/2}\beta b} A_{\alpha^a} \tau t)_{\epsilon\in\{-1,1\}, a\in\mathbb{Z}, b\in\mathbb{Z}, t\in\mathbb{Z}}$$

□

[2] See [Dahlke, Theorems 3.36 and 3.38].

Frequency Localization of Shearlet Frames

Remark: ψ is typically chosen as $\mathcal{F}\psi(\xi_1, \xi_2) = \mathcal{F}\psi_1(\xi_1)\,\mathcal{F}\psi_2(\xi_2/\xi_1)$ with $\operatorname{supp}\mathcal{F}\psi_1 \subseteq [-2, -1/2] \cup [1/2, 2]$ and $\operatorname{supp}\mathcal{F}\psi_2 \subseteq [-1, 1]$.

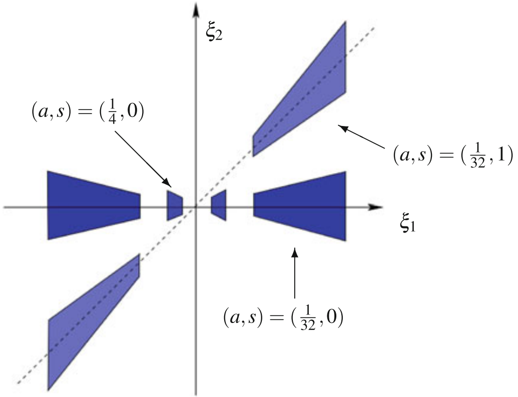

Figure: Support of ψ after scaling by a and shearing by $b := s$. [Dahlke e.a. (2015)]

Shearlet Frames for Edge Detection

Remark: The decay of $V_\psi f(a,b,t)$ for $a \searrow 0$ is

- Fast when t is a regular point of f, and
- Slow when t lies on an edge of f which is normal to $(1,b)$.

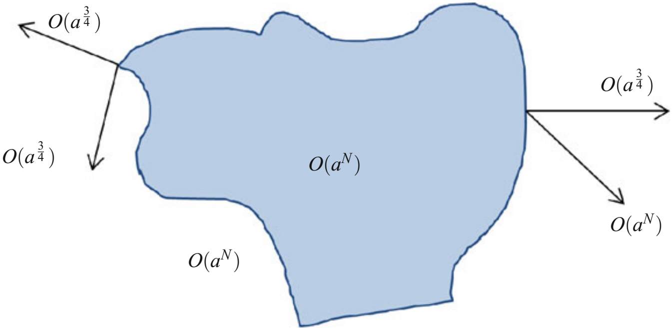

Figure: Indicator function f, points t with attached vectors $(1,b)$, and decay of $V_\psi f(a,b,t)$ for $a \searrow 0$. [Dahlke e.a., 2015]

Shearlet Frames for Edge Detection

Example: edge detection based on shearlet coefficients.

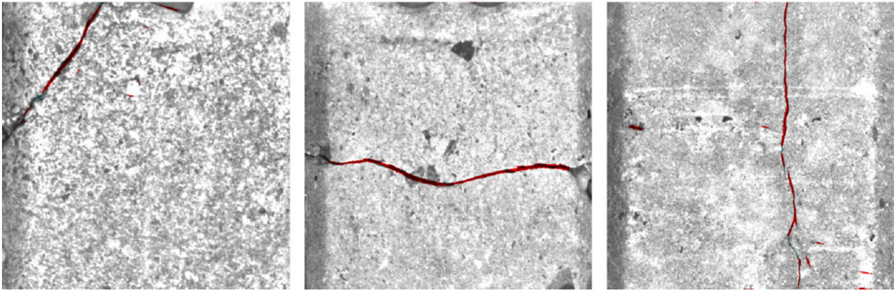

Figure: [Gibert (2014): Discrete Shearlet Transform on GPU with applications in anomaly detection and denoising]

Questions to Answer for Yourself / Discuss with Friends

- Repetition: Describe the construction of shearlet coorbit spaces.
- Check: Draw the locations of the scaling and shearing coefficients of the shearlet frame.
- Discussion: How could one redefine shearlets to achieve symmetry with respect to the horizontal and vertical axes in $\mathbb{R}^2$? Hint: define horizontal and vertical shearlets.
- Discussion: Are shearlets directional wavelets? In what sense?
- Background: Find out about ridgelets and curvelets and compare them to shearlets.

Mathematics of Deep Learning, Summer Term 2020

Week 6, Video 8

Wrapup

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Outlook on this week's discussion and reading session

- Reading:
  - Gröchenig (2001): Foundations of Time-Frequency Analysis
  - Mallat (2009): A Wavelet Tour of Signal Processing
  - Kutyniok and Labate (2012): Shearlets - Multiscale Analysis for Multivariate Data

Summary by learning goals

Having heard this lecture, you can now ...

- Describe Schrödinger, wavelet, and shearlet representations and the associated modulation, wavelet, and shearlet spaces.
- Explain the time-frequency tilings of the associated signal transforms.
- Implement these signal transforms by neural networks.